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Epistemological elements for teaching density and mass: exploratory tasks through integrals of one and more variables

Elementos epistemológicos para la enseñanza de densidad y masa: tareas exploratorias a través de integrales de una y más variables.

Éléments épistémologiques pour l'enseignement de la densité et de la masse : tâches exploratoires à travers les intégrales d'une et plusieurs variables

Elementos epistemológicos para o ensino de densidade e massa: tarefas exploratórias por meio de integrais de uma e mais variáveis

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Abstract

Differential and Integral Calculus (CDI) is an essential subject for teaching Mathematics and other sciences. Despite this importance, we observed student failure and high rates of failure or dropout, which justifies the relevance of considering epistemological aspects that make it possible to understand different phenomena in the teaching of this discipline. In this sense, we propose a study of epistemological elements of density and mass knowledge, through multivariational integrals, since integral is essential knowledge for the area of exact sciences. The objective of this study is to investigate the development and implementation of an intervention proposal, centered on study and research activities, that offers CDI students opportunities to explore the concept of the integral of one and more variables. To this end, we carried out an investigation through the creation and implementation of an intervention based on work with task-solving episodes, in order to analyze generalization movement(s) that students performed to define a multivariational defined integral from defined integrals of a variable. The results showed that the students mobilized a set of knowledge of multiple integrals, from the context of calculating mass in one, two and three dimensions. Expansive

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generalization was used to expand procedural issues in the calculation of an integral, while reconstructive generalization was used to understand structural aspects of the Riemann integral of more than one variable.

Keywords: Teaching differential and integral calculus, Multivariational integrals, Epistemology of knowledge, Density and mass, Exploratory tasks.

Resumen

El Cálculo Diferencial e Integral (CDI) es una asignatura imprescindible para la enseñanza de las Matemáticas y otras ciencias. A pesar de esta importancia, observamos fracaso estudiantil y altos índices de fracaso o deserción, lo que justifica la relevancia de considerar aspectos epistemológicos que permitan comprender diferentes fenómenos en la enseñanza de esta disciplina. En este sentido, proponemos un estudio de elementos epistemológicos del conocimiento de densidad y masa, a través de integrales multivariantes, ya que la integral es un conocimiento imprescindible para el área de las ciencias exactas. El objetivo de este estudio es investigar el desarrollo e implementación de una propuesta de intervención, centrada en actividades de estudio e investigación, que ofrezca a los estudiantes del CDI oportunidades para explorar el concepto de integral de una y más variables. Para ello, llevamos a cabo una investigación mediante la creación e implementación de una intervención basada en el trabajo con episodios de resolución de tareas, con el fin de analizar el(los) movimiento(s) de generalización que realizaron los estudiantes para definir una integral definida multivariacional a partir de integrales definidas de una variable. Los resultados mostraron que los estudiantes movilizaron un conjunto de conocimientos de integrales múltiples, a partir del contexto del cálculo de masas en una, dos y tres dimensiones. La generalización expansiva se utilizó para ampliar cuestiones de procedimiento en el cálculo de una integral, mientras que la generalización reconstructiva se utilizó para comprender aspectos estructurales de la integral de Riemann de más de una variable.

Palabras clave: Enseñanza del cálculo diferencial e integral, Integrales multivariantes, Epistemología del conocimiento, Densidad y masa, Tareas exploratorias.

Résumé

Le calcul différentiel et intégral (CDI) est une matière essentielle pour l'enseignement des mathématiques et des autres sciences. Malgré cette importance, nous observons des échecs chez les étudiants et des taux élevés d'échec ou d'abandon, ce qui justifie la pertinence de considérer les aspects épistémologiques qui permettent de comprendre divers phénomènes dans l'enseignement de cette matière. Dans ce sens, nous proposons une étude des éléments épistémologiques de la connaissance de la densité et de la masse, en utilisant les intégrales multivariables, étant donné que les intégrales sont des connaissances essentielles pour les sciences exactes. L'objectif de cette étude est d'examiner la conception et la mise en œuvre d'une proposition d'intervention, centrée sur des activités d'étude et de recherche, qui offre aux élèves du CDI des opportunités d'explorer le concept d'intégrale d'une ou plusieurs variables. À cette fin, nous avons mené une enquête basée sur la création et la mise en œuvre d'une intervention fondée sur des épisodes de résolution de tâches, afin d'analyser les mouvements de généralisation effectués par les élèves pour définir une intégrale définie multivariable à partir d'intégrales définies d'une variable. Les résultats ont montré que les élèves ont mobilisé un ensemble de connaissances sur les intégrales multiples à partir du contexte du calcul de la masse en une, deux et trois dimensions. La généralisation expansive a été utilisée pour élargir les questions de procédure dans le calcul d'une intégrale, tandis que la généralisation reconstructive a été utilisée pour comprendre certains aspects de l'intégrale.

Mots-clés : Enseignement du calcul différentiel et intégral, Intégrales multivariées, Épistémologie de la connaissance, Densité et masse, Tâches exploratoires.

Resumo

O Cálculo Diferencial e Integral (CDI) é uma disciplina essencial para o ensino da Matemática e outras ciências. Apesar dessa importância, observamos um insucesso dos estudantes e elevados índices de reprovação ou evasão, o que justifica a relevância de considerar aspectos epistemológicos que possibilitam compreender diversos fenômenos no ensino dessa disciplina. Nesse sentido, propomos um estudo de elementos epistemológicos dos saberes de densidade e massa, por meio de integrais multivariacionais, visto que integral é um saber essencial para a área das exatas. O objetivo deste estudo é investigar a elaboração e implementação de uma proposta de intervenção, centrada nas atividades de estudo e pesquisa, que ofereça aos estudantes de CDI oportunidades para explorar o conceito de integral de uma ou mais variáveis. Para tanto, realizamos uma investigação a partir da criação e implementação de uma intervenção baseada em episódios de resolução de tarefas, a fim de analisar os movimentos de generalização que os estudantes realizaram para definir uma integral definida multivariacional com base em integrais definidas de uma variável. Os resultados apontaram que os alunos mobilizaram um conjunto de conhecimentos de integrais múltiplas a partir do contexto de cálculo de massa em uma, duas e três dimensões. A generalização expansiva foi utilizada para expandir questões procedimentais do cálculo de uma integral, enquanto a generalização reconstrutiva foi utilizada na compreensão de aspectos estruturais da integral de Riemann de mais de uma variável.

Palavras-chave: Ensino de cálculo diferencial e integral, Integrais multivariacionais, Epistemologia do saber, Densidade e massa, Tarefas exploratórias.

Epistemological elements for teaching density and mass: exploratory tasks through integrals of one and more variables

The subject of Differential and Integral Calculus (DIC) has been associated with high failure and dropout rates for several decades, both in Mathematics and Physics courses and in Engineering courses (Silva, 2013; Zarpelon, 2022). One of the reasons for these high dropout rates can be attributed to the manner in which the subject of DIC has been traditionally approached in universities.

> Traditionally, mathematics subjects, both in Basic Education and (especially) in Higher Education, are " cartesian", supported by the example-exercise definition tripod, followed by a cumulative assessment that prioritises the reproduction of procedures. One of the effects of this tradition is to make these subjects, in the view of students, tedious and algorithmic, without objectives and demotivating, especially for engineering courses. (Couto; Fonseca & Trevisan, 2017, p. 51).

We support the idea that teaching and learning environments that differ from traditional methods are necessary for mathematics subjects in Higher Education, as they give students the opportunity to play a leading role, enhancing their understanding of mathematical concepts (Trevisan & Mendes, 2018; Trevisan; Alves & Negrini, 2021; Trevisan & Araman, 2021; Trevisan, 2022).

A potential approach involves the use of exploratory tasks (Ponte, 2005, 2014) with more open-ended characteristics, which can be solved intuitively, without a formal definition having been presented in advance, motivating students to think autonomously and, through the teacher's intervention, to explore mathematical concepts and not just reproduce them. In this way, collaborative work is prioritised (Granberg & Olsson, 2015; Carlsen, 2018), fostering interaction between classmates, where one student's contribution can influence the hypotheses of others, especially when a student is following an unfounded path.

In the specific context of the concept of the Riemann Integral via Riemann Sums, the focus of this study, understanding this concept is crucial not only in mathematics, but also in subsequent Science subjects (Haddad, 2013; Greefrath et al., 2021). However, many students, even after have finished the DIC subject, find it difficult to understand this concept, by mobilising only operational knowledge and disregarding mathematics subjects and properties. It is therefore essential to reflect on didactic situations, using epistemological and didactic concepts (Schneider & Job, 2016), which circumscribe the concept of integral.

According to Jones, Lim and Chandler (2017), several recent studies highlight that DIC students often experience difficulties when using the concept of integration, both in Mathematics subjects and in subsequent Science disciplines. In analysing these difficulties and their causes, it is recognised that "the ideas contained in the Riemann sum structure are fundamental to a robust understanding of definite integration" (Jones; Lim & Chandler, 2017, p. 1076).

In turn, Mateus-Nieves and Moll (2021, p.23), when reflecting on the epistemic complexity of the mathematical object integral, argue that "a strategy to ensure students' competence in using the integral to solve problems is to design sequences of tasks designed to present different interconnected partial meanings of the integral". In this sense, we have listed density and mass knowledge as epistemological elements to be studied by means of multivariable integrals, as a means of illustrating a possible intervention centred on research study activities that present some of the meanings of the integral in an interconnected way.

In particular, the development of an epistemological reference model (ERM) (Gascón, 2011) for teaching multivariable integrals involves reflecting on aspects or dimensions of the didactic problem of teaching this ICD content. Specifically, in this work we have taken this ERM as a didactic organisation, in the form of a proposal for exploratory tasks (Ponte, 2005), which we have experimented in two Engineering classes at the Federal Technological University of Paraná, Londrina campus, in the DIC 2 subject.

Therefore, *the aim of this study is to investigate the design and implementation of an intervention proposal centred on study and research activities that provides DIC students with opportunities to explore the concept of integral of one and more variables*. To this end, we carried out an investigation by designing and implementing an intervention based on working with episodes of task solving, taking into account the concepts of layers of knowledge related to the concept of definite integral (Sealy, 2006, 2014), together with the idea of Multiplicative Base Sums (Jones; Lim & Chandler, 2017), as well as the generalisation movements involved in the definition of multivariate integrals (Jones, 2015).

Epistemological elements for the teaching of density and mass by means of integrals

In an epistemological model in Mathematics, according to Almouloud (2007), the process of constructing scientific mathematical concepts must not only be related, but must also integrate their historical contexts - the evolution of knowledge - and their genesis, and analyse how the subject understands this process.

According to Mateus-Nieves (2021), there is a scarcity of literature related to studies on the integral as a scientific mathematical concept, which underlines the importance of a work such as this one, as it proposes to discuss some fundamental epistemological elements in the process of its theoretical constitution. As well as contributing to a better understanding of the concept of the integral itself, this study can offer tools that deserve to be taken into account in the construction of an epistemological reference model (ERM) for the teaching of ICD in general.

In this context, we present some epistemological elements for teaching density and mass using integrals of one and more variables. To do this, we go through the idea of how mathematical reasoning (MR) develops, considering its different processes, with an emphasis on generalisation and its types. Then, we argue how the use of integrals is fundamental for students to be able to restructure/reconstruct the concepts of density and mass.

Mathematical reasoning processes

The development of mathematical reasoning is an important objective within the context of mathematics teaching at all school levels (Goos & Kaya, 2020). According to Ponte, Quaresma and Mata-Pereira (2020, p.7), mathematical reasoning involves "making inferences in a reasoned way, in other words, starting from given information to obtain new information through a justified process". This perspective is shared by Jeannotte and Kieran (2017, p.7), who state that mathematical reasoning is "inferring mathematical statements from other mathematical statements".

However, understanding mathematical reasoning involves other nuances beyond its definition. Jeannotte and Kieran (2017) propose an approach to mathematical reasoning from two perspectives: structural and procedural. Within the structural framework, mathematical reasoning is categorised as deductive, inductive and abductive. Regarding the procedural aspect, the authors identify nine distinct but interrelated processes: generalisation, conjecture, pattern identification, comparison, classification, validation, justification, proof and formal proof (Jeannotte; Kieran, 2017).

In particular, the analysis of different mathematical reasoning processes in the context of working with exploratory tasks (Ponte, 2005) in DIC classes has been the focus of the research group in which this article is developed (Negrini; Trevisan & Araman, 2024, in press; Trevisan; Araman & Serrazina, 2023).

Our focus in this article is on the process of generalisation. Generalisation involves 'inferring a narrative about a set of mathematical objects, or a relationship between objects in the set, from a subset of that set' (Jeannotte; Kieran, 2017, p. 9). In this process, a pattern or property that is common to a set of objects is identified, allowing the domain of validity of that property to be extended to a larger set of objects (Ponte; Mata-Pereira, 2017).

In conjunction with these definitions, Harel and Tall's (1991) theory defines

> [...] the term 'generalisation' is used both inside and outside mathematics to refer to the process of applying a given argument in a wider context. However, the cognitive processes required for mathematical generalisation depend on the individual's current knowledge (Harel & Tall, 1991, p. 1).

Having this in mind, the authors distinguish three different types of generalisation: disjunctive generalisation, reconstructive generalisation and expansive generalisation. In the first case, disjunctive generalisation "occurs when, in moving from a familiar context to a new one, the subject constructs a new disjunctive schema to deal with the new context and adds it to the set of available schemas" (Harel & Tall, 1991, p.2). Expansive generalisation occurs "when the subject expands the scope of an existing schema without reconstructing it" (Harel & Tall, 1991, p.2); whereas "reconstructive generalisation occurs when the subject reconstructs an existing schema to expand its scope" (Harel & Tall, 1991, p.2). (Harel & Tall, 1991, p.2). Finally,

> [...] reconstructive generalisation is true generalisation in the sense that previous schemata are directly included as special cases in the final schema. Reconstructive generalisation differs in that the old schema is modified and enriched before it is subsumed into the more general schema (Harel & Tall, 1991, p.2,).

When applied to the concept of definite integrals, the disjunctive and expansive generalisations are not sufficient to extend the understanding of the integral of one variable to multivariable integrals, and the student needs to reach the reconstructive generalisation, as some modifications to the definite integral of one variable scheme are required to include more variables.

Multivariable integrals

The aim of this subsection is to present a discussion about the systematisation of the concept of multivariable definite integrals as a generalisation of the concept of definite integrals of a single variable. In the context of teaching these concepts, although it is clear to experts that multivariable calculus topics are natural extensions of single variable calculus topics, how students see the relationship between ideas such as function and rate of change in single and multivariable calculus is not well understood (Dorko & Weber, 2014, p. 2).

Therefore, extending the concept of an integral of one variable to an integral of multiple variables may not be trivial for students (Jones, 2015), as this process requires a restructuring/expansion of the concepts of integrals of one variable in order to conceptualise multivariate integrals. Jones (2015) suggests that students' understanding of multivariate integrals is strongly rooted in previous conceptions of integrals of one variable. These students 'may not be able to simply extend their knowledge to the multivariate domain, but they may need to revisit the ideas contained in their conceptions of single integrals and restructure them to create an understanding of multiple integrals' (Jones, 2015, p. 167).

The author bases this process of restructuring multiple integrals on the three different generalisation processes of Harel and Tall's (1991) theory presented above. Thus, the successive generalisations of the techniques for integrating an integral defined in R1, R2 and R3 are essentially a case of applying the same techniques to determine the value of the integral defined in each dimension. The algebraic aspects of this process will probably be an extended generalisation for most students. However, the geometric aspects of the definite integral in R1, R2 and R3 - the modification of geometric ideas in one, two and three dimensional space - are likely to require a reconstructive generalisation that few will be able to achieve (Jones, 2015).

Therefore, our proposal is to use contexts in which the definite integral of one variable is used to organise an intervention with a sequence of exploratory tasks that allow students studying DIC to establish an extension/reconstruction of the concept of integral from one to multiple variables. Given that most mathematics used in the real world involves functions of many variables, we will start from the context of Physics, choosing the concept of density and mass of a one-dimensional object in order to reconstruct/extend the concept of the definite integral to the two- and three-dimensional contexts.

Density and mass of a one-dimensional rod

In this and the next two sections, we will briefly discuss the concept of multivariable definite integral as an extension of the concept of definite integral of one variable, based on Stewart (2013, 2016).

Linear density is the measure of a quantity of any characteristic value per unit length. Let's consider a long, thin rod of mass m and length Δx. The density of this one-dimensional object is expressed as $\rho = \frac{m}{\Delta x}$. Therefore, the mass of this object is given by the formula $m = \Delta x \, . \, \rho$.

The above equation defines the mass as long as the density is constant. But what happens if the density is variable? In other words, $m = \Delta x \, . \, \rho(x)$. Suppose that a one-dimensional object is positioned along a coordinate axis between $x = a \; and \; x = b$, subject to a variable density $\rho(x)$ and that it is partitioned into five subintervals of $[a, b]$ (Figure 1).

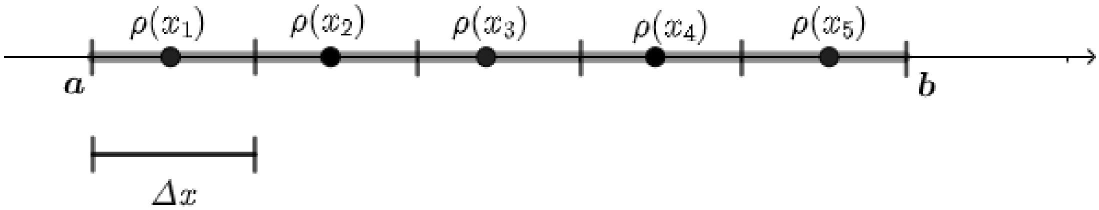


Figure 1.

Non-homogeneous rod partitioned into 5 subintervals (image by the authors, 2022)

If the density is variable, we assume a representative density, which we can also call sample density, in each subinterval in order to calculate an approximation of the mass of this one-dimensional object. Therefore, the approximation of the value of the mass of this object will be given by the sum of the product between the length and the representative density in each subinterval of $[a, b]$. In other words,

$$m \simeq \rho(x_1)\Delta x + \rho(x_2)\Delta x + \rho(x_3)\Delta x + \rho(x_4)\Delta x + \rho(x_5)\Delta x = \sum_{i=1}^{5} \rho(x_i)\Delta x$$

Now, if we partition the interval $[a, b]$ into 10 subintervals (Figure 2) and assume a representative density in each of these subintervals, as shown in the following figure, we have that the mass of this line is calculated by the sum of the products of Δx and the representative density in each subinterval:

$$m \simeq \rho(x_1)\Delta x + \rho(x_2)\Delta x + \rho(x_3)\Delta x + \rho(x_4)\Delta x + \rho(x_5)\Delta x + \rho(x_6)\Delta x + \rho(x_7)\Delta x$$
$$+ \rho(x_8)\Delta x + \rho(x_9)\Delta x + \rho(x_{10})\Delta x = \sum_{i=1}^{10} \rho(x_i)\Delta x$$

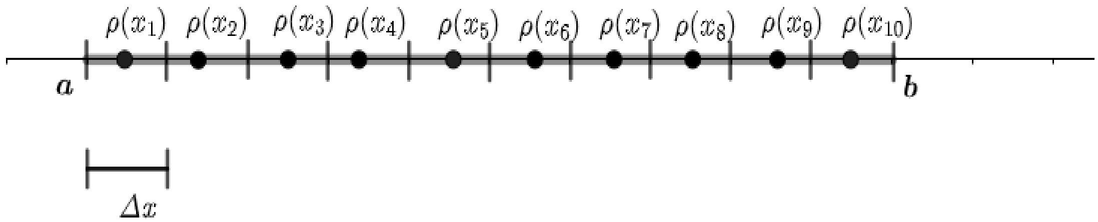


Figure 2

Non-homogeneous rod partitioned into 10 subintervals (image by the authors, 2022).

Now, let's assume that this rod is subdivided into *n* partitions and the density is approximately constant in each interval (Figure 3).

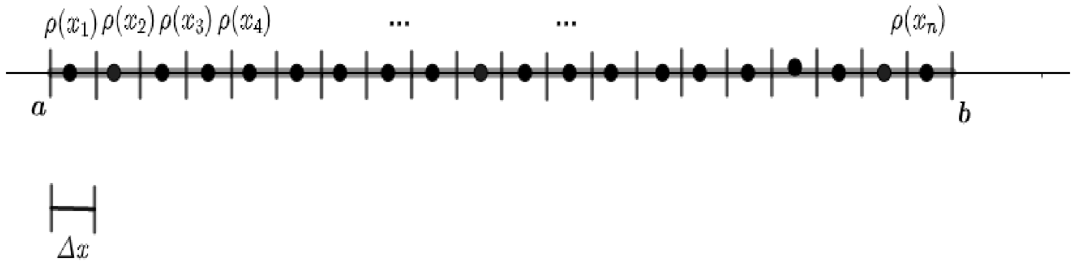


Figure 3.

Non-homogeneous rod partitioned into n subintervals (image by the authors, 2022).

The approximate mass of this rod is given by the following summation,

$$m \simeq \rho(x_1)\Delta x + \rho(x_2)\Delta x + \rho(x_3)\Delta x + \rho(x_4)\Delta x + \ldots + \rho(x_n)\Delta x = \sum_{i=1}^{n} \rho(x_i)\Delta x$$

that is,

$$m \simeq \sum_{i=1}^{n} \rho(x_i)\Delta x \quad (1)$$

The mass is therefore determined by a Riemann sum, in other words, a sum with a multiplicative basis. If we assume that Δx is infinitesimally small and n tends to a relatively large value, in other words, $n \to \infty$, we obtain the exact value of the mass of this rod of length b - a.

$$m = \lim_{n \to \infty} \sum_{i=1}^{n} \rho(x_i)\Delta x = \int_a^b \rho(x)\Delta x = \int_a^b \rho(x)dx$$

Let's now look at another context that leads us to interpret the concept of definite integrals based on the calculation of the area under a curve. Let's consider the area S of a region delimited by the graph of the function $y = f(x)$, bounded by $x = a$ and $x = b$ (Figure 4).

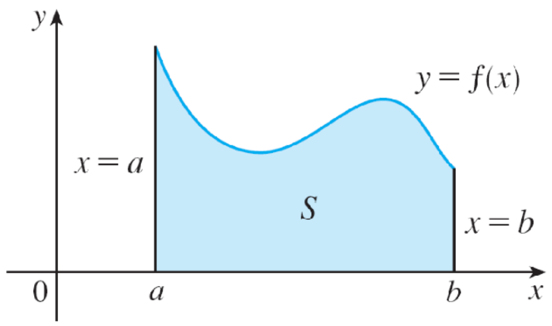


Figure 4.

Area under the $y = f(x)$ curve. (Stewart, 2013, p.326).

Suppose S is divided into n subintervals, $S_1, S_2, S_3, \dots, S_n$, of equal width, as shown in Figure 5.

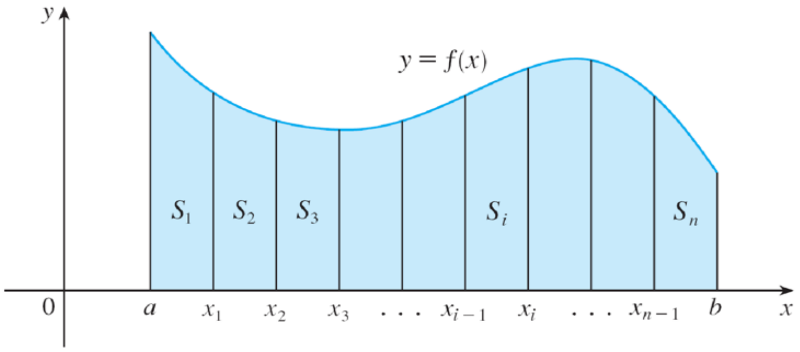


Figure 5.

Region S divided into n subintervals. (Stewart, 2013, p. 329).

We can approximate each interval by a rectangle with a base equal to the width of the interval and a height equal to the right side of the interval. In other words, the heights of these rectangles are the values of the function $y = f(x)$ at the right ends of the subintervals.

The width of the interval $[a, b]$ is $a - b$, so the width of each of the n bands is $x = \frac{b-a}{n}$. These bands divide the interval $[a, b]$ into n subintervals $[x_0, x_1], [x_1, x_2], [x_2, x_3], \dots, [x_{n-1}, x_n]$, where $x_0 = a$ and $x_n = b$.

Let's approximate the ith sheet $S_i$by a rectangle with width Δx and height $f(x_i)$, which is the value of f at the right end (Figure 6).

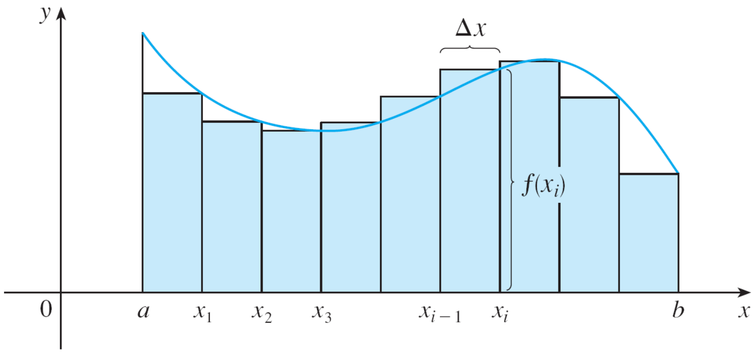


Figure 6.

Approximate the area of the region using rectangles. (Stewart, 2013, p. 330)

So the area of the *i*th rectangle is $f(x_i)\Delta x$. What we intuitively consider to be the area of S is approximated by the sum of the areas of these rectangles, which is

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$$

So let's define the area A of the region S as follows.

The area A of the region S lying under the graph of a continuous function is the limit of the sum of the areas of the approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x] = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

It can be shown that this limit always exists, since we assume that the function is continuous. It can also be shown that we get the same value by using the left ends of the approximants:

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty}[f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x] = \lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i)\Delta x$$

So instead of using the left or right edges, we can take the height of the *i*-th rectangle as the value of f at any number x_i^* in the *i*-th subinterval $[x_{i-1}, x_i]$. We call the numbers $x_1^*, x_2^*, \ldots, x_n^*$ sample points, as shown in Figure 7.

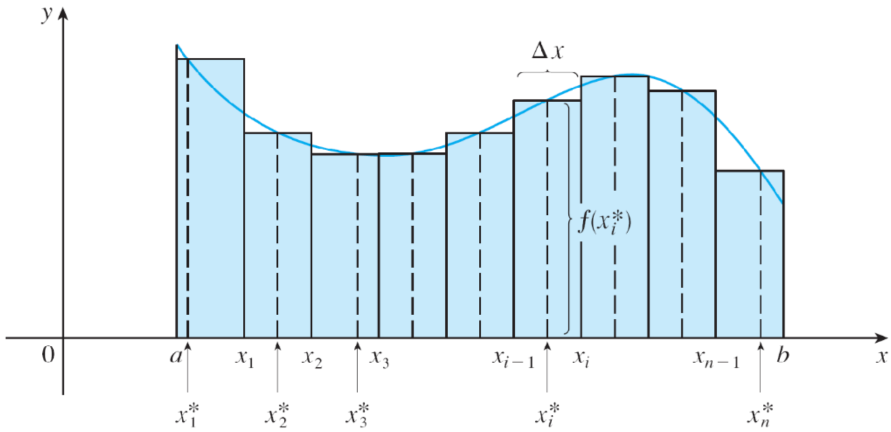


Figure 7.

Marking the sampling points. (Stewart, 2013, p. 331)

The figure shows that the approximate rectangles would be formed if the sample points were not chosen as edges. Therefore, a more general expression for the area S is

$$A = \lim_{n\to\infty}[f(x_1^*)\Delta x \ + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x]$$

Therefore,

$$A = \lim_{n\to\infty}[f(x_1^*)\Delta x \ + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x] = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

We have seen that this structure appears both when trying to find the mass of a one-dimensional non-homogeneous rod and when calculating the area of a region bounded by a curve and vertical rods. The same type of boundary appears in many other situations, such as calculating the work done by a force, the centre of mass of an object or the distance travelled by an object.

Thus, we can define the definite integral.

If f is a continuous function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal length $\Delta x = \frac{b-a}{n}$. Let $x_0(= a)x_1, x_2, \ldots, x_n(= b)$ be the ends of these subintervals, and let $x_1^*, x_2^*, \ldots, x_n^*$ be arbitrary sample points in these subintervals, so that x_i^* is in the i-th subinterval $[x_{i-1}, x_i]$. So the definite integral of f from a to b is

$$\int_R f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

as long as the limit exists and gives the same value for all possible choices of sample points. If it exists, we say that f is integrable on $[a, b]$.

Density and mass of a non-homogeneous sheet of sheet

Let's now consider an idealised flat object that is thin enough to be imagined as a flat two-dimensional region. We call such an object a sheet. A sheet is said to be homogeneous if its composition is entirely uniform, otherwise it is said to be inhomogeneous. The density ρ of a homogeneous sheet of mass m and area A is given by $\rho = \frac{m}{A}$. On the other hand, in an inhomogeneous sheet, the composition can vary from point to point. An appropriate definition of "density" must reflect this condition. To establish such a definition, suppose the sheet is placed on an xy plane. The density at the point (x, y) can be specified by a function $\rho(x, y)$, called the density function.

Suppose that the density of the sheet at the point (x, y) can be specified by a density function $\rho(x, y)$. Consider this non-homogeneous sheet bounded by the intervals $[a, b]$ and $[c, d]$, on the x and y axis respectively, and divided into n rectangles and, in each rectangle, the density is approximately constant, in other words, a representative density (Figure 8). Thus, the approximate total mass can be calculated by adding up the masses in each rectangle.

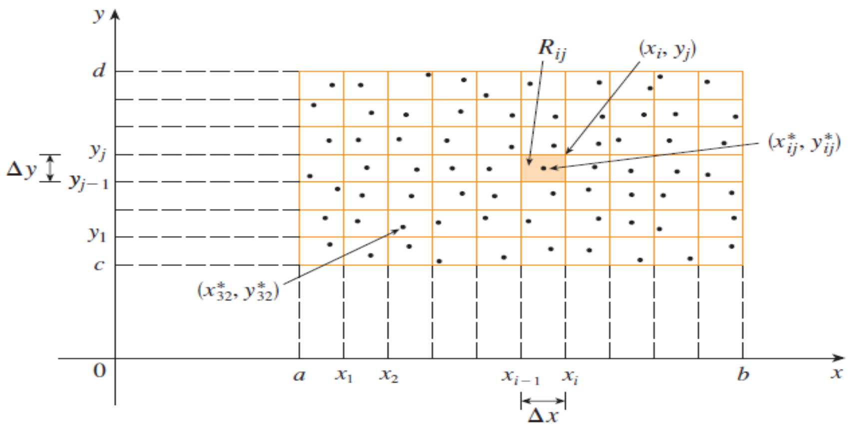


Figure 8.

Two-dimensional non-homogeneous sheet. (Stewart, 2016, p. 875)

If we choose a representative density at each $\rho(x_{ij}, y_{ij})$, the mass of this sheet is given by its density times the area of the base rectangle:

$$\rho(x_{ij}, y_{ij})\Delta A$$

If we do this for all the rectangles and add up the masses, we get an approximation to the total mass of the sheet:

$$m \simeq \sum_{i=1}^{m}\sum_{j=1}^{n} \rho(x_{ij}, y_{ij})\Delta A$$

This double sum means that for each sub-rectangle we calculate the value of ρ at the chosen point, multiply this value by the area of the sub-rectangle and then add the results together.

If we now increase n and m so that the dimensions of the rectangles tend towards zero, then it is plausible that the errors of our approximations will also tend towards zero,

$$m \simeq \lim_{m,n\to\infty} \sum_{j=1}^{n}\sum_{i=1}^{m} \rho(x_{ij}, y_{ij})\Delta A = \int_c^d \int_a^b \rho(x, y)\Delta A = \int_c^d \int_a^b \rho(x, y)\, dxdy$$

or

$$m \simeq \lim_{m,n\to\infty} \sum_{j=1}^{n}\sum_{i=1}^{m} \rho\big(x_{ij}, y_{ij}\big)\Delta A = \int_a^b \int_c^d \rho(x, y)\Delta A = \int_a^b \int_c^d \rho(x, y) dydx.$$

Let's now consider another interpretation of this result: a function f of two variables defined on a closed rectangle, $R = [a, b] \times [c, d] = \{(x, y) \in R^2 |\ a \leq x \leq b, c \leq y \leq d\}$ (Figure 9).

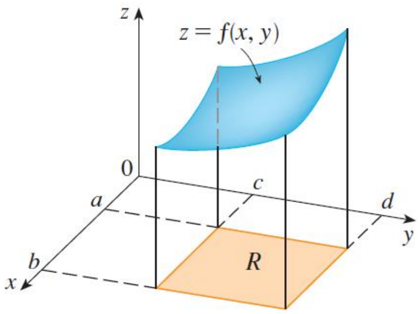


Figure 9

Solid S that is above the region R and below the graph of f. (Stewart, 2016, p. 874)

Let's first assume that $f(x, y) \geq 0$ and that the graph of f is the surface with equation $z = f(x, y)$. Let S be the solid that lies above the region R and below the graph of f, in other words, $S = \{(x, y, z) \in R^3 |\ 0 \leq z \leq f(x, y), (x, y) \in R\}$. Our objective is to determine the volume of S.

The first step is to divide the rectangle R into sub-rectangles. We do this by dividing the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal length $\Delta x = \frac{(b-a)}{m}$ and dividing the interval into m subintervals $\big[y_{j-1}, y_j\big]$ of equal length $\Delta y = \frac{(d-c)}{n}$. By drawing rods parallel to the coordinate axes, passing through the ends of the subintervals, as in Figure 10, we form the subrectangles.

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y)|\, x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j\}$$

each with an area of $\Delta A = \Delta x \Delta y$.

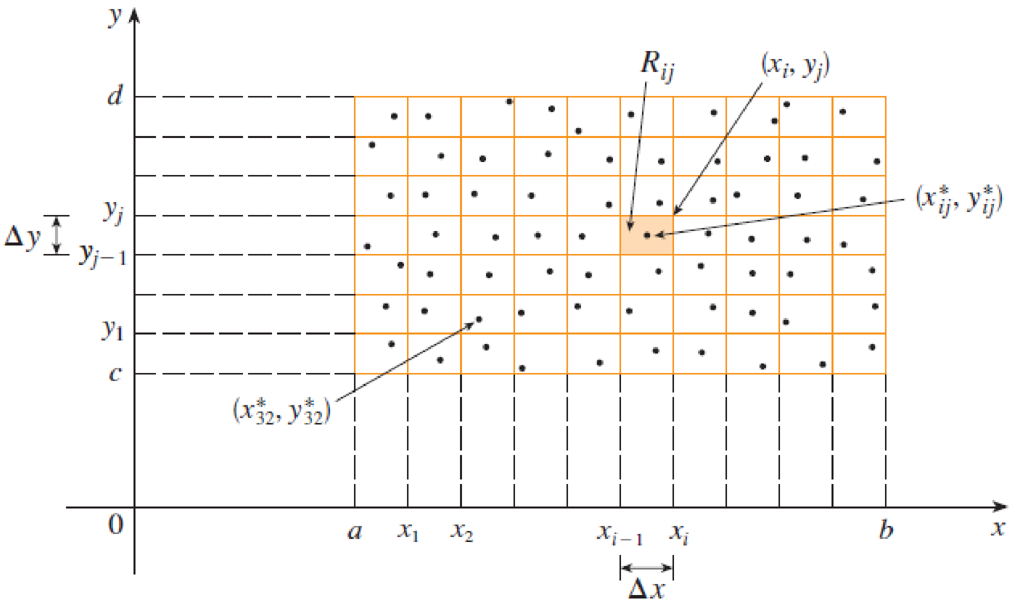


Figure 10.

Partitioned R area. (Stewart, 2016, p. 875)

If we choose an arbitrary point, which we'll call the sampling point, (x_{ij}^*, y_{ij}^*), we can approximate the part of S that is above each R_{ij} by a thin rectangular box (or "column") with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$, as shown in Figure 11. The volume of this box is given by its height times the area of the base rectangle, in other words, $f(x_{ij}^*, y_{ij}^*)\Delta A$.

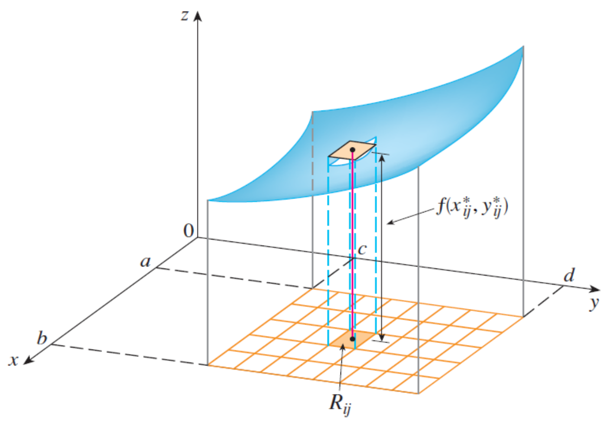


Figure 11.

Thin rectangular box. (Stewart, 2016, p. 875)

If we follow this procedure for all the rectangles and add up the volumes of the corresponding boxes, we'll get an approximation of the total volume of S:

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^{*}, y_{ij}^{*}) \Delta A$$

This double sum means that, for each sub-rectangle, we calculate the value of f at the chosen point, multiply this value by the area of the sub-rectangle and then add the results (Figure 12).

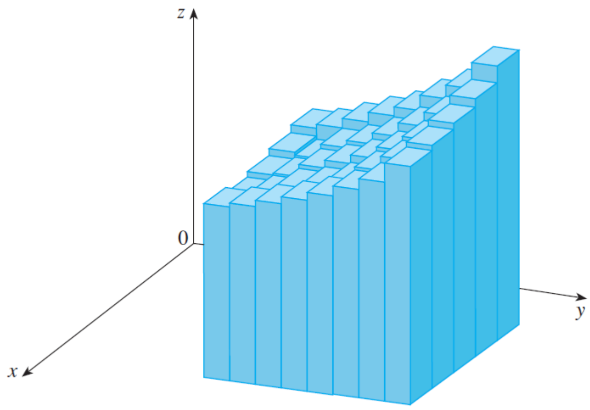


Figure 12.

Boxes (Stewart, 2016, p. 875)

Our intuition tells us that the approximation given in the previous equation improves when we increase the values of n and m and therefore we should expect that

$$V = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^{*}, y_{ij}^{*}) \Delta A$$

We used the above expression to define the volume of the solid S that corresponds to the region below the graph of f and above the rectangle R. Limits of this kind occur very often, not only when we are determining volumes, but also in various other situations. You can therefore define the double integral, or the integral of two variables, of a function f over a rectangular region R as

$$\int \int_{R} f(x, y)\, dA = \lim_{m,n \to \infty} \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_{ij}, y_{ij})\, \Delta A$$

if there is a limit.

Density and mass of a non-homogeneous solid

Consider a three-dimensional solid G. If G is homogeneous, then its density is defined as its mass per unit volume. Thus, if G is a homogeneous solid of mass m and volume V, its density is given by $\rho = \frac{m}{V}$. If G is inhomogeneous and is in an xyz coordinate system, its density at the generic point (x, y, z) is specified by the density function $\rho(x, y, z)$

Consider G a non-homogeneous solid with the shape of a rectangular box with variable density $\rho(x, y, z)$ (Figure 13).

$$G = \{(x, y, z) |\ a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$$

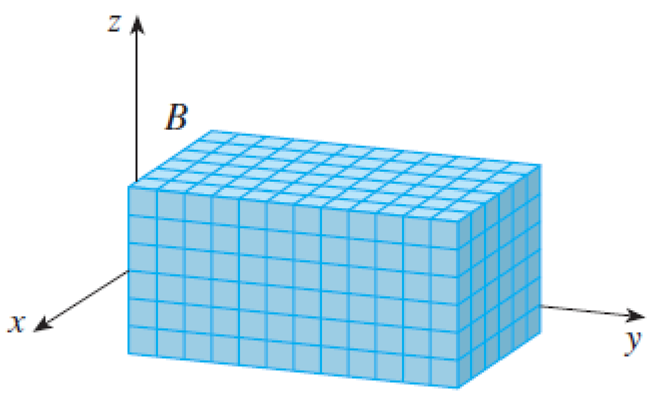


Figure 13.

Solid G. (Stewart, 2016, p. 913)

The first step is to divide G into subboxes. We do this by dividing the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal length Δx, dividing $[c, d]$ into m subintervals of length Δy and dividing $[r, s]$ into n subintervals of length Δz. The planes passing through the ends of these subintervals, parallel to the coordinate planes, subdivide the box G into lmn subboxes, as shown in Figure 14.

$$G_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

Each sub box has a volume, $\Delta V \ = \ \Delta x \Delta y \Delta z$.

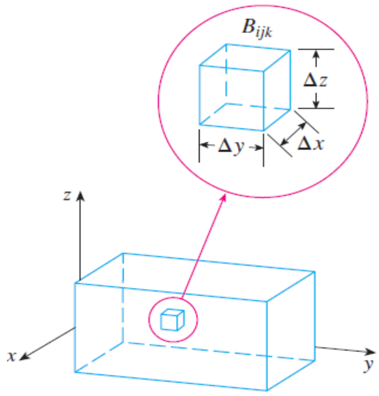


Figure 14.

Subbox (Stewart, 2016, p. 913)

So, to calculate an approximate value for the mass of solid G with variable density $\rho(x\ , y, z)$ we form the Riemann triple sum:

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}\rho\left(x_{ijk}, y_{ijk}, z_{ijk}\right)\Delta V$$

where the representative density $\rho(x_{ijk}, y_{ijk}, z_{ijk})$ is in G_{ijk}.

By analogy with the definition of the double integral, we define the triple integral as the limit of the Riemann triple sums in

$$m = \lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}\rho\left(x_{ijk}, y_{ijk}, z_{ijk}\right)\Delta V$$

$$= \int_a^b\int_c^d\int_r^s \rho(x, y, z)\Delta V = \int_a^b\int_c^d\int_r^s \rho(x, y, z) dzdydx$$

Therefore, you can define a triple integral or integral of three variables of a function f over a region G of space as

$$\int\int_G\int f(x, y, z)dV = \lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}, y_{ijk}, z_{ijk})\ \Delta V$$

if there is a limit.

Assumptions and planning of the intervention

This research is part of a series of studies on the teaching of DIC, focusing on the approach of working with episodes of solving exploratory tasks. Working with this type of task is "more effective in terms of learning if they [the students] discover their own method of solving a problem than if they are expected to learn the teacher's method and be able to recognise how to apply it in a given situation" (Ponte, 2005, p.9).

This approach is supported by the National Curriculum Guidelines for Undergraduate Engineering Courses (Brasil, 2019), which emphasise the need to develop students' ability to analyse and understand physical and chemical phenomena using symbolic models throughout their training. This analysis should involve the use of mathematical, statistical and computational tools, among others, to predict outcomes and design experiments that can produce concrete results.

This underlines the importance of planning an intervention that allows students to really understand the concepts of the DIC, and to make the connections and conjectures needed to solve problems in the context of their studies and, in the future, in the practice of their profession.

Learning in this work context takes on a collaborative character, as it "[...] encourages student participation in the learning process and makes learning an active and effective process" (Torres, Alcântara & Irala, 2004, p. 131). Collaborative work allows students to have their hypotheses questioned, promoting reflection on them, providing the possibility of validating them using valid propositions or refuting them if their hypotheses are inconsistent or false; in other words, colleagues in the group guide and assist them in the construction of knowledge.

Therefore, the teacher's role in classes structured around solving exploratory tasks and collaborative work is to guide students in their discussions so that they can achieve the learning objectives. Therefore, it is essential for the teacher to get to know their students and design exploratory tasks that promote peer discussion and learning.

In order to organise the intervention, we initially carried out a diagnostic assessment, so that we could get an idea of the prior knowledge of the students who would be studying Calculus of more than one real variable (a context that will be detailed in the next section), since this knowledge would help us to formulate tasks that would meet our objectives. With this in mind, we first looked for tasks that would help us revisit the concept of the definite integral of one variable, both in a purely mathematical context and in physical contexts, and that would allow us to explore the layers of the Riemann Sum structure in an intuitive way.

We then surveyed physics concepts in which it would be possible, in the same context, to extend the concept of definite integral of one variable to definite integrals of two and three variables. We then studied which physics contexts involved definite integrals and in which of them it would be possible to approach the definite integral of one, two and three variables. It was then that we arrived at the context of calculating the density and mass of an object, this concept being applicable in one, two and three-dimensional contexts.

Based on this context and our perceptions of the students' prior knowledge, we constructed a sequence of exploratory tasks in which we could achieve the objective of our work. The intervention was planned for engineering classes in which the second author and research supervisor was teaching DIC 2 in 2022.

We devised three exploratory tasks (Araujo, 2023), each of which was proposed to be solved in small groups, in one 50-minute lesson each. Two 50-minute lessons were then used to discuss the students' solutions to the tasks and systematise the concepts.

Once the concept of definite integral had been systematised, two more 50-minute lessons were dedicated to reviewing the techniques of integration of one variable, more specifically, the substitution rule and integration by parts. After this first part of the "revision" of definite integrals of one variable, another meeting with two 50-minute lessons was dedicated to exploratory task 3 (the focus of this article) - Box 1, which had the objective, within the context of the mass of an object, uni-, bi- and tridimensional, of defining the concept of integral of two and three variables.

Table 1.

Exploratory task 3 (The authors)

Exploratory task 3

1. **THE DENSITY AND MASS OF A ONE-DIMENSIONAL STEM**

Linear density is the measure of a quantity of any characteristic value per unit length. Consider a long, thin stem of mass m and length Δx. The density of this one-dimensional object is expressed as $\rho = \frac{m}{\Delta x}$. The mass of this object is then given by the formula $m = \rho \cdot \Delta x$.

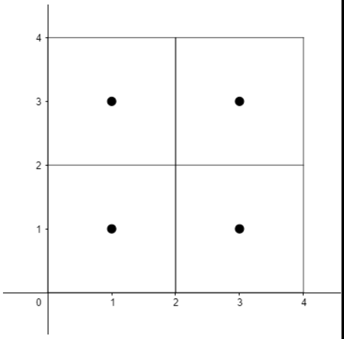
i. What is the mass of a homogeneous stem with a length of 1.5m and a density of 2.5kg/m?

ii. The above equation defines the mass as long as the density is constant. But what happens if the density varies? In other words, $m = \rho \cdot \Delta x$. Suppose this one-dimensional object is positioned along a coordinate axis between $x = a$ and , with variable density ρ(x). Explain how to find the total mass of the object, using words, drawings, writing formulas or any other type of record..

2. **DENSITY AND MASS OF AN INHOMOGENEOUS SHEET**

Let's consider an idealised flat object that is thin enough to be imagined as a flat two-dimensional region. We call such an object a sheet. A sheet is said to be homogeneous if its composition is entirely uniform, otherwise it is said to be non-homogeneous. The density ρ of a homogeneous sheet of mass m and area A is given by $\rho = \frac{m}{A}$. On the other hand, in a non-homogeneous sheet, the composition can vary from point to point, and an appropriate definition of "density" must reflect this condition. To establish such a definition, suppose the sheet is placed in an xy plane. The density at point (x, y)can be specified by a function $\rho(x, y)$, called the density function.

Consider a sheet with vertices (0, 0), (0, 4), (4 , 4) and (4 , 0), with variable density, in other words, a non-homogeneous sheet, with density $\rho(x, y) = x + y$ kg/m.



i. Consider this sheet subdivided into 4 rectangles, as shown in the figure below. Assuming that in each rectangle the density is constant and equal to the value at the representative point indicated, determine the approximate total mass of the sheet.

ii. Now consider subdividing this sheet into 8 rectangles of your choice. Assuming that in each rectangle the density is constant and equal to the value at a representative point also of your choice, determine the approximate total mass of the sheet.

Now suppose a more general case, where the density of the sheet at the point (x, y) can be specified by a density function $\rho(x, y)$. Explain how to find the total mass of this non-homogeneous sheet, using words, drawings, writing formulas or any other type of record.

3. **DENSITY AND MASS OF A SOLID**

Consider a three-dimensional solid G. If G is homogeneous of mass m and volume V, then its density is given by $\rho = \frac{m}{V}$. If G is non-homogeneous and lies in an xyz coordinate system, then its density at the generic point (x, y, z) is specified by a function $\rho(x, y, z)$.

Explain how to find the total mass of this non-homogeneous solid, using words, drawings, writing formulas or any other type of record.

Finally, two more 50-minute lessons were dedicated to discussing the solutions with the students and then systematising the concept of the definite integral of two and three variables.

Research methodology

We recognise that the objective of the research was to investigate a specific reality and, possibly, with this investigation to have enough information to make an intervention in this reality, which allows it to be characterised as qualitative (Bogdan & Biklen, 1994).

According to Gerhardt and Silveira (2009), qualitative research is not concerned with numerical representations of the context being researched, but rather with an in-depth understanding of a social group or organisation. For these authors, researchers who use qualitative methods "seek to explain why things happen, expressing what should be done, but they do not quantify values and symbolic exchanges, nor do they submit themselves to the proof of facts" (Gerhardt & Silveira, 2009, p.32).

The research was carried out in classes of Mechanical Engineering, with 35 students, and Materials Engineering, with 15 students, at the Federal Technological University of Paraná, Londrina campus, in the subject of DIC 2. The exploratory tasks were carried out in small groups in order to provide interaction between all the members of the group. All the students were informed about the research being carried out and could choose whether or not to make the material collected during the lessons available for research purposes. It is worth noting that the collection of written protocols from the groups, handed in at the end of each task, was part of the subject's work routine, being provided for in the teaching plan and subsidising the assessment carried out throughout the semester.

So it was up to the students to record the audios of the discussions of the tasks in groups and send them via Whatsapp, thus authorising their use for research purposes. In addition to this material, audios were also collected of the discussion and systematisation of mathematical concepts, conducted by the researcher (first author) with some interventions by the supervisor (second author).

The first stage of the work was to organise the audio material in Google Drive and select which of these materials could actually be analysed, discarding the audios that were inaudible and those in which the groups only systematised their resolution after completing the task instead of the entire discussion process - as the researcher had requested.

For exploratory task 3, we obtained 6 audios, 3 of which we considered to be audible and could be transcribed. We selected a group that we considered representative for analysis, allowing us to understand the mathematical reasoning used by the students to conceptualise a definite integral of two and three variables from the definite integral of one variable in order to identify associated generalisation movements (Harel & Tall, 1991; Jones, 2015). This data is presented and analysed in the next section.

Data analysis

In the first item of the task, the concept of the mass of a one-dimensional object was briefly explored and it was suggested that the students calculate the mass of a one-dimensional rod and then explain how to find the total mass of the non-homogeneous one-dimensional rod, in other words, with variable density. In general, the first question was solved without difficulty, first calculating the mass of the non-homogeneous rod and then setting up an integral to calculate the mass of the non-homogeneous rod, recognising the use of the integration of the density function (Figure 15).

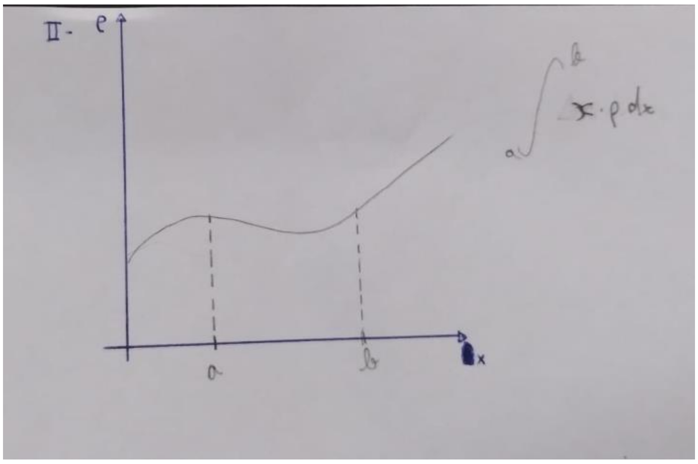


Figure 15.

Solving question 1, item (ii) of Exploratory Task 3 (The authors)

In question 2, an explanatory text was presented on the concept of an inhomogeneous two-dimensional sheet. In the first items of the task, in order to calculate the total mass, the students were asked to make some particular partitions in the sheet, assume a representative density in each partition and then determine the mass of the sheet from these partitions. Finally, in a third item, students were asked to explain how to find the total mass of this non-homogeneous sheet assuming a general case, in which the density of the sheet at the point (x, y) is a function $\rho(x, y)$.

The following excerpt is part of the group's discussion, based on their initial reading of the task statement, before they even began to solve it.

Student 3: Have you seen the balcony now?
Student 2: What?
Student 3: The integral can be a function of *x* and *y*, not just *x*.
Student 2: Yes, you have to integrate by part, it's like the derivative by part, the same as the derivative that integrates by part, you have to integrate by part.

Immediately after reading the statement, Student 3 draws the others' attention to the density function of a two-dimensional sheet, varying in terms of two variables. In other words, to determine the mass of the non-homogeneous sheet, the integral would have two variables and not just one variable, as in the previous item. Student 2 then conjectures that this integral will be solved in a similar way to a partial derivative (to which he refers with the expression "by parts"). He then concludes that a "piecewise" integration process will also be necessary (referring intuitively to the double integration process).

We recognise that this statement is an expansive generalisation of the procedural techniques for determining a double integral from a definite integral of one variable, based on the similarity to the partial derivation process. This is an expansive generalisation, since no existing model was modified in order to define the current one, because in order to define how to solve a definite integral of two variables, the students related it to the way in which a partial derivative is solved, in which one of the variables is considered as a constant in order to derive the other, and is repeated in an analogous way with the other variable. Therefore, there was no need to modify the model for solving a partial derivative to fit the model for solving a definite integral of two variables.

Student 3: So, is that right?

Researcher: Actually, we don't call it by parts, we call it interacted, but it's basically the same thought. In the first question, how many variables were we dealing with?

Student 2: The first one we were dealing with two.

Researcher: No, the first one [referring to question 1].

Student 3: One.

Researcher: When we calculate the integral, what are we calculating?

Student 1: The area under the curve.

Student 2: That's not all.

Student 1: No, but in the example it is.

Student 2: Yes, in this example it is.

Student 2: But the area under the curve would be the sum of the integrals.

Researcher: Not the integrals. If it's the area under the curve, what would it be the sum of?

Student 1: The rectangles.

Researcher: What of the rectangles?

Student 2: Riemann sum.

Researcher: Yes.

Student 2: The area of the subdivisions of the interval.

Researcher: So you were calculating the sum of all the products. Because you were calculating the area under the curve. And when we talk about the area under the curve, I'm talking about when you calculate, all the little areas, and the areas you get through the product.

Student 2: Yes.

Researcher: In the first box, what is the structure for calculating the mass?

Student 2: In the first case? A product.

Researcher: So we're adding up the masses in all the very small intervals and what is the structure of this sum? It's a product. It's a Riemann sum, a multiplicative sum. That's why you use integral. And in the second case?

Student 2: In the second case, it's two variables.

Researcher: An integral for one variable. That means I'm calculating the sum for that variable. When I have two variables? Is one integral sign enough?

Student 2: I don't think you'd have to integrate over two variables, it's as if you were deriving in both x and y. Then integrate in x and y.

In this section of the dialogue, the researcher's contributions, based on interactions with the group members, facilitated an investigation into the concept of integral, extending beyond the conventional notion of an area under a curve to encompass a more comprehensive understanding of multiplicative summation. Consequently, there is a progression towards an expansion of the integral concept from a single variable to multiple variables.

The students then engage in a discussion regarding the most appropriate symbol to represent the concept of "integrating in two variables" as a fundamental step in determining the mass of a two-dimensional sheet. It is therefore concluded that the most appropriate method for calculating the definite integral of two variables would be to use two integral symbols. In this instance, we may perceive the expansive nature of this systematisation, given that the students ultimately determined that, in the case of a single variable, a single integral symbol would suffice; however, in the event of two variables, a symbol representing both variables would be required.

The dialogue continues, now without the intervention of the researcher:

Student 3: Look here and if we calculate the integral of x, the area will be the integral of y, the area will be here.
Student 2: Yes.
Student 3: That must be it.
Student 2: Yes.
Student 3: x will be up and $\rho(y)$ to the right, right?
Student 2: We have to arrive at a function of x and then a function of y.
Student 1: So, that's it. If you take the integral in x, won't it give this area here? Because this area is exactly the area of the square.
Student 3: Wait, why the other way round?
Student 2: Because if you integrate y, x will be a constant, x won't change, it'll be the same. So I don't know if y will really change directions. We're assuming it will.
Student 1: Let's try to put it on paper then.
Student 2: Come on.
Student 3: But let's do the first one, it's good.
Student 2: OK. Consider a sheet subdivided into 4 rectangles, as shown in the figure, assuming that each rectangle has the same density. Okay, in all four, each density here is the same, right? Determine the mass...
Student 3: Mass is density times area.
Student 2: Yes. Area why?
Student 3: Because when it's homogeneous you just do that. Density times area.
Student 2: Yeah?
Student 1: Man, it makes you angry when you know the answer but you have to do it the way you have to.
Student 2: Yeah.
Student 3: Look at the fourth line of the text [student reads the statement].
Student 2: But do we know the density? But in order to calculate ρ, we'll have to calculate this backwards, $x + y$ to give density per kilogram per square metre.
Student 1: Is ρ the density?
Student 2: Yes, and ρ is $x + y$.
Student 3: OK, but wait. When you were talking to the researcher. Do you have to do the mass or the product first?
Student 2: The integral is like the business of a product. It's the area of a product.
Student 3: So, I get it. The density will be in the dots there.

Student 2: Yeah. Is it?

Student 3: Yes, he says: assuming that in each rectangle the density is constant and equal to the value at the representative point.

Student 2: So we're going to do it for each part, right, so we're going to have to do the density for each one. That's going to take a lot of work.

Student 3: No.

Student 2: No, the job I'm talking about is to get, like, the density of each of them.

Student 3: Yes.

Student 2: So the mass is ρ times A, and ρ is $x + y$, right?

Student 3: Put, $\rho_1 a_1 + \rho_2 a_2 + \rho_3 a_3$.

Student 2: Sum of ρ, easier.

Student 3: From 1 to 4.

Student 2: Times A, right?

[...]

Student 3: 4.4 + 4.4 + 6.4.

Student 2: It's 8, it's 16, and it's 16, and 24. That's 64.

Student 3: We've divided it into 4 squares, so if we integrate we'll divide it into infinite squares.

Student 2: True

Student 4: So you shouldn't have used the sum?

Student 3: Then the integral would be when the sum to infinity.

Student 2: That's right, but then it's per part, right?

Student 2: Got it. You know what's going to happen in the integral, we're going to integrate over dx and dy. Then the moment dy is left, x will become a constant, and x will come out of the integral, and then only dy will be left inside.

From this excerpt of the students' discussion, it is evident that the concept of integral from one to two variables is recognised as being reconstructive in nature. It is not simply an extension of the geometric interpretations of simple integrals to double integrals; rather, it incorporates a fundamentally new concept. In reconstructive generalisation, the student modifies an existing scheme in order to broaden its applicability. With regard to the context of analysis, we can see that the student uses the geometric interpretation of a definite integral to extend the understanding of a definite integral to the context of the mass of an object (Harel & Tall, 1991).

In this task, students were required to consider how an appropriate integral should be configured in order to calculate the mass of an inhomogeneous sheet. They did so by interpreting the definite integral of a variable beyond the concept of the area under the curve, thereby demonstrating an understanding of the idea of multiplicative summation. This is evidenced in the solution proposed by Student 3, which suggested that the mass would be $\rho_1 a_1 + \rho_2 a_2 + \rho_3 a_3 + \rho_4 a_4$.

Student 3 proceeds to acknowledge that in a more general case (which ultimately leads to an integral), the sheet would be divided into an infinite number of rectangles. In the event that infinite representative densities are assumed, the mass (m) of the inhomogeneous sheet can be calculated by multiplying the area of the infinite rectangles by the representative density of each rectangle.

It is acknowledged that this movement is defined as reconstructive generalisation, whereby the "old schema is modified and enriched before being incorporated into the more general schema". (Harel & Tall, 1991, p.2)

In conclusion, the analysis of this task reveals that the students employed a number of generalisations as they extended the concept of the definite integral of one variable to more than one variable, as illustrated in Table 2:

Table 2.

Generalisation movements made possible by Investigative Task 3. (The authors)

Expansive Generalisation	Generally relating procedural questions of calculating a definite integral of two and three variables, since in order to set up the integral and define how to solve it, the students used the procedures of the definite integral of one variable and partial derivatives.
Reconstructive Generalisation	● Determine conceptual aspects of the definite integral. ● Incorporating the concept of mass into the geometric interpretation of the definite integral in order to calculate the mass of an object. ● Relate the division of infinitesimally small intervals to infinitesimally small regions.

Final considerations

In order to analyse a didactic situation, integrating epistemological and didactic concepts (Schneider & Job, 2016) linked to the mathematical object integral, we propose the creation and implementation of an intervention based on working with task resolution episodes. (Trevisan & Mendes, 2018; Trevisan, Alves & Negrini, 2021; Trevisan, 2022) in order to provide DIC students with opportunities to explore this concept. The intervention comprised the organisation of a series of tasks (Mateus-Nieves & Moll, 2021) of an exploratory nature (Ponte, 2005), with the objective of elucidating the generalisation processes employed by students when extending the concept of the definite integral of one variable to multiple integrals.

In order to facilitate the organisation of the sequence of tasks and the subsequent analysis of the data, we drew upon the work of Jones and Dorko (2015), which discusses the generalisation processes employed by students to expand the concept of the definite integral for two and three variables, commencing with the definite integral of one variable. In our analysis, we sought to identify the potential of the task and the students' discussions to promote these diverse generalisation processes.

The initial epistemological element entailed the identification of promising applied contexts that would facilitate the generalization of the concept of definite integrals from one to multiple variables for students. Once the aforementioned context had been identified, a second epistemological element was considered: a task capable of addressing the one-, two- and three-dimensional definite integral, beginning with the concept of the mass of an non-homogeneous one-dimensional body, a non-homogeneous two-dimensional body and a non-homogeneous three-dimensional body.

The analysis of the generalisation movements revealed the use of expansive generalisation to address procedural issues in the calculation of an integral. Reconstructive generalisation was employed to comprehend the structural aspects of the Riemann integral with multiple variables. This included the utilisation of the geometric structure of the definite integral to develop a methodology for calculating the mass of an object.

We emphasise the importance of recognising these generalisation movements as a fundamental aspect of understanding the epistemic complexity of the concept of integral, a field that remains relatively unexplored in the context of Mathematics Education research (Mateus-Nieves, 2021). Additionally, the integral should not be regarded as a complex entity to be taught or learned, but rather as a mathematical construct comprising diverse meanings that must be deconstructed for analysis (Mateus-Nieves, 2021, p. 1611). Consequently, the findings indicate that the integral possesses epistemological characteristics that facilitate comprehension of certain pedagogical phenomena. This provides DIC instructors with the opportunity to reevaluate their instructional strategies, focusing on the processes of generalising the integral from one to multiple variables, in consideration of its inherent epistemic complexity.

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