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The development of algebraic thinking associated with polynomial operations in Mathigon

El desarrollo del pensamiento algebraico asociado a las operaciones polinómicas en Mathigon

Le développement de la pensée algébrique associée aux opérations polynomiales à Mathigon

O desenvolvimento do pensamento algébrico associado às operações polinomiais no Mathigon

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Resumo

Este trabalho, desenvolvido no âmbito do Grupo de Pesquisa Educação Matemática e Tecnologias Digitais – EMATED, tem por objetivo apresentar e analisar o desenvolvimento do pensamento algébrico movimentado na realização de tarefas sobre polinômios e suas operações, usando o recurso Ladrilhos da Álgebra na plataforma *Mathigon*. Para isso, partiu-se da ideia de área de regiões retangulares para a representação de polinômios com grau inferior a três. Foram aplicadas tarefas sobre representação e as quatro operações com polinômios. Os participantes são estudantes da terceira série do Ensino Médio. Usamos o laboratório de informática em um período de seis horas-aulas. Na perspectiva teórica que adotamos nesta pesquisa, compreender o pensamento algébrico pressupõe uma posição epistemológica de natureza histórica. Para tanto, essa base epistemológica descreve três condições caracterizantes desse tipo de pensamento matemático: o objeto, a sua representação simbólica e a analiticidade. Tal

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compreensão sobre o pensamento algébrico e também a Teoria da Objetivação nos deram o suporte epistemológico para a análise dos dados, registrados por áudio e vídeo. Os dados foram analisados tendo como foco os processos de generalização e abstração, presentes no pensamento algébrico. O resultado aponta o desenvolvimento do pensamento algébrico relativo às operações com polinômios tanto de grau dois, exploradas nas tarefas, quanto de grau superior a dois.

Palavras-chave: Pensamento algébrico, Polinômios, Teoria da objetivação.

Abstract

This work, developed within the framework of the Mathematics Education and Digital Technologies Research Group (EMATED), aims to present and analyze the development of algebraic thinking involved in performing tasks on polynomials and their operations, using the Algebra Tiles resource on the Mathigon platform. To do this, the idea of the area of rectangular regions was used to represent polynomials of degree less than three. Tasks on representation and the four operations on polynomials were applied. The participants were students in the third year of secondary school. We used the computer laboratory for six hours. From the theoretical perspective adopted in this research, the understanding of algebraic thinking presupposes an epistemological position of a historical nature. To this end, this epistemological basis describes three conditions that characterize this type of mathematical thinking: the object, its symbolic representation and analyticity. This understanding of algebraic thinking and objectivation theory provided us with the epistemological support for analyzing the data recorded by audio and video. The data was analyzed with a focus on the processes of generalization and abstraction present in algebraic thinking. The result shows the development of algebraic thinking in relation to operations with polynomials of both degree two, as explored in the tasks, and degree greater than two.

Keywords: Algebraic thinking, Polynomials, Theory of objectification.

Resumen

Este trabajo, desarrollado en el marco del Grupo de Investigación en Educación Matemática y Tecnologías Digitales (EMATED), tiene como objetivo presentar y analizar el desarrollo del pensamiento algebraico implicado en la realización de tareas sobre polinomios y sus operaciones, utilizando el recurso Algebra Tiles de la plataforma Mathigon. Para ello, se utilizó la idea del área de regiones rectangulares para representar polinomios de grado inferior a tres. Se realizaron tareas de representación y de las cuatro operaciones con polinomios. Los participantes eran alumnos de tercer curso de secundaria. Las sesiones de trabajo se realizaron en el laboratorio de informática durante un período de seis horas. Desde la perspectiva teórica adoptada en esta investigación, la comprensión del pensamiento algebraico presupone una posición epistemológica de carácter histórico. Desde esta base epistemológica, se describen tres condiciones que caracterizan este tipo de pensamiento matemático: el objeto, su representación simbólica y la analiticidad. Esta comprensión del pensamiento algebraico y de la Teoría de la Objetivación nos proporcionó el soporte epistemológico necesario para analizar los datos grabados en audio y vídeo. Los datos se analizaron centrándose en los procesos de generalización y abstracción presentes en el pensamiento algebraico. El resultado muestra el desarrollo del pensamiento algebraico en relación con las operaciones con polinomios, tanto de grado dos, como las exploradas en las tareas, como de grado mayor que dos.

Palabras clave: Pensamiento algebraico, Polinomios, Teoría de la objetivación

Résumé

Ce travail, développé dans le cadre du Groupe de Recherche sur l'Enseignement des Mathématiques et les Technologies Numériques – EMATED, a pour objectif de présenter et d'analyser le développement de la pensée algébrique dans l'exécution de tâches sur les polynômes et leurs opérations à l'aide de la ressource *Algebra Tiles* sur la plateforme *Mathigon*. À cette fin, l'idée de l'aire des régions rectangulaires a été associée à la représentation de polynômes, avec un degré inférieur à trois. Les tâches sur la représentation et les quatre opérations avec des polynômes ont été appliquées. Les participants sont des élèves de troisième année du secondaire. Nous avons utilisé le laboratoire informatique pendant une période de six heures de cours. Dans la perspective théorique que nous adoptons dans cette recherche, la compréhension de la pensée algébrique présuppose une position épistémologique de nature historique. À cette fin, cette base épistémologique décrit trois conditions qui caractérisent ce type de pensée mathématique : l'objet, sa représentation symbolique et l'analyticité. Une telle compréhension de la pensée algébrique, ainsi que la théorie de l'objectivation, nous ont fourni le support épistémologique pour l'analyse des données, enregistrées par audio et vidéo. Les données ont été analysées en se concentrant sur les processus de généralisation et d'abstraction, présents dans la pensée algébrique. Une telle analyse indique le développement de la pensée algébrique liée aux opérations avec des polynômes, à la fois de degré deux, explorés dans les tâches, et de degré supérieur à deux.

Mots-clés : Pensée algébrique, Polynômes, Théorie de l'objectivation.

The Development of Algebraic Thinking Associated with Polynomial Operations in Mathigon

The development of Digital Technologies (DT) has brought about significant changes in society, mainly of a cultural nature and in the relationship with work. Their use in everyday life has become useful for communication and the expression of thought. Consequently, the ways of teaching and learning add to and are impacted by DT, since teaching and learning environments are places where culturally historical knowledge meets social and ethical reflection, with the aim of humanistic and Omni lateral education. This human development involves the conscious and non-alienated use of DM. In this configuration, we understand that DM is a human product incorporated into our society, becoming an artifact with historical voices, going beyond the materialization of thought, but making it "thought-with-and-through-artifacts" (Radford, 2011a, p. 324).

In fact, mathematical thinking is a mediated reflexive social praxis, in which the individual's cognitive processes are related to the meanings attributed to the objects of knowledge. These processes deepen procedurally until the individual reaches an understanding of concepts. During this process, DM can act as a tool to signify abstract mathematical content, such as polynomials.

Lautenschlager and Ribeiro point to a reduction, throughout history, in the emphasis on topics related to the content of polynomials in basic education. This represents a loss, since working with this knowledge can develop forms of algebraic generalization and deepen important mathematical concepts, such as the Fundamental Theorem of Algebra:

> This is a worrying fact, given that the macro-assessments show that most students have great difficulty in understanding not only the concepts, definitions, theorems and applications involving polynomials, but also the factoring process and its relationship with the roots of polynomials (Lautenschlager & Ribeiro, 2017, p. 238–239).

Through a literature review using the Buscad tool[4], it is possible to conclude that the same reduction manifests itself in Brazilian academic papers that relate the content of polynomials to algebraic thinking, as can be seen in Table 1. It shows the number of studies found on different platforms using the descriptors "algebraic thinking" and (AND) "polynomials".

[4] A search tool for scientific papers developed by Daniel Redinz Mansur in collaboration with Renan Altoé. Mansur and Altoé. Available at https://bit.ly/buscad_form.

Table 1.

Counting academic papers (authors using the BUSCAd spreadsheet)

Combination of Descriptors	Platforms					
	Capes: T&D	Scielo	Journals Capes	DOAJ	BDTD	Total
‘Algebraic Thinking' AND Polynomials	1	0	3	1	1	6

In the hope of making the study of polynomials a sensory experience, the Mathematics Education and Digital Technologies (EMATED) research group studied the Mathigon platform[5]as an artifact for teaching and learning this content. This article is the product of this challenge and aims **to present and analyze the development of algebraic thinking involved in performing tasks on polynomials and their operations using the Algebra Tiles resource on the Mathigon platform**. The tasks were formulated from the perspective of the teaching-learning activity of the Objectivation Theory, in the intervention for learning recovery, in the mathematics classroom, of a third-grade high school class. It should be noted that the content of polynomials is included in the curriculum for the third year of secondary school in the school where the research was carried out. We found that the students had little or no knowledge of polynomials. What's more, they had difficulty working with algebraic thinking.

In order to characterize this type of thinking, we considered Radford's perspective and Theory of Objectification, which in this article is the epistemological basis and analytical tool for the data constructed in the investigation. The principles of this theory are presented below.

Theory of Objectification and Algebraic Thinking

Theory of Objectification - TO - originated in a movement that began in Mathematics Education in the 90s. According to Luis Radford (2021b), author of the theory, it is based on culture as the main influence on the formation of the subject and their understanding of the world:

> Theory of Objectification is situated in a different educational project: it sees the goal of mathematics education as a political, social, historical and cultural endeavor aimed at the dialectical creation of reflective and ethical subjects who position themselves critically in historically and culturally constituted mathematical discourses and practices, and who consider new possibilities for action and thought (Radford, 2021b, p. 38).

[5] Mathigon, created by Mathigon Ltd in 2009, is an interactive learning platform for mathematics and can be accessed at: https: //mathigon.org.

For TO, teaching and learning are inseparable processes that achieve knowledge and the formation of subjects, in other words, they work on the dimension of knowing and being. In this way, learning is understood as a social process in which students encounter historically constructed forms of action and thought in a given culture. When they encounter these forms, they are transformed. In other words,

> are the active, embodied, discursive, symbolic and material processes through which students encounter, notice and critically familiarize themselves with culturally and historically constituted systems of thought, reflection and action. In this encounter, students are confronted with the unknown: the other. This encounter is felt as something that objectifies (etymologically speaking, something that is set against or opposes) the Objectification, in TO, is a dialectical, transformative and creative process between subject and object that influences each other and goes beyond the updating of knowledge and reaches the transformation of being. In this way, it can be understood that math classrooms not only produce but also reproduce knowledge (Radford, 2021b, p. 61).

TO therefore understands that objectification is a process linked to activity. Activity, by affecting being, actualizes it in a continuous movement. In this movement, in the context of the classroom, students and professors operate in the form of working together, shoulder-to-shoulder, as subjects who affect each other and collaborate, through mutual cooperation, solidarity, collective responsibility and ethics.

Ethics, in TO, takes on a leading role, and its concept is community ethics, based on the reflexive and critical constitutions of what Marx (2004, p.110) classified as human capacities. In this logic, Radford (2020, p. 35) proposes a concept centered on responsibility, commitment and care for others, and states that these three vectors make up the essential structure of subjectivation.

The processes of subjectivation refer to the transformations that subject undergo in the encounter with the cultural objects, where the focus is not on the cognitive processes of learning (objectivation) but on how the subject has been affected in the social dimension. In short, the processes of objectification update knowledge, while those of subjectification update being. Both happen simultaneously, dialectically and inseparably through activity.

In a collaborative movement between professors and students, activity and the vectors of community ethics form the ontological unity of TO, called joint labor. The learning of new knowledge takes place through activity in the classroom, in a non-alienating way from historically constructed life, in which matter, body, movements, signs and artifacts bring out activity as joint labor. Radford explains:

> In order to meet their needs (survival needs and artistic, spiritual and other needs created by/in society), human beings actively launch themselves into the world. They expose themselves and, by exposing themselves, they produce. What they produce to satisfy their needs is produced in a social process that is both the process of inscribing individuals in the social world and the production of their own existence. The name of this process [...] I have called it joint labor (Radford, 2020, p. 23).

In TO, thought is considered to be "a mediated reflection according to the form or mode of activity of individuals" (Radford, 2011a, p. 316). Mediation takes place through activity in the realization of social practices. In this dynamic, artifacts are constitutive and consubstantial parts of thought - we think with and through cultural artifacts (Radford, 2011a, p. 261). In other words, for the author, artifacts are an integral part of human thought and activity. Specifically in mathematics education, the author states that:

> Artifacts can no longer be considered as a means to access mathematical objects and mathematical forms of reasoning, as these are not conceived of as transcendental entities. Artifacts, rather, are considered part of mathematics as material practice. Within this context, mathematics appears as a collective activity, spatially situated, which unfolds in a certain span of time, where the historical voices embedded in artifacts and the voices of students and teachers merge. Let us note, *en passant*, that in this perspective, the discussions about mathematical proofs assisted by computers (Devlin, 1992) take a different turn. The computer is not helping the mathematician carry out some calculations. Both become part of one chorus singing a polyphonic song. (Radford, 2012, p. 287).

From this TO perspective, we planned and presented a teaching-learning activity for a group of third-year high school students. This activity focused on polynomials and their operations, in a more concrete and materialized way using Mathigon, making the students think, develop their hypotheses and conclusions about this content.

In this context, algebraic thinking stands out, which, like scientific and mathematical thinking in general, originates from the combination of various social processes in a movement of production, modification and reconstruction in the various interactions in which it occurs. Thought, and mathematical thought in particular, is a factor that emerges in the interaction between the individual and society, in order to construct, modify and reconstruct the thoughts of the individual, others and society.

Specifically with regard to algebraic thinking, Radford (2006) states that human beings do not appropriate this cognitive process naturally and that appropriation does not depend on genetic maturation. The author points out that "algebraic thinking is a very sophisticated type of cultural reflection and action, a way of thinking that has been refined successively over centuries before reaching its current form" (Radford, 2011b, p. 319).

From a theoretical perspective, algebraic thinking is characterized not only by the nature of the magnitude, but also by the type of reasoning that is done with the magnitude. More specifically, three conditions characterize this thinking: (i) indeterminacy of quantities - indeterminate quantities, such as unknowns, variables and parameters used in problem-solving, we will refer to these quantities simply as indeterminate; (ii) denotation - the use of signs, gestures, natural language or/and a mixture of all to symbolize indeterminate quantities; and (iii) analytical - the manipulation of unknown quantities, i.e., thinking algebraically makes it possible to operate deductively, even if an indeterminate quantity is not known.

For Radford (2021a), when thinking algebraically, students reason with determinate and indeterminate quantities, signifying relationships between both types of quantity, treating unknown quantities as if they were known, operating with other quantities. In doing so, they re-conceptualize the operations involved. These actions of re-conceptualizing reveal another important factor present in algebraic thinking, the semiotic system, because it configures the way of signifying. In TO, the author uses Vygotsky's reframing of the semiotic system, which conceptualizes signs as tools for reflection that allow individuals to organize thoughts and behaviors, plan actions, communicate and express, giving meaning to objects of cultural knowledge.

According to Radford (2003, p. 41), the objectification of mathematical objects is associated with the subjects' mediated and reflective efforts to achieve the goal of their activity. To do this, subjects use a range of means, such as manipulating concrete objects, drawings/schemes, gestures and linguistic categories, analogies and metaphors. In short, in order to achieve the objective, the subjects involved articulate various tools, signs and linguistic devices through which they organize their actions in space and time. These objects, tools, linguistic devices and signs, used intentionally in processes of constructing social meanings to achieve a stable form of consciousness, make their intentions apparent and carry out their actions to achieve the goal of their activities, are defined in TO as semiotic means of objectification.

Another algebraic element that addresses the semiotic dimension in TO is generalization. The author follows the line of Cultural-Historical Theory, pointing out that “since concepts do not derive from logical rules - as suggested by rationalism - nor from external impressions - as suggested by empiricism - the origin of all concepts, Vygotsky argued, is to be found in generalizations” (Radford, 2008, p. 83).

From this perspective, generalization is the ability to understand similarities in a particular and extend the logic of those similarities to subsequent concepts or identities, being able to use common elements to provide a direct expression of any concept or identity.

Radford (2008) categorizes three types of generalization and examines them in terms of the semiotic means of objectification that students use in mathematical generalization processes. These are: factual generalization, contextual generalization, and symbolic generalization. The first type, factual generalization, has reasoning linked to the concrete, is characterized by perception, feelings, and spatial and temporal elements of the student's physical environment or problem, and is demonstrated by gestures, concrete objects, natural and embodied language, as well as linguistic elements related to a current situation/enunciation. Contextual generalization is characterized by the introduction of the semiotic system. In other words, the student still uses material sensitivity, but in a way that is mediated by the use of signs, introducing the first elements of symbolic language. Symbolic generalization, on the other hand, no longer relies on reasoning embedded in the spatial and temporal elements of the environment or the problem/statement. In this category, thinking is about the object, is located in abstract and general space, and involves the use of semiotic systems.

Finally, the author highlights one of the most important cognitive elements in the formation of mathematical concepts, abstraction, and states that this process is not instantaneous. Moreover, it is what allows us to go beyond a few particular cases towards something more general. Thus, he defines

> Abstraction is a process. During this process, the student mobilizes ideas already acquired and arrives, using language, symbols and cultural artifacts, to make connections that they did not make before and, therefore, constitutes a new idea. From the point of view of teaching and learning mathematics, the question is to determine the didactic actions that allow students to engage in processes of abstraction (Radford et al., 2009, p. 7).

According to Radford, Demers and Miranda (2009), mathematical abstractions start from a concrete sensory experience to create general categories. These categories are then quickly related not to concrete objects, but to symbols that represent them, which are then concatenated, giving rise to new abstractions in a continuous process. In search of this concrete sensory experience, we saw the potential for working with polynomials using the Mathigon artifact.

Methodology

This research is qualitative in nature and was developed within the collective of professors - researchers from the Mathematics Education and Digital Technologies Research Group - EMATED. This research group is made up of professors linked to the Postgraduate Program in Science and Mathematics Education - Educimat, at the Federal Institute of Espírito Santo, master's and doctoral students from this program, undergraduate students and mathematics professors and pedagogues working in the state and federal spheres.

This article presents a study with third grade high school students on forms of algebraic thinking, through the content of polynomials and their operations, using the potential of the Mathigon platform as an artifact. In this context, we designed an intervention with tasks[6] in which the Algebra Tiles[7] resource is a semiotic system for signifying polynomials.

The intervention was necessary because the students had difficulties with the meaning of polynomials and the generalizations of the operations for a polynomial expression of degree n. Another difficulty was working with polynomials with more than one indeterminate, for example expressions of the $p(x,y) = x^2y - 3xy + xy^2$,, which are used to model problems in technical logistics subjects.

The field research took place in the computer lab room, with a group of six students from the third year of the technical course in Logistics integrated into secondary education at the Federal Institute of Espírito Santo - advanced campus of Viana and three professor-researchers, the authors of this article, over a period of five lessons, each lasting fifty minutes, organized into three meetings. The classes were the environment in which the data was produced, recorded in audio and video format, thus constituting the research's unit of analysis. To maintain consistency, the analysis of the data produced went through processes of "apprehension of reality". This was the procedure, since

> to overcome both the empirical immediacy of the phenomenon (its immediate singular condition) and its abstract genericity (its formal generic condition), it is necessary to apprehend the phenomenon in its constant and objective movement between these singular and general traits that constitute it (Araujo & Moraes, 2017, p. 61).

This double view, based on the singular and general dimensions, determines the "apprehension of reality". From this perspective, the entity is constituted as part of the complex whole. It is "the product of analysis which, unlike the elements, retains all the fundamental

[6] The tasks can be accessed in full at http://educapes.capes.gov.br/handle/capes/746183.
[7] Available at https://polypad.amplify.com/p#algebra-tiles

properties of the whole and cannot be divided without losing them" (Vygotsky, 2002, p. 11). In this way, we tried to use units to grasp the totality of the phenomena we were studying - the encounter with algebraic thinking.

In order to process the data, we structured two categories of analysis based on Objectivation Theory, specifically the characterization of semiotics and joint labor as ontological elements of the teaching-learning activity. Semiotics refers to the recognition of the semiotic structure for the representation of polynomials, the use of this structure for polynomial operations and the adoption of meanings in the process of communication, expression and teaching and learning by students and professors. Labor Conjunto refers to the perception of characterizations of activity with algebraic thinking, in the form of materialization of processes of generalization and abstraction, through the use of the artifact, in dynamic systems of cooperation and community ethics between students and professors. Below we present Mathigon, the tasks performed, the interactions during the tasks and our analysis of the data produced.

Studying polynomials using *Mathigon*

Launched in 2009 by *Mathigon* Ltd, *Mathigon* is, according to the developers, an interactive math-learning platform that promises to make learning more personalized. One of the tools available in *Mathigon* is the *Polypad*. This tool offers resources related to geometry, numbers, fractions, algebra, probability and data, as well as games and applications. In addition to this tool, the platform also offers courses on mathematics content for the last years of primary and secondary school; activities - tasks on various topics; and lessons - puzzles, activities, and lesson plans.

According to Takinaga and Manrique (2023, p. 197), "the position that the task occupies in the structure of the teaching-learning activity brings it in line with the objectives of the activity" and outlines three conceptual levels of the tasks, aiming at a gradual and progressive encounter with cultural-historical knowledge:

- The first level is associated with a concrete sensory experience, i.e. experimentation and reflection through the use of concrete materials;
- The second level of conceptualization involves a theoretical reflection based on the use of concrete objects that could highlight possible emergent links that give meaning to mathematical objects;
- The third level of conceptualization appears with the manipulation of mathematical symbols with which students elevate previous experience (sensory,

concrete experience) to another level of consciousness (Takinaga & Manrique, 2023, p. 198-199).

The task proposal we are presenting develops the three levels indicated by the teaching-learning activity. At the first level, the Algebra Tiles, from the *Mathigon* platform, are artifacts of this sensory and concrete experience, which can be linked to a meaning (Figure 1). From there, the second level is reached, in joint work (professors and students together), to characterize mathematical objects (Figure 2). Finally, there is an expansion to broader and more formal concepts (Figure 3).

Figure 1

Representation of polynomials with Algebra Tiles

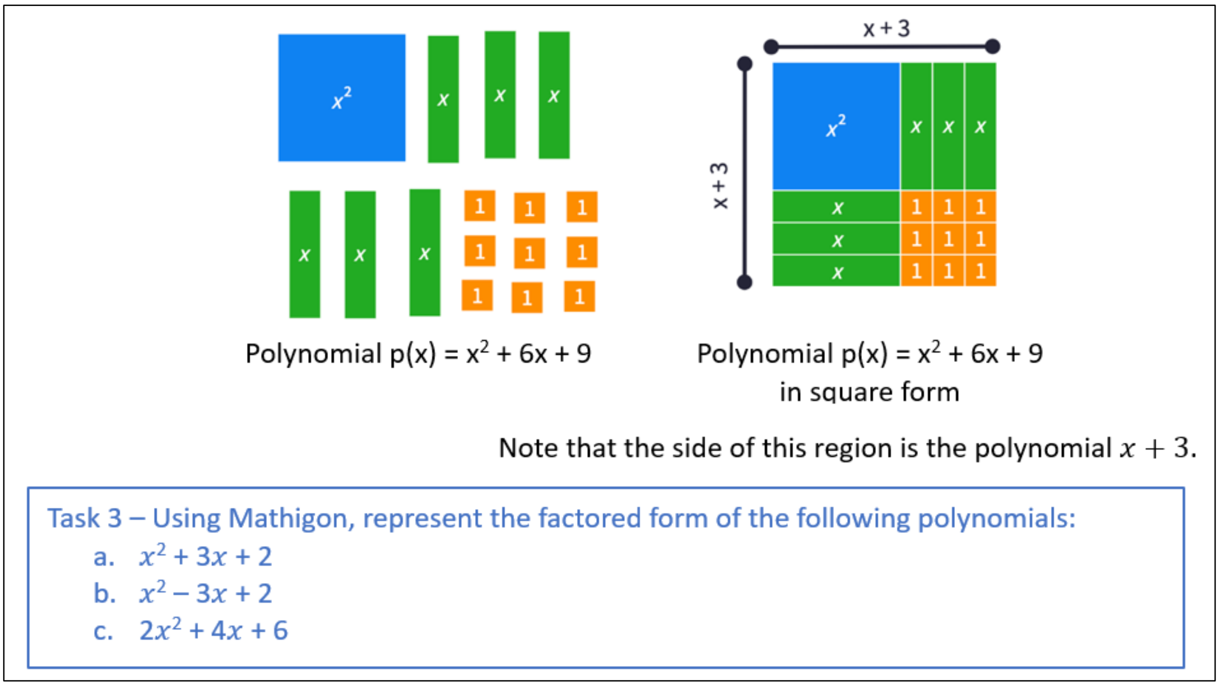

Figure 2

Cutout of the prepared material

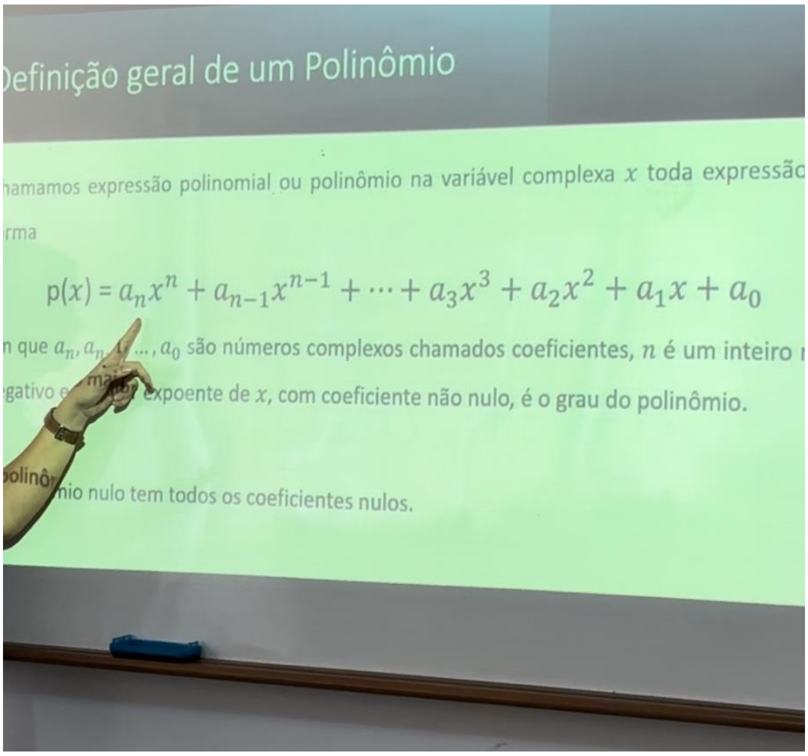

Figure 3

General discussion about a generalization of a polynomial expression

In our assignments, we explored the set of algebra tiles that can be found in the algebra resources. In particular, we believe that the platform offers a great potential for developing tasks on different mathematical content, as long as professors and students produce them jointly.

With this in mind, we built a set of tasks using Objectivation Theory on the content of polynomials and their operations. The tasks were divided into five stages: (i) representing polynomials with the algebra tiles, (ii) factoring, (iii) addition and subtraction, (iv) multiplication, and (v) division of polynomials based on the representations with the tiles. All the work was done by organizing the students into small discussion groups so that each group used a computer to perform the tasks.

The representations and operations were limited to polynomials of degree less than three, since they are associated with the concept of area. Therefore, the goal of the study was to show students *how to understand polynomial operations with polynomials of degree less than three using Mathigon's* Algebra Tiles feature. However, to the researchers' surprise, it was possible to extend the understanding to polynomials of degree greater than two.

Mathigon's Algebra Tiles group is made up of indeterminate rectangular regions that can be dragged into the construction region and operated on (editing the legend and colors, partitioning, etc.), highlighted in Figure 4.

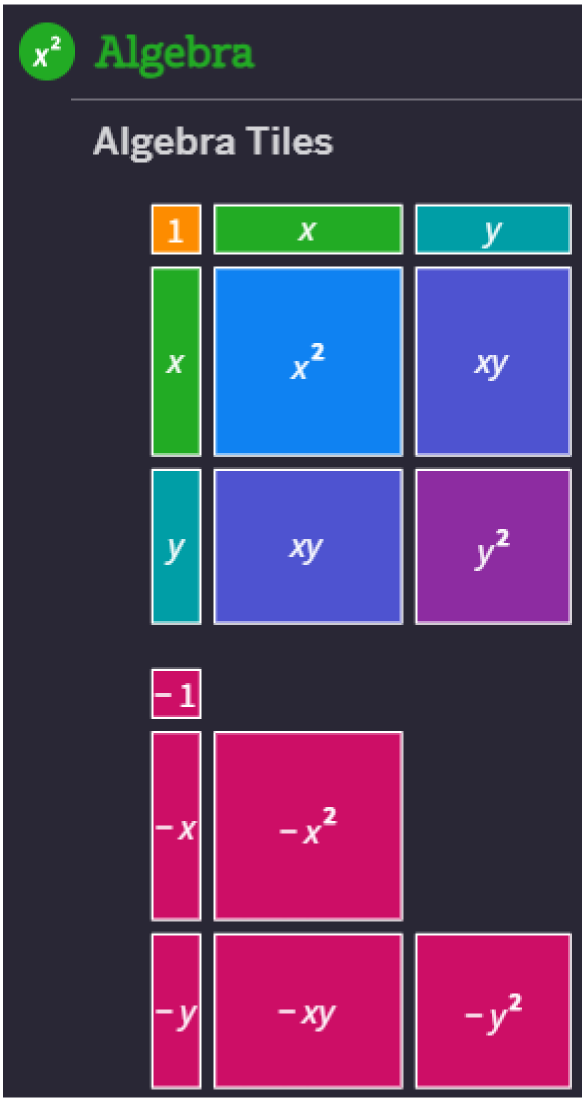

Figure 4.
Algebra tiles (Mathigon)

To construct the signs, we presented Figure 5 to the students, with the rectangular regions and their respective markers depicted on the inside of the regions, representing their unit areas.

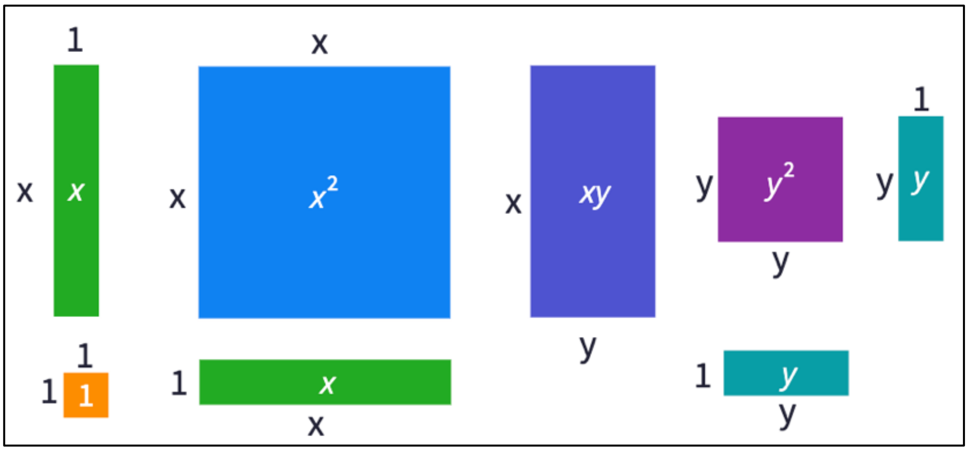

Figure 5.

Signs and meanings of the tiles

In addition to these blocks, we can represent negative blocks as the negative of the areas, i.e. the area times -1. The x and y measures are indeterminate and can be changed using sliders, but we would point out to the students that they would be working with unknown length measures. In this sense, we can see the presence of the indeterminate, an element that characterizes algebraic thinking, according to Radford (2021a). Also, a semiotic system that operates both the denotation of the indeterminate (x and y) and the adoption of concrete symbologies (rectangular regions) to signify the areas. In our experience, the negative forms did not represent an epistemological obstacle for the students, as they easily understood the algebraic forms linked to the semiotic system and not to the areas.

The following is a transcript of the dialog and gestures during the task of representing polynomials, when the professor was visiting the pair of students Ême and Jony[8] .

With the re-signification of the regions, the students were asked to represent the polynomial $p_1(x) = 4x + 6$ and $p_2(x) = x^2 + 3x$, according to intuition, considering only the established system of signs (Figure 6).

[8] The names of the students are fictitious in order to preserve their identities.

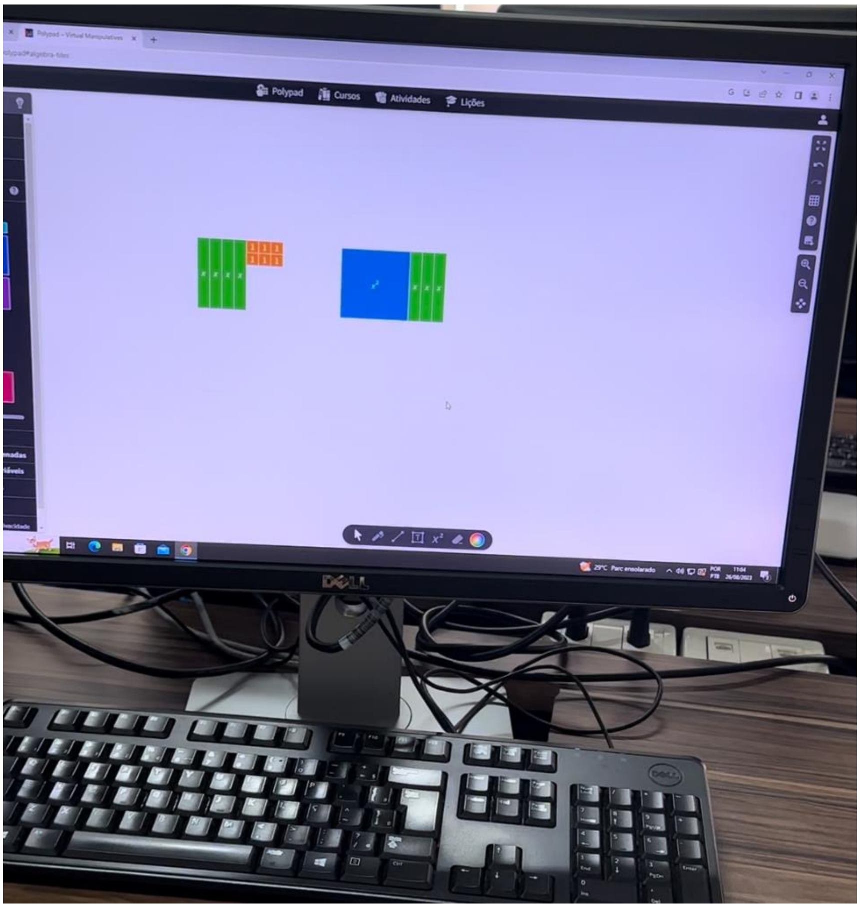

Figure 6.

Representation of polynomia

Figure 6 shows the student using his finger to show the representation of polynomials to the professor who was visiting the group. Their conversation is transcribed below.

Prof. Al: Can you explain to me why you represented the polynomial like that?

Student Ême: When we take $4x + 6$ we put four green rectangles and six orange 'ones', while for $x^2 + 3x$, we put one blue square, which is x^2 and three green [rectangles].

Prof. Al: But why did you put these charts and rectangles together, stuck next to each other?

Student Ême: Yeah... (looking at Jony) Why Jony?

Student Jony: Because it's a sum, look (with his finger he points to the paper with the description of the polynomial in the form $p_1(x) = 4x + 6$. Then we put it together. This part is $4x$ (points to the green rectangles) and this part is 6 (points to the orange rectangles).

StudentÊme: So this sum gives $4x^2$? (points to the representation of the polynomial $p_2(x) = x^2 + 3x$).

Prof. Al: Do you think we can add up the blue and green regions? (The students were silent for a while). Why did you represent the $3x$ part with three green regions?

Student Jony: Because they are 1 times x, which gives x. So we took 3 green rectangles, which gives x plus x plus x, $3x$.

Prof. Al: And why did you take a blue square?

StudentÊme: Because it's x times x, which is x^2. (pause) Oh, I see. You can't put them together, because one is 1 times x and the other is x times x. They're different.

Prof. Al: That's right! Very good. This refers to the degree of the polynomial, which is why we can't add them together (the students nod, validating the professor's summary).

Regarding the semiotic representation, we see in line (2) thatÊme operates with the semiotic system, using colors and signs (x) as a system of communication and thought for the representation of the polynomial. In this state, the tiles are part of the thought for the materialization of knowledge. We can see that the semiotic resources of the tiles play a crucial role in learning, asÊme thinks with and through the artifact.

However, in lines (4) and (6) we see thatÊme is still unable to grasp the totality of the representation to which the task is directed. She still doesn't understand that when she connects the regions, the sum that Jony explains is $4x$ and 6, and she ends up assuming that this sum is a sum of regions.

Jony doesn't understand, but she still can't formalize a verbal answer. Professor Al practices community ethics by staying in the process and inviting reflection. In this way, there are signs of activity between lines (7) and (10), as Al doesn't give the answer and doesn't take the product of thinking for himself. Al leads in a way that collaborates withÊme and Jony, taking responsibility for the students as he participates in their process of objectification.

In this collective moment, we can see signs of the process of objectification with algebraic thinking.Ême generalized because she understood the similarities in a particular region - the green regions (line (2)), and extended the logic to the region represented by the area x^2. ('blue square').

In line (10), we notice thatÊme shows signs of the process of objectifying the abstraction, which is still factual (associated with an element from her context), when she explains that she can't add the blue region (x^2) to the region represented by the three green rectangles ($3x$) because the dimensions are different, and when she agrees with the professor's synthesis.

Given the space available for this article, we will not present here the analysis of the data related to the operations of addition, subtraction and division, but we will now present an analysis of the data related to the operation of multiplying polynomials.

Polynomial multiplication using Algebra Tiles in *Mathigon*

First of all, it is necessary to clarify that the students understood addition and subtraction using the semiotic form of the tiles in Mathigon. This includes the idea of canceling areas, which consists of overlapping tiles of the same area with opposite signs, as illustrated in Figure 7.

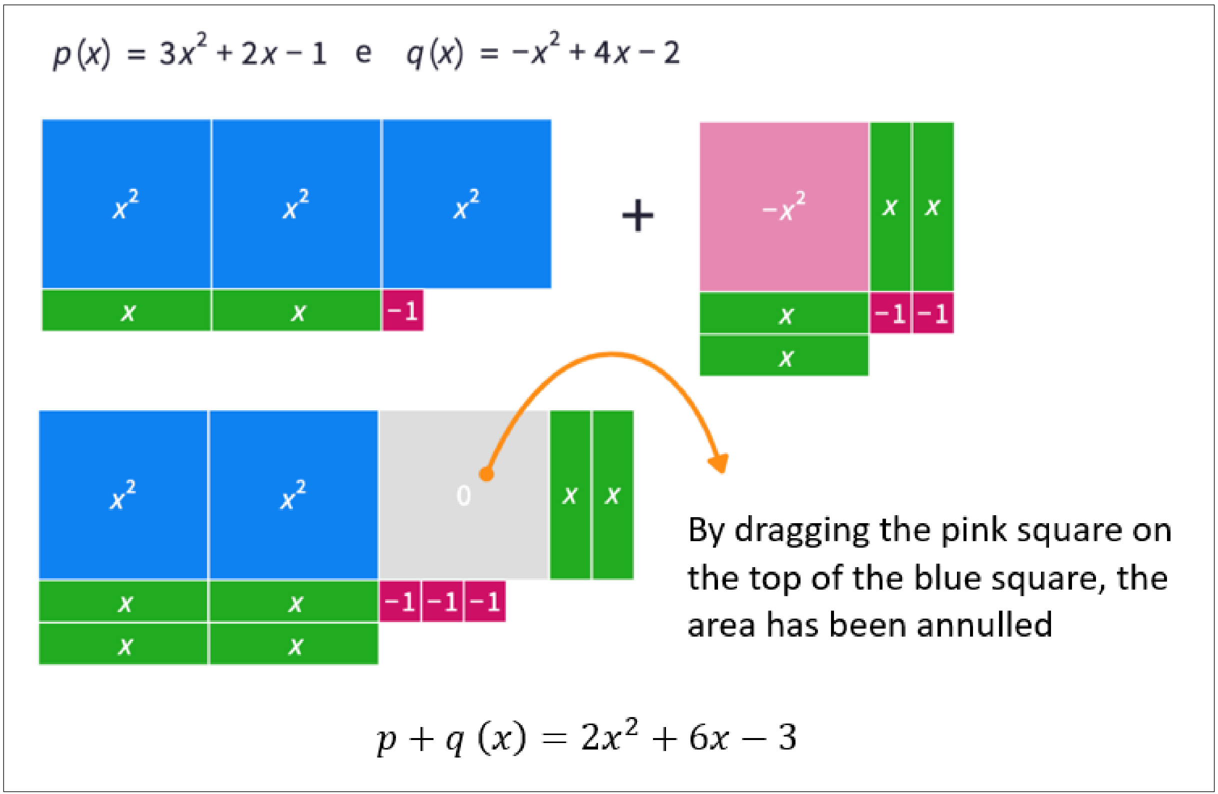

Figure 7.

Overlapping areas annulled

The data to be analyzed on the polynomial product refers to the following task:

Using the Mathigon algebra tiles, calculate the products by means of the geometric representations.

a. $(\boldsymbol{x} - \mathbf{3}) \cdot (\boldsymbol{x} - \mathbf{2})$
b. $(\boldsymbol{x} - \boldsymbol{y})^{\mathbf{2}}$

The dialog takes place in a group of four students, for whom we have chosen fictitious names, with the consent of the individuals. They are: Amora, Duda, Tetê and Sara.

(12) Amora: The area is side by side, so do this one here (indicating with your fingers, horizontally across the screen, the polynomial $x - 3$) and the second one, here (again indicating with your fingers on the screen, this time vertically, the polynomial $x - 2$).

(13) Sarah: I don't understand.

(14) Amora: Because the polynomial isn't a rectangle?

(15) Tetê: Remember how we represented it before? It was a rectangle, remember?

(16) Duda: The coffee in the canteen is R$2.00, I drink a lot of coffee, old man.

(17) Sarah: Ah Duda, pay attention. I remember. It's the area, that's why you want to put the sides and then this is already the result of the area? (points to the writing of the task on the paper).

They nod their heads, except for Duda. Then Sara, controlling the mouse, builds the following representation (Figure 8):

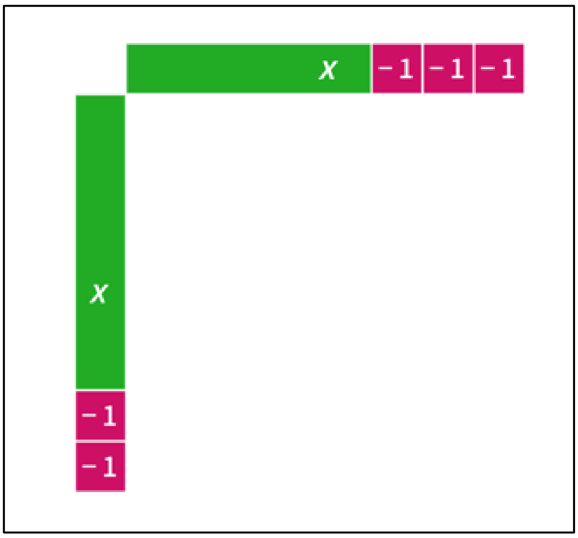

Figure 8.

Geometric formation of the operation of multiplying polynomials

In the representation of the $x - 3$ polynomial, the students placed the three -1 tiles on top of the x tile. The same goes for the $x - 2$ polynomial, on the left-hand side. The students went on to complete the rectangular region, as shown in Figure 9.

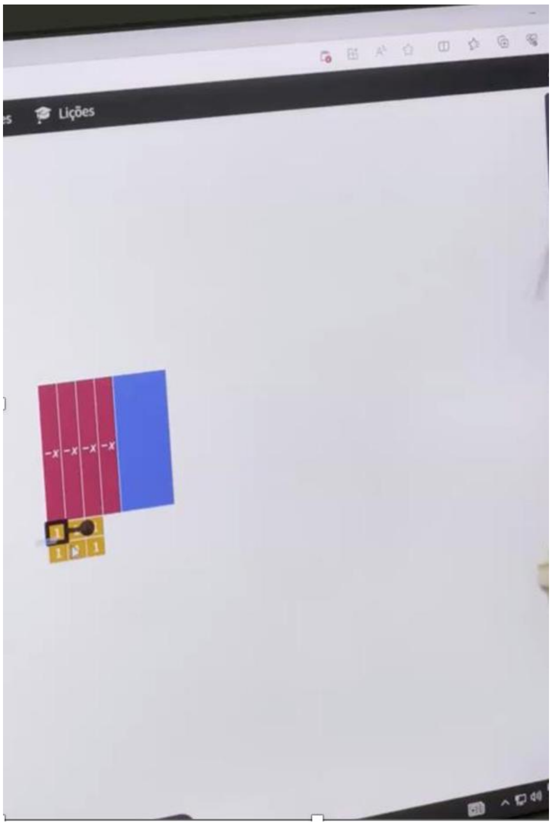

Figure 9.

Representation of the product

(18) Professor Al: Duda, can you explain what your group has created in this figure? (Duda is startled and shows attention to task)

(19) Duda: You put $x - 3$ here and $x - 2$ here. (He points to the tile representations of the polynomials on the screen). Then they put the blue [square] on top, you know? x times x? And those red [rectangles] there.

(20) Professor Al: And why like that? Can you interpret that for me?

(21) Duda: I don't know. I think it's wrong. In the area x times 1, there's a hole of -3. This piece is $x - 3$ (with his finger on the screen, he shows the horizontal measurement for the green part in Figure 8).

(22) Amora: What's wrong, Duda?

(23) Duda: I think you should make these squares outside the green [rectangle].

(24) Professor Al: I also think it's easier, girls.

(25) Sara, Tetê and Amora: But you said that if it's negative, one part cancels the other...it makes a hole...and there's no area.

(26) Professor Al: That's right, but in the representation. I don't know if you can do multiplication that way. I haven't tried.

The students then follow the form that the professor pointed out and complete the rectangle by calculating the polynomial product, as shown in Figure 7.

(27) Professor Al: Try to think, even though it's a hole, try to think of the idea of the rectangle by not putting the red in the blue.

(28) Sara: By not putting it in?

The students follow the professor's instructions and assemble the rectangle, realizing the result of the polynomial product (Figure 10).

Figure 10.

Solving the task

(29) Professor Al: The way you were thinking... using the idea of the hole... (overlapping areas) we'll do it on the graph and then everybody will see it, you'll help.

The professor goes to the graph to set up the polynomial product using the idea of overlapping. At first, the plot didn't work (Figure 8 - left image). However, the professor and the students insisted on the idea, using the representation in Figure 10, but with overlapping layers (Figure 11 - right image).

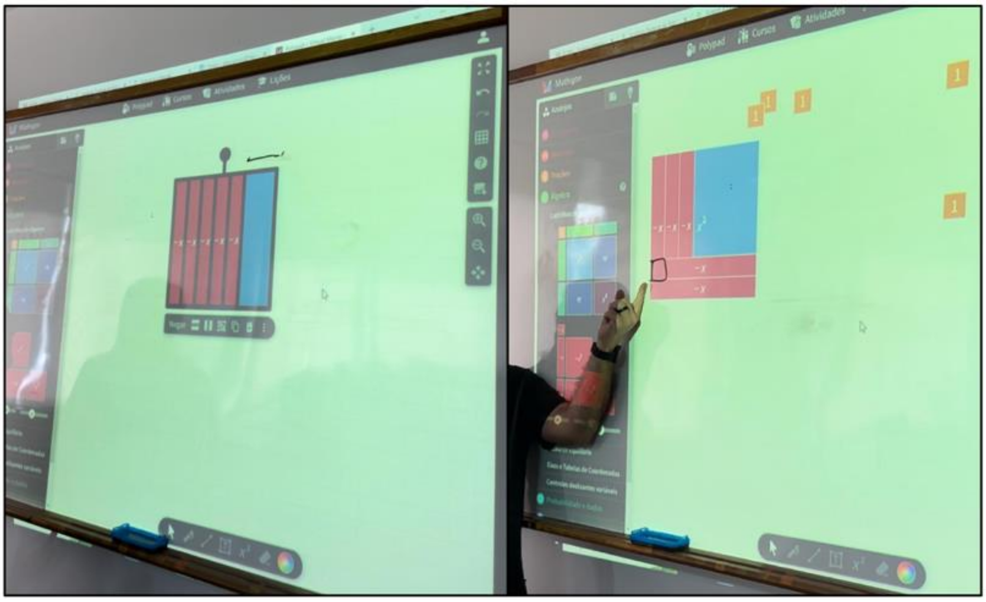

Figure 11.

Calculating the polynomial product using over position

(30) Professor Al: In this little square, we have the blue part of x^2 canceled out by the red part of $-x$, but then there is another layer of $-x$, so we are owed a 1 by 1 square here.

(31) Jony: Then you just have to add orange [squares], right?

(32) Professor Al: Exactly. That is very nice. I hadn't thought of that, I didn't know how to do it like that, so we put it all together (Figure 12 - left image). Now, let's interpret the answer to the product, the calculation.

Here, the professor writes down the dimensions and checks with the students that the rectangular region has dimensions $x - 3$ by $x - 2$. He then systematizes the calculation (Figure 12 - right image).

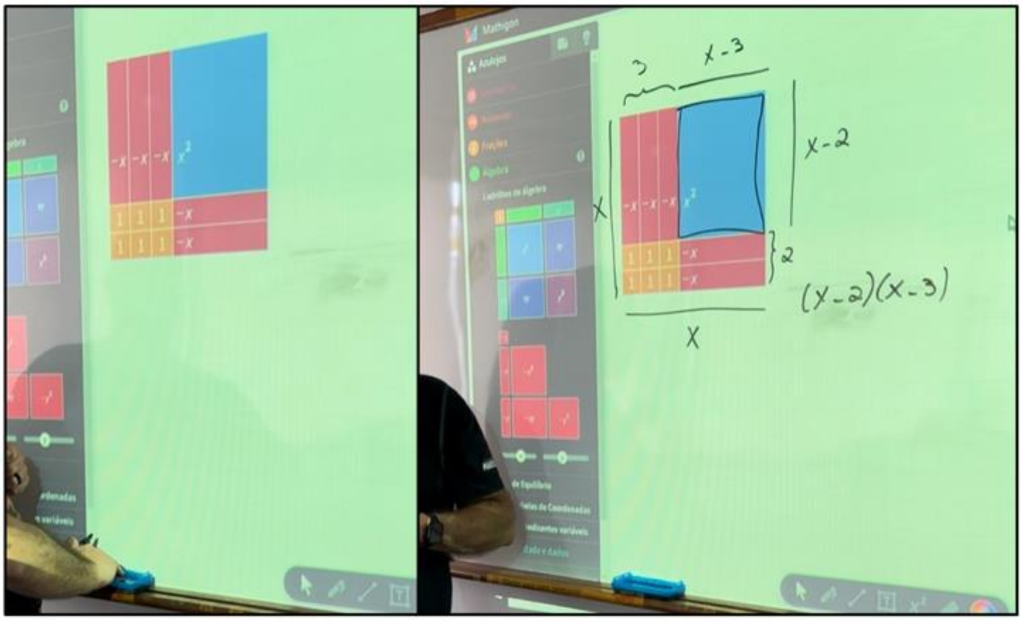

Figure 12.

Systematization of the polynomial product

(33) Professor Al: Which rectangles do I have here? The blue one...

(34) Students: x^2

(35) Professor Al: How many red ones?

(36) Students: 5.

(37) Professor Al: But do I write 5 here?

(38) Jony: No, it's $-5x$, that's 6 areas of x by negative 1.

(39) Professor Al: Yes. And that's it?

(40) Students: 6 more.

(41) Professor Al: That's it... 6 more. So, it's like the polynomial of the product of $x - 3$ and $x - 2$?

(42) Students: $x^2 - 5x + 6$.

(43) Professor Al: Which is the result we found.

(44) Jony: Professor, I saw that if we do the distributive, we get this result. Is it like that for a product of polynomials of any degree?

(45) Professor Al: What do you think? What would it be like if I made the product of a polynomial of degree 2 with one of degree 1?

(46) Some students: It would be a parallelogram... of degree 3. Because of x times x^2

(47) Professor Al: And algebraically?

(48) Students: Do the distributive.

In light of this data, we would like to highlight two categories. The first is the use of a sophisticated semiotic system to develop polynomial representations and their appropriation as a language for algebraic thinking, since the tiles are characterized as indeterminate elements and are used for operations. In this configuration, semiotics is characterized by dealing with the areas of rectangular regions, the lateral dimensions of these regions and the use of these elements to think about the operation of multiplying polynomials.

You can see in lines (12) to (17) that student Duda was not active, as she was not involved in solving the task. For this reason, her classmate Sara reprimands her. The professor invited her to the activity, acting responsibly and concerned about Duda's learning (line (18)).

Using the Mathigon algebra tiles, Sara and Amora represent the polynomials $x - 3$ and $x - 2$ as areas. They also position them vertically and horizontally to do the multiplication. However, when answering the professor, Duda describes what she sees on the screen (line (19)). In addition, it is possible to see that the student shows signs of thinking, using the distributive property, when she puts x times x. In this configuration, the tiles act as semiotic means of objectification. Radford explains

> The objects, tools, linguistic devices, and signs that individuals intentionally use in the processes of creating social meanings to achieve a stable form of consciousness, to make their intentions clear and to carry out their actions in order to achieve the object of their activity, are called semiotic means of objectification. These are semiotic insofar as they are key to the production of meanings embedded in the processes of objectification (Radford, 2021b, p. 136).

As the dialogue continues, in line (20), the professor shows interest in Duda's understanding of the process. We can see that Duda is expressing an opinion about her classmates' responses, which creates a situation of conflict. Duda didn't agree with the form of representation her classmates were constructing (lines (22) and (23)) and preferred not to express her way of thinking or participate in the process. The professor's intervention in the collective process was very important for Duda to feel part of the group's solution and to be heard. In this episode, we can see signs of the community ethic, collectivity, and activity that characterize collaborative work.

But there are still tensions, because in lines (24) to (28) it is possible to see that the professor causes a rupture in the way the students Sara, Tetê and Amora investigate, directing a method that was familiar and comfortable to him.

However, from line (29), the professor's move to develop the investigation based on the students' provocation turns the movement of knowledge on its head and expands it to include the participation of all the students and not just the group of students who were accompanied at the time. Here we see the professor's commitment to the vectors of community ethics and collaboration.

In addition to ethical relations, there are signs of activity in the development of algebraic thinking, specifically in the understanding of the operation of cancelling domains that is, overlapping domains that are canceled. It should be noted that Mathigon itself facilitated the understanding of the context. The solution to the task, shown in Figure 10, was the driving force behind the design of the solution using the overlapping areas method. Radford (2021b) classifies this movement as semiotic contraction.

Semiotic contraction, for the author of TO, is a movement that characterizes learning in which the individual develops strategies more quickly, making more refined connections to previous semiotic activities, which highlights it as a resource susceptible to judging its relevance: "The significance of this process is that it reflects a deeper level of awareness and understanding of the problem at hand. I see this as evidence of learning. The name of this process is semiotic contraction" (Radford, 2021b, p. 139). It is important to note that the learning represented in these data was an activity of both the students and the professor.

Still on the subject of method, the students, together with the professor, constructed a semiotic reading of the elimination of overlapping areas. In line (31), we see that Jony presents a solution to the lack of area, understanding and constructing a new solution method, which characterizes an activity of mathematical abstraction.

The students show that they were part of the collaborative work by participating in the synthesis, interpreting the calculation of the product of polynomials using the method of overlapping areas (lines (33) to (42)).

In line (44), student Jony asks a question, demonstrating an activity of algebraic generalization, since from a more specific situation with polynomials of degree less than 3, the student saw a possible generalization of the calculation for polynomials of degree greater than 3.

The operations of adding and subtracting polynomials have already been elements in the development of algebraic generalization, in terms of the method of calculation in algebraic form. The processes of objectification or generalization for addition, subtraction and multiplication with polynomials of degree greater than 2 took place based on the understanding of operations using tiles, limited to polynomials of degree less than 3. The generalization process was not expected by the researchers and was a pleasant surprise, which we present in summary form here in this article.

Only the operation of dividing polynomials was limited to a method of calculating with tiles that the students were unable to generalize, and which they questioned. The professor, however, explained that the algebraic calculation of polynomial division was done using an algorithm of its own, which would be worked on in class later.

Conclusion

This work presents a pedagogical intervention for the development of algebraic thinking, specifically addressing the content of polynomials and the four operations, through the representations of algebra tiles on the platform Mathigon.

In these terms, in the analysis of the data, in the light of Theory of Objectification and algebraic thinking, according to Luis Radford, the three characterizations pointed out by the author for the development of this type of mathematical thinking can be seen. With regard to the three elements, the indeterminate, the semiotic structure and the analyticity.

The presence of the indeterminate is characterized by the tiles, the areas of which are given by the indeterminates x, y, xy, x^2 and y^2. These elements are part of a semiotic structure of meaning that assigns a geometric representation to polynomials. The lateral dimensions and areas of the regions are components of meaning for analytical development, with the indeterminate.

All these characteristics have been taken into account in the design of the material, and it can be seen that the interaction with the tool *Mathigon*, when working with the students to develop these meanings, was efficient in terms of learning in the light of the theory, as well as providing an environment for the development of the processes of objectification of knowledge, which is algebraic thinking itself.

In carrying out the tasks in an environment proposed by TO - in smaller groups of students and with the interaction of the professor - it is possible to see signs of joint work, with the observation of the relationships of community ethics and activity in relation to algebraic thinking, specifically aimed at the development of abstraction and generalization, which were present in the work with each of the polynomial operations whose data were not analyzed in this article.

In conclusion, we believe that the tasks fulfilled the objective of developing both the processes of objectification of learning about algebraic thinking and the processes of subjectification. This shows that the students developed beyond the researchers' teaching-learning expectations. This development became evident when the students generalized the algorithms for adding, subtracting, and multiplying polynomials of degree greater than two, going beyond the proposed task, which focused on operations with polynomials of up to second degree. The only operation limited by the tile representation was the division of polynomials of degree greater than two, since the algorithm is more sophisticated in this case.

Finally, we point out that *Mathigon*, as a teaching artifact, offers possibilities for developing algebraic thinking when combined with tasks that can provoke a process of objectification and subjectification in students.

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