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Epistemological obstacles in learning limit of real functions of a real variable

Obstáculos epistemológicos en aprendizaje de límite de funciones reales de una variable real

Obstacles épistémologiques à l'apprentissage de la limite des fonctions réelles d'une variable réelle

Obstáculos epistemológicos na aprendizagem de limite de funções reais de uma variável real

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Abstract

The objective of this article is to identify epistemological obstacles manifested in learning the Limit of real functions with a real variable and associate them with the categories proposed by previous studies. Thereunto, we collect productions from the Theses and Dissertations Catalog (CTD in Portuguese acronym) by Capes, with filters for academic master's and doctorate productions published in the last ten years. We developed a theoretical framework based on researchers who categorize epistemological obstacles within the content of Limit, which were also a reference for the works selected in CTD. Categories proposed by the authors were associated with each other, and based on this joint discussion, a specific categorization was created for grouping epistemological obstacles discussed by these authors. This categorization supported the analysis of difficulties reported in five dissertations and one thesis, which described students' difficulties regarding various aspects involving Limit. From the analysis carried out, it was concluded that the most common difficulties are associated with obstacles categorized

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into E1, E2 and E4: *Complexity of basic mathematical objects, Notion and formalization of Limit* and *Calculus Ruptures*.

Keywords: Differential and integral calculus, Limit learning difficulties, Mathematics education.

Resumen

El objetivo de este artículo es identificar los obstáculos epistemológicos que se manifiestan en el aprendizaje del Límite de funciones reales con una variable real y asociarlos con las categorías propuestas por estudios previos. Para ello, Recopilamos producciones en el Catálogo de Tesis y Disertaciones (CTD) Capes, con los filtros para producciones académicas de maestría y doctorado publicadas en los últimos diez años. Elaboramos un marco teórico basado en investigadores que categorizan los obstáculos epistemológicos dentro del contenido de Límite, que también fueron referencia para los trabajos seleccionados desde el CTD. Las categorías propuestas por los autores fueron asociadas entre sí, y desde esa discusión conjunta, se creó una categorización específica para agrupar los obstáculos epistemológicos discutidos por esos autores. Esa categorización apoyó el análisis de las dificultades reportadas en cinco disertaciones y una tesis, quien describieron dificultades de los estudiantes, relacionadas con diversos aspectos que involucran Límite. A partir del análisis realizado, se concluyó que las dificultades más comunes están asociadas a obstáculos categorizados en E1, E2 y E4: *Complejidad de los objetos matemáticos básicos, Noción y formalización de Límites* y *Quiebres en Cálculo*.

Palabras clave: Cálculo diferencial e integral, Dificultades en aprendizaje de Límite, Educación matemática.

Résumé

L'objectif de cet article est d'identifier les obstacles épistémologiques manifestés dans l'apprentissage de la limite des fonctions réelles avec une variable réelle et de les associer aux catégories proposées par les études précédentes. Pour ce faire, nous avons fait une recherche dans le Catalogue des Thèses et Masters de la Capes (Coordination pour l'Amélioration du Personnel de l'Enseignement Supérieur) (CTM), en utilisant les filtres pour les productions académiques de master et de doctorat publiées au cours des dix dernières années. Nous avons élaboré un cadre théorique basé sur les chercheurs qui catégorisent les obstacles épistémologiques liés au concept de limite, ce cadre théorique a également servi de référence pour les travaux sélectionnés dans le CTM. Les catégories proposées par les auteurs ont été associées les unes aux autres et, sur la base de cette discussion commune, une catégorisation propre a été élaborée afin de regrouper les obstacles épistémologiques discutés par ces auteurs.

Cette catégorisation a servi de support à l'analyse des difficultés rapportées dans cinq mémoires de master et une thèse, qui décrivent les difficultés des étudiants liées à divers aspects du concept de limite. L'analyse a montré que les difficultés les plus fréquentes sont associées aux obstacles catégorisés E1, E2 et E4: *Complexité des objets mathématiques de base, Notion et formalisation des Limites* et *Ruptures du Calcul*.

Mots-clés : Calcul différentiel et intégral, Difficultés d'apprentissage du concept de limite, Éducation mathématique.

Resumo

O objetivo deste artigo é identificar obstáculos epistemológicos manifestados na aprendizagem de Limite de funções reais com uma variável real e associá-los às categorias propostas por estudos anteriores. Para isso, coletamos produções no Catálogo de Teses e Dissertações (CTD) da Capes, com os filtros para produções acadêmicas de mestrado e doutorado publicadas nos últimos dez anos. Elaboramos um quadro teórico a partir dos pesquisadores que categorizam obstáculos epistemológicos dentro do conteúdo de Limite, que também foram referência para os trabalhos selecionados no CTD. As categorias propostas pelos autores foram associadas entre si, e a partir dessa discussão conjunta, uma categorização própria foi elaborada a fim de agrupar obstáculos epistemológicos discutidos por esses autores. Essa categorização subsidiou a análise das dificuldades relatadas em cinco dissertações e uma tese, que descreveram dificuldades dos estudantes relacionadas a diversos aspectos envolvendo Limite. A partir da análise realizada, concluiu-se que as dificuldades mais comuns estão associadas aos obstáculos categorizados em E1, E2 e E4: *Complexidade dos objetos matemáticos básicos, Noção e formalização de Limite* e *Rupturas do Cálculo*.

Palavras-chave: Cálculo diferencial e integral, Dificuldades em aprendizagem de limite, Educação matemática.

The high failure rate in Differential and Integral Calculus has been studied both in Brazil and abroad (Rezende, 2003). Over the years, the subject has undergone changes since its implementation, such as reorganization of contents and curriculum to overcome learning obstacles and reduce the significant failure rate (Lima, 2013). Often, difficulties with the Calculus have their genesis associated with academics' lack of knowledge regarding contents studied in Mathematics Classes in Elementary and High School. According to Rezende (2003), some institutions suggest teaching students the mathematics necessary for technical completion of Calculus. However, delays in academics education from those who have recently graduated from high school should not be a specific problem in Calculus teaching because it is equally important for teaching other higher education subjects, and their results are not as worrying as when compared to the Calculus subject (Rezende, 2003).

We understand that several factors can influence the failure in the subject, and among these factors are learning difficulties related to epistemological obstacles (Brousseau, 1986). Epistemological obstacles are those that arise from conflicts that occur throughout the historical development of mathematical concepts, resulting from interaction between old and new knowledge (Sierpinska, 1985). The concept emerged within the context of discussions on the development of science by Bachelard (Mendes & Moraes, 2020), understanding error as intrinsic to knowledge. The author asserts that it is 'in the very act of knowing, intimately, where hindrances and confusions appear, by a kind of functional necessity,' which highlights 'the causes of stagnation and even regression, [...] where we will discern causes of inertia that we will call epistemological obstacles' (Bachelard, 1947, p. 15).

In the context of Calculus not only as a subject, we investigate epistemological obstacles surrounding the subject of Limit of real functions of a real variable, a definition that we consider based on Augustin-Louis Cauchy (1789-1857), enhanced by Karl Weierstrass (1815-1897) and Bernhard Riemann (1826-1866), who gave Calculus a rigorous basis, using the already existing Algebra of inequalities, building a logically connected structure of theorems on the concepts of Calculus (Grabiner, 1983).

Then, we propose to identify epistemological obstacles manifested in the learning of limits of real-valued functions with one real variable and to associate them with the categories proposed by previous studies. Thereunto, we carried out a bibliographical review based on theses and dissertations that report students' difficulties in learning this concept, published between 2014 and 2024, available in the Theses and Dissertations Catalog by CAPES (CTD in its Portuguese acronym). We use the word 'Limit' with an initial capital letter when referring to the mathematical concept, and with a lowercase initial when referring to the term as a mathematical operation or respecting the notation defined by the cited author. From the references used in these works, we identified authors who discuss epistemological obstacles in learning Limit, which became part of our theoretical framework, as detailed in the section that follows.

Theoretical-methodological procedures

This research is characterized as a literature review based on the collection[4] carried out in CTD by Capes with publications between 2014 and 2024. We recognize that many academic papers are developed from dissertations and theses, which allows us to obtain a comprehensive understanding of the topic. The time frame considered was the last ten years, because several software and digital resources were incorporated into teaching during this period.

The first stage of the search consisted of entering, in the Capes CTD search box, the following expressions: *Teaching Differential and Integral Calculus* and *Difficulties in teaching Differential and Integral Calculus*. The findings cover a wide range of productions that aim at investigating, from the use of different methodologies for teaching Calculus, the use of digital technologies, to teaching the notions of Calculus in High School.

Filters used were the indicated time frame (2014 to 2024) and doctorate and academic master’s degree productions in Education. This selection resulted in 51 productions, of which 12 did not have authorized dissemination, i.e., the files cannot be accessed through the Capes CTD. After reading the abstracts, we discarded all those that did not carry out a specific discussion of difficulties related to learning Limit.

To identify the sections that discussed difficulties, the works’ keywords, abstract, summary and conclusions were read. We focused on investigating learning difficulties in higher education, therefore excluding research conducted in high school, since the content of Limits is not included in the Base Nacional Comum Curricular[5]. Were also excluded investigations on Limit with real functions of more than one variable, works whose results discuss only methodological problems, and students’ personal reasons for failure or insufficient performance, such as lack of time or motivation to study. We sought to identify sections that

[4] Collection carried out in January 2024.

[5] Normative document for public and private education networks. It serves as a reference for the development of school curricula and pedagogical proposals for Early Childhood Education, Elementary Education, and High School in Brazil (Brasil, 2017).

report, present or discuss difficulties in learning Limit. This initial selection was organized in a .xlsx file which contained the main data of each work, such as title, author, institution, publication year, search filters and link to access the work.

After these described procedures, we list the five dissertations and one thesis that discuss, in a specific and more detailed way, students' difficulties in learning Limit, organized in Table 1.

Table 1.

List of thesis and dissertations selected from CTD

Reference	**Title**	**Type**
Araújo, 2020	A construção do conceito de Limite através da resolução de problemas	Dissertation
Carvalho, 2016	A análise dos erros dos alunos em Cálculo I como estratégia de ensino	Dissertation
Eckl, 2020	Ensino do conceito de Limite: aplicação de UEPS para identificar indícios de aprendizagem significativa com estudantes de ciências contábeis	Dissertation
Moraes, 2013	Um estudo sobre as implicações dos obstáculos epistemológicos de limite de função em seu ensino e aprendizagem	Dissertation
Muller, 2015	Objetos de aprendizagem multimodais e ensino de Cálculo: uma proposta baseada em análise de erros	Thesis
Costa Neto, 2017	O ensino e a aprendizagem de Cálculo 1 na universidade: entender e intervir	Dissertation

Dissertation by Moraes (2013) was not found in this search, it was shared by the author herself after reading a published paper that presents didactic-pedagogical implications in construction of the concept of real function limit of a real variable (Moraes & Mendes, 2016). The discussion starts from the study of Limit's epistemological obstacles presented by Cornu (1983, apud Moraes; Mendes, 2016), Sierpinska (1985) and Rezende (2003). After Reading the paper, we contacted Moraes (2013) asking for the dissertation file to be made available for inclusion in our analysis.

Theoretical fundations used in these dissertations (Table 1) identified after initial reading of these works were the studies by Artigue (1995), Cornu (2002) and Sierpinska (1985; 1987), which talk to each other. Furthermore, these authors propose categories for epistemological obstacles to learning Limit, and therefore support the theoretical framework used in our analyzis. Works by Cornu (1981), Rezende (2003) and Tall (1993, 2012) expand these authors' discussions about learning obstacles in Limit.

Next, we created a table that approximates the obstacles categorized by the authors (Table 5), according to our reading, used in analyzis to identify and understand difficulties highlighted in Brazilian theses and dissertations. We understand, however, that they are related to each other and may be intrinsically linked. Then, the proposed categorization is not intended to limit or end discussions but to facilitate articulation in the analysis intended in this paper.

Finally, we discuss epistemological obstacles according to Artigue (1995), Cornu (2002) and Sierpinska (1985; 1987).

Epistemological Obstacles in Learning Limit

We assume as epistemological obstacles in learning Limit what Sierpinska (1985) characterizes as challenges faced by mathematicians, throughout history, to define concepts within a mathematical content and which even today present a high degree of difficulty in understanding for students in their learning process. For the author, these are situations that may be identified as causes of the process of slowing down the understanding of a mathematical concept. Epistemological obstacles may involve particularities of concept and specificities necessary for development of that concept (Sierpinska, 1985).

We denote the categories of obstacles defined by the author with the initial letter of their name, Artigue (A), Sierpinska (S) and Cornu (C), followed by the cardinal number (1, 2, 3, 4 or 5) to distinguish categories by the same author. Colors indicate which are discussed together and which were associated in Table 5.

Artigue (1995) defines difficulties in learning Limit in three categories[6] (Table 2). Cornu (2002) and Sierpinska (1985, 1987) elaborates 4 and 5 categories respectively to organize specific difficulties of learning Limit (Tables 3 and 4).

[6] It might be translated as groups, types, classes, or categories. This last term was adopted throughout this work.

Table 2.

Difficulties in Calculus according to Artigue (1995).

Category	Difficulty	Description
A1	Complexity of basic mathematical objects	Misunderstanding of different mathematical objects important for individual's mathematical training, such as real numbers, sequences, and functions.
A2	Notion and formalization of Limit	Conceptualization, formalization of notion of Limit, difficulties with language, use of terms (*tending to, as small as one wishes*).
A3	Algebra/Calculus Disruption	Necessary breaks from purely algebraic modes of thought, very familiar with specificities of technical work in Calculus.

Table 3.

Epistemological obstacles related to Limit according to Sierpinska (1985, 1987).

Category	Obstacle	Description
S1	Horror to Infinity	It refers to refusal of set of infinity and mathematical operation status attributed to Limit.
S2	Notion of functions	Difficulties associated with understanding and applying the notion of function in the context of Limit.
S3	Geometric obstacles	Challenges linked to geometric interpretation of quantities and continuity.
S4	Logical obstacles	Difficulties of logical nature in understanding the concept of Limit.
S5	Symbolic obstacles	Difficulties related to interpretation and manipulation of mathematical symbols in the context of Limit.

Table 4.

Epistemological obstacles related to Limit according to Cornu (2002).

Category	Obstacle	Description
C1	Numerical transposition	Passage from geometric to numerical concept.
C2	Infinite	Obstacle associated with the notion of *infinitely big* and *infinitely small*.
C3	Metaphysics	Introduction of ideas that are not just calculations or logical deductions, with introduction of Limit discussed more in the field of metaphysics and philosophy than Mathematics.
C4	Limit reaching or not	Discussion associated with Limit reaching or not reaching the value it tends to.

Difficulties of category A1 are related to objects that students should have had some contact with throughout their elementary and high school education, such as real numbers and functions (Artigue, 1995). The author cites as examples the lack of clarity about differences between numerical sets, as students tend to confuse different numerical categories, such as whole numbers, rational numbers, and irrational ones.

Regarding functions, specific obstacle of category S2, Artigue (1995) cites the results obtained in researches such as those carried out by Eisenberg (2002), Leinhardt et al. (Artigue, 1995), and Dubinski e Harel (Artigue, 1995), who made evident a set of difficulties in learning this content that would be difficult to solve in the Calculus discipline. Eisenberg (2002) affirms that the lack of skills for understanding graphical functions and the loss of meaning through the intellectual process of mathematical abstraction are some examples of common difficulties in learning functions.

Tall (1993) highlights that, in addition, Calculus students may construct a restricted notion of the concept of function, not understanding examples different from those covered in class. According to the author, students must not only know specific cases of functions, but also must understand the concept, so that such difficulties can be overcome.

Regarding obstacles A2, Artigue (1995) mentions that, for teaching Calculus, "the concept of Limit occupies an essential place, given the central position of the concept in this field" (p. 112). In this context, for the author's perspective, associated with understanding the concept of Limit is the meaning attributed to the word *Limit*, which may suggest an insurmountable and unreachable barrier, a static conception of Limit. Cornu (1981) associates this difficulty with the discussion about whether the Limit may be reached or not, in the category C4.

Tall (1993) describes problems related to different symbolic registers, such as graphic and algebraic representation, which Artigue (1995) associates with obstacles in notion and formalization of Limit. The author suggests that working with computers and graphing calculators can be useful to alleviate difficulties in understanding different records, citing research with promising results involving the use of software to represent Limits in different ways.

Additionally in the category A2, there are problems related to the dual operational and structural situation of Limit, which is evident in the difficulty to separate the algebraic process to obtain the Limit of its meaning (Artigue, 1995). Tall (1993) states that there are difficulties surrounding the language and the use of terms such as *tending to, approach, as small as one wishes*, among others. The author reminds that such terms are loaded with meanings in colloquial language and might conflict with formal and operational concepts throughout teaching and learning processes.

Cornu (1981) proposes a discussion regarding the multiple meanings which are attributed to the terms used to deal with Limits. The author considers that students have what he calls *spontaneous models,* which are the previous meanings, distinct or not from the mathematical model presented by the teacher. Throughout learning process, these two models will conflict, generating adaptations that will result in development of their own models, which may be multiple for the same concept (Cornu, 1981).

For Cornu (1981), the models are neither totally wrong nor totally right in terms of mathematical concepts, and in the case of the *own models*, they are often mathematically inaccurate. *Own models* evolve as they are used in classes or studies, becoming more precise and accurate, although they may remain far from the *mathematical model*. An example of model conflict cited by the author is associated with the word *Limit,* which is most common in students' everyday language. The word almost always refers to something static, fixed, like a geographic boundary, a limit that must not be exceeded (moral or regulatory), a limit that we prohibit ourselves from exceeding. This is the notion of difficulty in capturing the idea of approach, not a value to be achieved and therefore the notion of *indefinitely approach*. The insurmountable meaning (mark) of Limit is predominant, because in common meaning, limit is what separates two things, and this understanding will have consequences for mathematical activity. In common sense, the notion of a Limit does not contain the idea of variation, movement, or approach (Cornu, 1981).

Therefore, we understand that this obstacle allows us to discuss not only the difficulties associated with the category C4, but also those derived from language. Understanding the word *Limit* as barrier to be respected, something that cannot be overcome is an example of different meanings between the mother tongue and the mathematics language. We understand that the use of certain expressions to teach Limit content arises from the need for offering an intuitive approach, but it is important to consider that there are multiple meanings attributed by students' *spontaneous models*. Considering that in oral communication we use the mother tongue, which differs from the mathematics language, both have their own symbols and meanings that must be observed.

An example cited by Sierpinska (1985) makes evident the presence of spontaneous models in her research with students. She came across the absence of quantifiers for the definition of Limit used by them.

> The students never say that the values of the function $\frac{sin\,(x)}{x}$ differ as small as we wish for values of x sufficiently close to zero; they just say that if x is close to zero, then $\frac{sin\,(x)}{x}$ is close to 1, or that the difference between the values of x and $sin\ x$ becomes smaller and smaller (Sierpinska, 1985, p. 54).

For the author, the absence of quantifiers and the non-use of symbols to denote the passage to the Limit in the process of building *own models* represents the logical obstacle, represented by category S4 in this work.

We can also associate A2 with difficulties in understanding complex concepts and new ideas in a very short time (Tall, 1993). Students are often confronted by a multiplicity of definitions, and according to Robert and Boschet (1984 *apud* Tall, 1993), students who are most successful in Calculus are those who find it easier to switch between different representations. According to Tall (1993), these students

> need to develop an ability to deal with the complexity of the subject, turning almost intuitively to the representation that will prove useful in the specific cause. It may be that calculus works for those who can think flexibly and fails for those who seek guidance in more procedural methods to help them overcome their problems (p. 8).

The obstacle associated with the Algebra/Calculus rupture (A3) is described as the need to change reasoning to deal with problems that depend on algebraic manipulation, which does not satisfy the necessary solutions to Calculus problems. For instance, knowing that there is not a well defined limit for $\lim\limits_{x \to 1} \frac{1}{x-1}$ does not end the investigation, since it is possible to look at the behavior of infinite Limits, Limits at infinity, asymptotes, among other aspects. For Artigue (1995), "the modes of reasoning underlying this work are new to students, and [...] mathematical techniques involved are delicate" (p. 115). In that regard, knowing how to manipulate Limits algebraically is important to understand the contents of Calculus but it should not be the only objective of the subject.

The rupture between Algebra and Calculus is aggravated when the student's task is limited to solving lists of exercises. Artigue (1995) argues that traditional teaching, based on expository explanation and solving lists of exercises, does not help to minimize students' difficulties, maintaining an illusion of progressive and continuous learning. Usually, demonstrations are the responsibility of the teacher, they receive greater focus in classes and assessments, and the application of techniques, calculations of Limits, Derivatives, Antiderivatives, and Integrals prevails, prioritizing the study of techniques to the detriment of meaning, generating pedagogical conflicts (Rezende, 2003).

According to Artigue (1995), it is also important for students to know how to operate with formal definitions when dealing with problems, because at each stage of the process it is necessary to understand what information can be considered and the level of precision required to be able to advance in the resolution. This requires not only familiarity with expressions and size orders, but a clear understanding of concepts being addressed in the problem as well.

In that regard, as it is in Calculus that students are confronted for the first time with the concept of Limit, they will need a different approach to traditional teaching, as this concept does not just involve calculations developed through a simple arithmetic or algebraic treatment (Tall, 1993). In a later work, Tall (2012) proposes a *sensitive approach* to Calculus, based on teaching and learning processes on human perceptions and the use of senses to formulate and understand mathematical concepts. This approach seeks to develop Calculus ideas from sensitive sources[7], allowing fundamental concepts to be understood intuitively. The goal is to provide a solid foundation for understanding more advanced concepts, such as the concepts of Limit and Infinitesimals. Instead of introducing abstract concepts in isolation, Tall (2012) suggests an approach that seeks to relate mathematical learning to real-world experiences, making concepts more tangible for students.

Categories associated with the passage of geometric interpretation of Limit, called geometric obstacle or numerical transposition, are indicated as C1 and S3. This obstacle originates from the significant time that passed between the Exhaustion method, developed by Greeks in the 5th century BC, and the precise definition of Limit, which appeared in the 17th century. According to Sierpinska (1985), it is possible to identify this epistemological obstacle in the excessive use of the geometric interpretation of Limit.

Cornu (2002) states that the solutions developed by Greeks to calculate the area of a circle, for instance, offered an opportunity to develop tools very similar to the concept of limit. The possibility of successively reducing something to the point of reaching as small a size as wished, as in the case of the Exhaustion principle, is very close to the idea of Limit, but it cannot be said that it is the same conception that we have today. Both Sierpinska (1987) and Cornu (2002) assume the Exhaustion method as purely geometric.

This geometric and non-numerical treatment generates another distinct epistemological obstacle, denoted by S1 and C2, which concerns exclusively the problems surrounding the inadequate understanding of infinity. For Sierpinska (1985),

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[7] Tall (2013) considers *sensitive origins* those situations that enable the use of human perceptions and senses as a starting point for understanding mathematical concepts.

> although it is paradoxical, we cannot understand the notion of limit without having understood the notion of real number, and real numbers are only really understood when we understand the notion of limit itself, but this cannot serve as the basis for a definition. This paradox cannot be resolved unless we accept the existence of infinite sets, of different infinities, accepting them as actually, and not just potentially infinite (p. 35).

The lack of understanding infinity also extends to the treatment of infinitely small quantities. Cornu (2002) highlights (category C2) that the concept of infinitesimal drove progress in development of the notion of Limit. In this epistemological obstacle, the author describes the difficulty to understand infinitely small and infinitely large quantities, on the question that arises from the possibility of "having quantities so small that they are almost zero, and yet not having a specific 'assignable' size" (Cornu, 2002, p. 160).

Still on difficulty in dealing with infinity, we bring an example discussed by Rezende (2003) about common procedures performed in the computation of Limit. The author discusses the recurring error of students in stating that the limit $\lim_{x\to\infty} x \cdot sen\left(\frac{1}{x}\right)$ results is zero. The students' procedure to reach this inconsistent conclusion is a clear case of misunderstanding indeterminacies. The students' resolution exemplified by the author is that when $x \to \infty$ then $\frac{1}{x} \to 0$ and $sen\left(\frac{1}{x}\right) \to 0$; hence, from these statements, we obtain that $x \cdot sen\left(\frac{1}{x}\right) \to \infty \cdot 0 = 0$, because every number multiplied by zero results zero. Examples like this brought by Rezende (1994), which the author calls *algebra of infinity,* constitute an epistemological obstacle to understanding infinite, which we can include in the categories S1 and C2.

Finally, the category C3 is related to metaphysical aspects of Limit, and the category S5, related to problems with symbology. In C3, Cornu (2002) states that, by introducing the concept of Limit, there is a significant paradigm shift, going beyond calculations or logical deductions. The author argues that, in classical mathematics, infinity was not considered part of the study field. Both infinity and the notion of Limit were seen as more related to metaphysics or philosophy than to Mathematics.

In category S5 is the barrier of symbols. Sierpinska (1985) states that the creation of a symbol to represent the Limit operation was late introduced by Augustin-Louis Cauchy (1789-1857). This obstacle manifests, for example, in the lack of understanding of the notation used to denote Limit operations by students.

There are certainly other epistemological obstacles not mentioned here. We chose to briefly discuss those cited by the reference authors and categorize them, but we agree with Sierpinska (1987) that this list should not be definitive. Categories must be explored, investigated, and discussed in the classroom should be modified and deepened. Cornu (2002) states that errors made by students may be sources of investigation to identify epistemological obstacles. We agree with the author that epistemological obstacles must be considered by the teacher when thinking on the process of teaching these concepts, not preventing students from encountering these obstacles, but if they occur, provide opportunities for discussions with the aim of overcoming them (Cornu, 2002).

Table 5 summarizes these epistemological obstacles related to learning Limit, serving as a reference for the analysis of theses and dissertations collected, helping to organize and articulate difficulties encountered in learning Limit of real functions of a real variable.

Tabela 5

Epistemological obstacles

Category	**Obstacle**	**Description**
E1	Complexity of basic mathematical objects	Difficulties associated with understanding elementary school math contents, such as real numbers, algebraic operations, sequences, functions, etc.
E2	Notion and formalization of Limit	Conceptualization, formalization of the notion of Limit, difficulties with language, use of terms (*tending to*, *as small as one wishes*). Discussion associated with Limit reaching or not the value it tends to.
E3	Infinite	Refers to refusal of the set of infinity and the mathematical operation status attributed to Limit. Obstacle associated with the notion of *infinitely big* and *infinitely small*.
E4	Calculus Ruptures	Necessary breaks from purely algebraic modes of thought, very familiar with specifics of technical work in calculus. Challenges linked to geometric interpretation of quantities and continuity. Difficulties of a logical nature to understand the concept of Limit. Introduction ideas that are not just calculations or logical deductions, with introduction of Limit discussed more in the field of metaphysics and philosophy than Mathematics.
E5	Language obstacles	Difficulties related to interpretation and manipulation of mathematical symbols in the context of Limit. Differences between mother tongue and mathematical language.

Categories are referenced with the letter E followed by the corresponding cardinal number. Names of obstacles have been reorganized to include those defined by Artigue (1995), Cornu (2002) and Sierpinska (1985, 1987), cited in this work and indicated by the color used as highlight. Next, we analyze and discuss theses and dissertations collected for this paper.

Learning Difficulties concerning Limits

From the selected theses and dissertations (Table 1), we analyze the main difficulties in learning about Limit, establishing relationships with epistemological obstacles of Table 5, elaborated considering the theoretical framework adopted, and seeking to establish epistemological difficulties in Brazilian research. We denote association with obstacles by indicating the category and its cardinal number in parenthesis.

Epistemological obstacles identified through Brazilian research

We started from the research by Araújo (2020), who investigated errors made by Calculus students and how these errors may help in teaching and learning processes concerning Limit. The analysis is based on work and activities carried out by Calculus students in 2018 at a public university. Its conclusion points out that the concept of Limit is not well understood by students, in addition to pointing to difficulties in algebraic aspects of Basic Education (Araújo, 2020).

The main errors identified are algebraic and interpretative (E1). Araújo (2020) affirms "that students evoke a conceptual image alluding to the idea that Limit works as a resource for approach" (p. 96). The author identifies that students understand *tending to some value* as a numeric value to be replaced (E2). They assume that the purpose of equality in the context of Limits is only to indicate a calculation to be carried out from an operational viewpoint (E2).

There are indications, in the students' responses, that Limit was not understood in its entirety, whether due to its definition and/or geometric representation in cases of an infinite Limit (E4). A case that highlights this difficulty is demonstrated by the student's incoherent responses stating that the limit $\lim_{x \to -3} \frac{2}{|x+3|}$ exists and is equal to -3. This error may be associated with both E1 and E4 because the student was unable to correctly understand the graph of this function, which prevented him from understanding its behavior when x tends to -3. It is possible to consider that the student associated the Limit with a value to which the points on the graph tend to, but without distinguishing whether this approximation occurs in the domain of the function or in the image set.

Araújo (2020) states:

> the Limit of a function is seen [by students] as an algebraic operation, where we operate with values and find a numerical value; understanding of Limit [is seen] as an intuitive notion; then it was possible to reach the conclusion that students' understanding of the

concept of Limit is not correct, since this understanding is not evidenced and conflicts with conceptual definition of Limit (p. 102).

The author brings examples in which students have difficulty recognizing patterns (E1) and identifying indeterminacies (E4), recognizing ∞ as a number (E3) (Araújo, 2020). An example that illustrates this obstacle is the achievement of what Rezende (2003) calls *Algebra of infinity*: when trying to find the Limit of a function when *x* tends to infinite, in the process of solving, the student is faced with $\infty - \infty$, and his answer is ∞.

Araújo (2020) concludes stating that technical teaching of Limits, devoid context based on solving long lists and using Calculus textbooks as the only source plays a significant role in students' low performance in the subject, without being able to promote understanding of various topics in elementary education. This reaffirms what Tall (2012) discusses on a teaching Calculus that favors an intuitive approach, as a consistent alternative to teaching based solely on proofs and solving exercise lists.

The research by Carvalho (2016) aimed identifying the most frequent mistakes made by students in Calculus classes in the 2015 second semester in Civil and Production Engineering faculties. The analysis sought to understand the most common difficulties, proposing teaching approaches and strategies, in addition to suggesting possible adjustments to the course syllabus (Carvalho, 2016).

The author developed an organization for types of errors described in her research, in which type 1 refers to when the student does not have specific knowledge of the subject Calculus, and he may neither correctly use derivation techniques nor identify the concept of Limit in a function or relate lateral Limits with continuity of functions. Type 2 errors refer to a lack of basic mathematical knowledge, such as factorization techniques, properties of powers and interpretation of graphs of functions. Type 3 errors are those committed by distraction and derived from arithmetic manipulation (Carvalho, 2016).

Based on the analysis of errors made in two tests, it was possible to highlight the prominence of type 1 and 2 errors among student responses. Type 1 errors represented approximately 55% of mistakes made, and type 2 and 3, 45% and 5%, respectively. Carvalho (2016) highlights, with the analysis of the students' resolutions, that they did not understand lateral limits and limits of the function at a specific point (E2). The author also identified errors when factoring polynomials (E1).

Geometric representation of Limit (E4) was a challenge in the research by Carvalho (2016) as well. The students were unable to notice discontinuity of functions (E1) by inspecting the graph and they made mistakes analysing lateral limits of functions (E2). Furthermore, some students were unable to understand the idea, even if intuitive, of function limit, lateral limit, continuity (E2), wrongly using the notion of Increasing Functions (E1).

Dissertation by Eckl (2020) had as research aim the teaching and learning of mathematical knowledge related to the concept of Limit of a function of a real variable. The author investigates the development of a Potentially Significant Teaching Unit (PSTU, UEPS in its Portuguese acronym) and its possible contributions to learning concepts related to Limits. The qualitative research was carried out with students from a higher education course in Accounting Sciences.

In analyzing the created PSTU development, the author identified, through the answers to the first proposed tasks, that students had prior knowledge about functions, but a large proportion provided only partially correct answers (E1) with results that differed from how the variables were defined in the problem (Eckl, 2020). Analyzing student responses to initial functions tasks, the author states that the answers do not comply with what was initially questioned, but that subsequently it was possible to identify signs of maturing understanding of variables, with a coherent interpretation of dependency relationship in the subsequent tasks.

In the second task, according to the author, it was possible to conclude that a partial number of students understood the growth behavior of terms in the sequence presented (E2 and E4). The problem situation was stated as follows: "Dividing 100 bales of barley between five men in arithmetic progression, so that the sum of the two smallest is one-seventh of the sum of the three largest" (Eckl, 2020, p. 103-104). The problem presented the sequence $\frac{10}{6}, \frac{65}{6}, \frac{120}{6}, \frac{175}{6}, \frac{230}{6}$, but the majority were unable to correctly explain the occurrence of increases and the trend present in the analyzed behavior, indicating difficulties in expressing their interpretations in written form (E1 e E2).

The students' responses related Limit to the idea of analyzing functions through approximate values without attributing an explanation for possible trends, partial interpretations of behavior, and the context of approach appears, but without elaborate precision in the answers, as can be read in the following excerpt.

> The limit is something that we cannot say where it goes, but there may be an ending in a given equation or situation. Its objective is to determine the behavior of the function when there are values. The limit helps in understanding various functions through points such as minimum and maximum, or even the halfway points between the functions. The limit is infinity, unlimited [students' response] (Eckl, 2020, p. 107).

Students demonstrated poorly elaborated interpretations. Therefore, it is understood that students had difficulties understanding/interpreting the question and/or the problem situation (E2).

The research carried out by Moraes (2013), in her dissertation, aimed at identifying which epistemological obstacles are present in the process of constructing the concept of Limit. The author starts from theoretical framework by Cornu (1983), Sierpinska (1985) and Rezende (1994) to analyze whether obstacles are present and how they manifest in student responses, through questionnaires.

The author identified that the definition of Limit, when stated by students based on the 'Weierstrass' formal definition, presents a symbolic confusion (E2). The student, when questioned, conveys the idea of Limit in the following way: "Given a function and a point defined in the function or not, there is the limit of the function at this point, when there are values close to that point, their images tend to a number which is the limit of the function at that point", and adds: "$\forall\, \varepsilon > 0, \exists\, \delta > 0;\ |x-a|\delta > \Rightarrow |f(x)-L| > \varepsilon$" [student's response] (Moraes, 2013, p. 95). The student demonstrates understanding the intuitive idea of Limit, but not its definition (E2).

Another difficulty identified by Moraes (2013) was regarding the use of terms, such as *approach to, tend to, try to approach, infinitely approach, approach*. The author associates the use of these terms with the dynamic aspect of Limit, highlighting that students did not understand the static nature of Limit (E4). In one of the answers, the author noticed that the student only considers the case in which the function is defined in a ($\lim_{x \to a} f(x) = f(a)$): "The value $f(x)$ tends to a limit. The value of this limit is obtained when a point x (that has the image $f(x)$) tries to approximate to a point y which approximates infinitely close to a value L. The Limit is the value of $f(x)$ when x tends to y" (Moraes, 2013, p. 96). The student considers a definition dynamically, not just statically, but disregards the case in which $\lim_{x \to a} f(x) \neq f(a)$. The author connects difficulties such as those in this example and the use of terms inappropriately with the Geometric obstacle by Sierpinska (1985) and the Numerical Transposition by Cornu (2002) (E4).

Regarding geometric obstacles (E4), the author brings excerpts with the answers regarding what they understood by Limit: “Limit of the function is the maximum point, as the name suggests, a limit, maximum where the function can reach”, or “Studying the behavior of graph in relation to a point belonging to the domain” (Moraes, 2013, p. 98). They denote difficulties related to the concept of Limit (E2).

Difficulties associated with infinity become more evident in the second question analyzed by Moraes (2013), in which the author requests, in the questionnaire, a brief description of what is understood by infinity. Students’ responses characterize infinity as something immeasurable, extensive, without beginning or end (E3). Some students see it as an unreachable number: “Infinity is defined as some unreachable place, space or even number, where one knows it will always exist, but which one cannot reach. In limit, it becomes an extremely large or extremely small number” (Moraes, 2013, p. 100). Other answers associate infinity with something that can be discussed within the scope of philosophy, as can be read in the following excerpt.

> Infinity is not a real number, infinity is beyond concepts, which is why we have no control over the operations of addition, subtraction, multiplication, division when it involves infinity. Philosophically, infinity is beyond human understanding, therefore, it can be abstracted but does not exist in the concrete. Mathematically it is an incommensurable value [student’s response] (Moraes, 2013, p. 100).

One of the answers that highlights the difficulty in understanding how the Limit behaves in infinity is the one that associates it with the idea of indeterminacy: “$\lim_{x\to\infty} f(x) =$ an indeterminacy occurs. Infinity is denoted as an approximation of the graph going from $+\infty$ to $-\infty$” [student’s response] (Moraes, 2013, p. 101). Therefore, the student does not assimilate the existence of the Limit with x tending to infinity (E3). There are answers in which infinity is assumed only as a symbol: “Infinity is a symbology that represents a large-scale value”, or “Infinity is when one cannot determine a value because it is too large, ∞ is a symbol, not a number” (E3) (Moraes, 2013, 101-102).

Moraes (2013) carries out a discussion, in her analysis, regarding these difficulties associated with understanding infinite. The author identifies that students present variations in their conceptions of infinity, which she classifies into physical or symbolic aspects. These two interpretations are cited by Rezende (2003), when discussing introduction of symbol ∞ for infinitely large quantities or as a number subject to arithmetic operations. In the students’ responses, the first interpretation is mainly observed, which for Moraes (2013) presents an *arithmetization* of Calculus.

The fourth question proposed to verify the students' conceptions regarding accuracy of the value of the Limit of the function. Only one student answered the question correctly, out of the thirty-three participants. The author justifies the choice of incorrect alternatives to the obstacle of associating Limit with a strictly dynamic aspect, obstacle C4 by Cornu (2002). Another inconsistency explained was the students' choice to write alongside the graphic representation "$\lim_{x \to a} f(x) = L$", but they chose the alternative that describes the values of coordinates of the function, when x tends to a from the left or from the right, get closer, but does not reach L (Moraes, 2013). The author associates this difficulty with the obstacle of numerical transposition described by Cornu (2002) (E4).

The fourth question asked students to explain the differences between two functions, f and g, represented only graphically, in which f is a continuous function defined in $\mathbb{R}$, and g its restriction on the set of reals, except at the point a. On the axes, the values were projected with dotted segments the values of a and $L = f(a) = \lim_{x \to a} g(x)$.

The differences between the functions were expressed in different ways, with the use of terms *continuous* and *discontinuous* being recurrent. Among the students' responses, it is possible hightlight: "Function f is continuous and function g is discontinuous, as there is a 'hole' in the graph", and "One is defined in a and the other is not, that is, f is defined in a and g is not, but both are continuous" [student's response] (Moraes, 2013, p. 105). In the answers, it is noted that the first student considers continuity as a *graph without holes;* and in the second answer, it is observed that the concept of continuity of a function is not understood (E1).

Other indications for differences between the functions were to assume the existence of the Limit only for f or that the Limit is getting close to L: "Limit of $f(x)$ is exactly L when x tends to a. Limit of $g(x)$ tends to L but never will be L" [student's response] (Moraes, 2013, p. 106). Here, we consider that they were unable to interpret the Limit based on its graphical representation (E4), transferring, as cited by Moraes (2013), notions of Limit into function properties (E1). The author relates these difficulties to the obstacle of the Limit being reached or not (Cornu, 2002).

In a second item of the same question, the author seeks to verify whether the student understands what is necessary and sufficient for the existence of the Limit of a function at a point. Some students stated that, in "$\lim_{x \to \infty} f(x) = L$, limit does not exist, because it exists when x gets closer to a and not when x is a" [student's response] (Moraes, 2013, p. 108). The author assumes it was a writing error x tending to infinite in the answer. Most of responses state the non-existence of Limit in g, justified by discontinuity of function in a (E4). In this item, the author identifies that an obvious obstacle in the answers is the aspect of movement present in the kinetic obstacle discussed by Rezende (1994), that we associate with the ruptures in the calculation present in the category E4 in this work.

The last question was intended to verify the understanding of Limits at infinity. It presented the functions $f(x) = 2x + 3$ and $g(x) = -x + 1$, and requested responses to the following items: a)$\lim_{x \to \infty} f(x)$; b)$\lim_{x \to \infty} g(x)$; c)$\lim_{x \to \infty} f(x)$+$\lim_{x \to \infty} g(x)$; d)$\lim_{x \to \infty} f(x) - \lim_{x \to \infty} g(x)$; e)$\lim_{x \to \infty} \frac{g(x)}{f(x)}$. The main difficulties identified in the answers to this question were related to notation errors (E5), such as symbols (E5) and indeterminacies (E4). A student considered $\infty + 3$ as an indeterminacy, which the author related to the obstacle defined by Sierpinska (1995), *Horror to infinity* (E3).

Moraes (2013) identified, in the answers to this question, difficulties arising from algebraic manipulation (E1), in which infinity was considered a number, and manipulated without any specific interpretation by students. The largest number of errors made was in the item e), in which few were able to use Calculus properties to find the Limit correctly (E3), assuming the result as indeterminate or one of multiple answers "$\lim_{x \to \infty} \frac{-\infty}{\infty} = -\infty$, $\lim_{x \to \infty} \frac{-\infty}{+\infty} = -\infty$, $\frac{-\infty}{\infty} = -\infty$, $\frac{-\infty}{+\infty} = +\infty$" [student's response] (Moraes, 2013, p. 113).

Muller (2015) aimed to analyze learning difficulties presented by Calculus students, and to test possibilities for reducing these difficulties through technological resources. In the research first phase, the author analyzed errors made by Calculus students from two different classes, one in the Information Systems course and the other in an Engineering course, when taking a test. The results from this phase showed that the greatest difficulties were related to basic mathematics content (E1). The questions presented in the first questionnaire were related to the calculation of Limits, Derivatives, and analysis of behavior of functions. In the second test to probe these difficulties, the author focused the questions on the contents in which she identified the most insufficient understanding: algebraic fractions, distributive properties, equations, functions, and trigonometric ratios. There was no specific discussion regarding the difficulties with Limits.

Costa Neto (2017) investigated failure rates in Calculus at University of Brasília between 2010 and 2016. The objective of his research was to investigate the need to implement a Pre-Calculus subject in mathematics department at the university, proposing the use of apps, such as Excel and Calc, to work on more intuitive aspects of Calculus tools to later insert the concepts in a formalized way (Costa Neto, 2017). What matters for our research was the analysis carried out by the author, on questions that were asked in previous Calculus tests, carried out using two psychometric theories: Classical Test Theory (CTT) and Item Response Theory (IRT).

The three questions analyzed about Limit and Continuity required the student to formulate and articulate arguments appropriately about ordinary Limits and Limits involving infinity, in which two questions were considered medium level and one difficult (Costa Neto, 2017). All questions had multiple alternatives.

The first question requested a condition so that $\frac{x}{x+1} > 1 - 10^{-3}$, with $x > 0$ as hypothesis. Although the question theme is considered Limits tending to infinity, correct solution required only knowledge of basic mathematics. Only 37,5% answered, and out of them, 62,5% got it wrong (E1) (Costa Neto, 2017).

The second question asked for the limit $\lim_{t \to \infty} \frac{sen(t)}{t}$, in which the correct answer would be the existence of the limit, given that $sen(t)$ is bounded and the quotient $\frac{1}{t}$ tends to zero. The options that attracted the most choices were the incorrect alternatives B, C and E, with 30%, 25% and 21% respectively (Costa Neto, 2017). Option B dealt with Fundamental Limit, which is the Limit of the same quotient mentioned in the question but tending to 0 (E2). Option C mentions the Lower and Upper Limits of the sine function: the student failed to distinguish this interval, for which the sine function is defined, with the notion of Side Limit (E2). Option E presents an error that revealed the difficulty in distinguishing the image and domain from the sine function (E1).

The third question asked to discuss the non-existence of the limit $\lim_{x \to 0} x \cdot sen\left(\frac{1}{x}\right)$, requiring knowledge of fundamental properties of Limit (Costa Neto, 2017). It was necessary to identify that sine function is bounded in the closed interval between -1 and 1, and the limit $\lim_{x \to 0} x = 0$. The students understood that every Limit denotes continuity, mistakenly replacing $x = 0$ (E2). Alternative B is considered implausible, as the student would replace $x = 0$ in the function and would incorrectly determine the maximum value of the sine function. Option C, chosen by approximately 15% of students, presents an error like that described in option A. Item E indicates an error related to basic mathematics, due to multiplication by 0 (E1) (Costa Neto, 2017).

Finally, Costa Neto (2017) highlights that this question reveals two common difficulties among students: tendency to confuse the maximum points of sine function with those of cosine function, which is indicated by the choice of option E by 21% of students (E1), and the difficulty in understanding the exact numerical value of a function at a given point and the approximate value, as is the case with Limit (E2).

Concluding the analysis of theses and dissertations above, we realized that the most present obstacles were those associated with the proposed categories E1, E2 and E4. We noticed that different categories were mentioned in the same question, and from this we infer correlations between different epistemological obstacles in learning Limit.

Conclusion

Throughout the research, we sought to establish connections between the specific difficulties mentioned in the works and the broader concepts addressed by the proposed theoretical framework. This process made it possible to articulate the challenges faced in the learning process presented by the research (Table 1) and relate them to epistemological obstacles listed (Table 5).

It was possible to identify several difficulties faced by students in the Limit learning process, which are intrinsically linked to epistemological obstacles discussed. The research helped us realize that there are many difficulties in different types of subjects, and that they do not just end with the analysis carried out. The questions presented and analyzed by the authors were of great help, offering an idea of concepts related to Limit, which students face difficulties to understand.

Based on the analysis, we identified that the most recurrent difficulties involve the complexity of basic mathematical objects (E1), the notion and formalization of Limit (E2), and ruptures in Calculus (E4). Students may face difficulties related to understanding real numbers, sequences, and functions, which are fundamental to understanding the concept of Limit. We understand, however, that the objective of Calculus subject is not to make up for the gap in the content not assimilated by students throughout their basic mathematical education in Elementary and High School. Nonetheless, we cannot disregard such difficulties, since it is evidently an obstacle that prevents the student from effectively understanding when learning Limit.

The very notion and formalization of Limit (E2) is aggravated if the student presents difficulties in the categories E1 and E5. The lack of understanding of mathematical language and basic mathematical objects may lead to difficulties regarding conceptualization of Limit and its mathematical formalization (E2), in which understanding definitions is incomplete or partial. The student does not understand the difference between infinite Limits or infinity (E3), does not perform a graphical analysis or interpretation of behavior (E4), assuming Limit simply as a value to be found through algebraic manipulation, which is not enough to answer Calculus questions.

In addition to these specific difficulties, it is important to highlight that transition between different representations of Limit, the very changes, and ruptures that Calculus historically proposes, understanding complex concepts in a short time, and the need to develop flexibility in mathematical thinking can also represent significant obstacles in learning Limit. We understand that it is essential to understand that categories of obstacles presented should not be considered definitive. The research aimed at proposing a classification for difficulties reported in literature, but which can and should be explored, investigated, and debated not only in subsequent research, but also in the classroom.

We understand that errors related to Limits are not explained only by epistemological obstacles of Calculus, but the proposed association allowed classifying these difficulties, often resulting from one or two obstacles simultaneously. By categorizing epistemological obstacles, it is possible to identify specific difficulties faced by students when learning about the concept of Limit. This allows for a more targeted and effective approach to overcoming such obstacles.

We conclude that the categorization proposed by Artigue (1995), Cornu (2002) and Sierpinska (1985; 1987) allows to expand understanding difficulties and errors made by students in the process of learning Limit of real functions of a real variable. We recognize the limitations of this work and propose, after reflecting on the issues discussed, the expansion of error analysis for future research, investigating different content related to Limit within Calculus for a more detailed understanding, making it possible to comprise implications of epistemological obstacles in learning Limit.

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