EMP Educação Matemática Pesquisa
ISSN 1983-3156
Qualis A1
http://dx.doi.org/10.23925/1983-3156.2024v26i3p399-421

Construction of a reference epistemological model as the foundation of a new didactic paradigm for the study of differential calculus

Construção de um modelo epistemológico de referência como fundamento de um novo paradigma didático para o estudo do cálculo diferencial

Construction d'un modèle épistémologique de référence comme fondement d'un nouveau paradigme didactique pour l'étude du calcul différentiel

Construcción de un modelo epistemológico de referencia como fundamento de un nuevo paradigma didáctico para el estudio del cálculo diferencial

Catarina Oliveira Lucas[1]
Instituto Superior Politécnico Gaya (ISPGAYA); CEOS.PP, ISCAP, Politécnico do Porto, Portugal
Doctor en Matemática
https://orcid.org/0000-0002-0645-2169

Josep Gascón Perez[2]
Universitat Autònoma de Barcelona
Doctor en Matemáticas
https://orcid.org/0000-0001-5570-1144

Abstract

This work is underpinned by the *anthropological theory of the didactic* (ATD). The aim is to formulate, in a coordinated way, an outline of a *reference epistemological model* for *functional modelling*, REM(FM), which gives to the *elementary differential calculus* (EDC) a new raison d'être. After describing the general criteria to be met by the REM, a redefinition of what is meant by functional modelling is proposed by means of an *activity diagram* that integrates the role assigned to EDC in this mathematical activity. In coherence with this way of interpreting the role played by the EDC in mathematical activity, new educational ends associated with its study are proposed, as well as didactic means that we consider appropriate to achieve these ends. In short, we propose a new *modality for the study* of EDC in the transition from Secondary education to University, that is, a new *reference didactical paradigm* (RDP) for the study of EDC. Taking this DP as a reference, a brief analysis is made of the current mode of study of EDC, highlighting the didactic phenomena that emerge when the study of EDC in the transition from Secondary education to University is governed by this DP.

[1] clucas@ispgaya.pt
[2] josep.gascon@uab.cat

Keywords: Anthropological theory of the didactic, Reference epistemological model (REM), Elementary differential calculus, Functional modelling, Didactic paradigm.

Resumo

Este trabalho é sustentado pela *teoria antropológica do didático* (TAD). O objetivo é formular, de forma coordenada, um esboço de um *modelo epistemológico de referência* para a *modelação funcional*, MER(MF), que dá ao *cálculo diferencial elementar* (CDE) uma nova razão de ser. Depois de descrever os critérios gerais a que o MER deve obedecer, propõe-se uma redefinição do que se entende por modelação funcional através de um *diagrama de atividade* que integra o papel atribuído ao CDE nesta atividade matemática. Em coerência com esta forma de interpretar o papel desempenhado pelo CDE na atividade matemática, são propostos novos fins educativos associados ao seu estudo, bem como os meios didáticos que consideramos adequados para os alcançar. Em suma, propomos uma nova *modalidade para o estudo* do CDE na transição do Ensino Secundário para a Universidade, ou seja, um novo *paradigma didático de referência* (PDR) para o estudo do CDE. Tomando este PD como referência, faz-se uma breve análise do atual modo de estudo do CDE, destacando os fenómenos didáticos que emergem quando o estudo do CDE na transição do Ensino Secundário para a Universidade se rege por este PD.

Palavras-chave: Teoria antropológica do didático, Modelo epistemológico de referência, Cálculo diferencial elementar, Modelação funcional, Paradigma didático.

Résumé

Ce travail s'appuie sur la *théorie anthropologique du didactique* (TAD). L'objectif est de formuler, de manière coordonnée, les grandes lignes d'un *modèle épistémologique de référence* pour la *modélisation fonctionnelle*, MER(MF), qui donne au *calcul différentiel élémentaire* (CDE) une nouvelle raison d'être. Après avoir décrit les critères généraux auxquels doit répondre le MER, une redéfinition de ce que l'on entend par modélisation fonctionnelle est proposée au moyen d'un *diagramme d'activité* qui intègre le rôle assigné au CDE dans cette activité mathématique. En cohérence avec cette façon d'interpréter le rôle joué par le CDE dans l'activité mathématique, de nouvelles finalités de l'éducation associés à son étude sont proposés, ainsi que les moyens didactiques que nous considérons appropriés pour atteindre ces finalités. En résumé, nous proposons une nouvelle *modalité d'étude* du CDE dans la transition de l'école secondaire à l'université, c'est-à-dire un nouveau *paradigme didactique de référence* (PDR) pour l'étude du CDE. En prenant ce PD comme référence, nous analysons brièvement la modalité d'étude actuel du CDE, en mettant en évidence les phénomènes didactiques qui émergent lorsque l'étude du CDE dans la transition de l'école secondaire à l'université est régie par ce PD.

Mots-clés : Théorie anthropologique du didactique, Modèle épistémologique de référence, Calcul différentiel élémentaire, Modélisation fonctionnelle, Paradigme didactique.

Resumen

Este trabajo está sustentado por la *teoría antropológica de lo didáctico* (TAD). El objetivo consiste en formular, de manera coordinada, un esquema de un *modelo epistemológico de referencia* en torno a la *modelización funcional*, MER(MF), que asigna al *cálculo diferencial elemental* (CDE) una nueva razón de ser. Después de describir los criterios generales que deberá cumplir el MER, se propone una redefinición de lo que se entiende por modelización funcional mediante un *diagrama de actividad* que integra el papel que se asigna en dicha actividad matemática al CDE. En coherencia con esta forma de interpretar el papel que desempeña el CDE en la actividad matemática, se propugnan unos nuevos fines educativos asociados a su estudio, así como unos medios didácticos que consideramos adecuados para alcanzar dichos fines. En definitiva, se propone una nueva *modalidad de estudio* del CDE en el paso de Secundaria a la Universidad, esto es, un nuevo *paradigma didáctico de referencia* (PDR) para el estudio del CDE. Tomando dicho PD como referencia, se analiza de manera somera la modalidad de estudio vigente en torno al CDE poniendo de manifiesto los fenómenos didácticos que emergen cuando el estudio del CDE en el paso de Secundaria a la Universidad está regido por este PD.

Palabras clave: Teoría antropológica de lo didáctico, Modelo epistemológico de referencia, Cálculo diferencial elemental, Modelización funcional, Paradigma didáctico.

Every didactic problem (in the sense of a research problem in didactics of mathematics) involves a field of mathematical activity which, in some extreme cases, can be the whole of the mathematical activity carried out in a given institution. In our case the mathematical activity involved is *functional modelling* (FM), initially characterised in Ruiz-Munzón (2010) and reformulated by means of an *activity diagram* that appears later in this paper. Within this field we are particularly interested in the role that *elementary differential calculus* (EDC) plays and could play, interpreted in the sense in which it is characterised below.

In the *anthropological theory of the didactic* (ATD), it is postulated that the formulation of a didactic problem always assumes and takes into consideration, more or less explicitly, an interpretation of the field of mathematical activity involved. Thus, for example, when one speaks of the teaching, learning or construction of the *concept of the derivative*, one is inevitably supporting an interpretation (a model, however imprecise) of the mathematical activity that accompanies this notion in the institution in question. The extent and type of cut-off of the field of mathematical activity taken into consideration may be very varied, but they always condition to a large extent the nature of the model. In the particular case of the mathematical activity around the derivative, it is clear that the type of didactic problems that can be formulated will depend on whether one considers a scope restricted to the notion of the derivative, whether one extends it to what is usually considered as elementary differential calculus (or a part of it) or whether one takes the scope of functional modelling and the role of differential calculus in this field. In short, the types of didactic problems that can be formulated and tackled are strongly conditioned by the *unit of analysis* of the didactic processes taken into consideration. Bosch and Gascón (2005) describe in detail that the *minimum* unit of analysis in ATD must include a *relatively complete local praxeology* (Fonseca, 2004).

In the research carried out in ATD, we consider that it is essential to make explicit the epistemological model that is inevitably assumed and used as the *reference epistemological model* (REM). The aforementioned clarification constitutes, on the other hand, the basis on which the *modality of study* proposed for this field in the institution in question is sustained. This modality of study, or *reference didactic paradigm* (RDP), constitutes a scientific hypothesis that can be formulated in our case as follows: if a study community (that meets certain conditions) carries out a process of study of EDC governed by the RDP, then it will avoid the didactic phenomena that emerge when this study is governed by the current modality of study (in the transition from secondary school to university) and which are considered ‘undesirable’ from the perspective of ATD.

The RDP should therefore be interpreted as a *tentative hypothesis or conjecture* which, as such, is open to review and, if necessary, modification. A RDP is a creative scientific hypothesis that we must subject to the test of contingency. This test is carried out through the design and experimentation of *study and research paths* (SRP) based on the REM that forms the basis of the RDP.

In short, the methodology of ATD requires the explicitness of a RDP, i.e. a modality of study of the field of mathematical activity involved in order to be able to formulate didactic problems with precision (and to give meaning to the possible answers). It is important to insist on the fact that a RDP is associated, as we shall see in what follows, with a *didactic phenomenon* that constitutes the starting point, the initiator, of the construction of the RDP by the scientific community.

Outline of an alternative REM to the current epistemological model: redefinition of functional modelling by an activity diagram

We will begin by describing the *fundamental features* or *general criteria* that the structure of a REM must fulfil, which we will complete by means of the explicit construction of *a priori mathematical paths*, based on this structure. These paths will take the form of functional modelling processes designed to answer certain questions. Here we will outline only their main features structured in an *activity diagram* as a schematic map capable of supporting different mathematical paths. Subsequently, we will use this REM to support the construction of a RDP for the study of the EDC in the transition from Secondary School to University. Finally, and from the perspective of this RDP, we will carry out a brief didactic analysis of the current modality of study in the transition from secondary school to university in relation to the study of EDC.

The following criteria can also be considered as conditions which we impose on the alternative REM.

- *In our proposed REM, different processes of construction, use and comparison of functional models, the relationship between them and the role of the EDC in them must be made explicit in detail.*
- *This REM should take into account the relationships between discrete and continuous functional models and, therefore, relatively complete the one presented in Ruiz-Munzón (2010).*

- *As a prior step to the construction of the continuous functional models, we will start with discrete data and, therefore, we will initially work with discrete models expressed in terms of sequences and finite difference equations.*
- *If starting from discrete data, different types of regression will be used to move from discrete to continuous models either starting, depending on the nature of the system to be modelled, from the raw data, the average rate of change (ARC) or the relative average rate of change (RARC), to build functional models fitting a set of discrete data.*
- *The process of approximating discrete models (formulated in terms of finite difference equations) by continuous models (given by differential equations) will be justified and evaluated.*
- *It will be shown that, depending on the nature of the system to be modelled, the regression by approximation on the ARC or RARC (sequences obtained from the raw data) provides functional models that are relatively tighter and, above all, with better predictive capacity than those obtained by directly approximating the raw discrete data.*
- *The technical economy of moving from discrete to continuous will be demonstrated by showing, by means of explicit calculations, in what sense and to answer what kind of questions EDC techniques are more economical than the algebraic techniques of discrete mathematics. It will be shown, for example, that when it comes to 'controlling a model', i.e. predicting its long-term behaviour or constructing a model that fulfils certain conditions given in advance, EDC techniques are much more powerful and economical than algebraic techniques.*
- *Different types of variation (both between discrete and continuous quantities) will be constructed and articulated by defining the universe of types of variation to be considered. Each of these types will be characterised by imposing conditions (hypotheses) on the ARC or on the RARC, although the latter can also be expressed as conditions on the ARC. A certain universe of elementary types of variation will thus be delimited.*
- *Using EDC techniques, the meaning of the parameters of a functional model will be interpreted in terms of the system and, more specifically, in terms of the variation of one variable of the system with respect to another.*
- *EDC will be used to study all the local properties of the constructed functional models (which will then be interpreted in terms of the variables defining the modelled system).*
- *If continuous data are used, the model function itself or its derivative will be constructed using algebraic techniques. In the latter case, the 'exact' functional model (which may be a family of functions) is constructed by integrating a differential equation.*
- *In all cases, the functional modelling processes will be developed with the aim of providing an answer to a sufficiently general and relatively ambiguous generating question in the sense that it must be a question formulated with open 'parameters' that only progressively have to be converted into concrete data.*

It will be shown in what follows that this set of conditions will help to refine the notion of FM as conceptualised in Lucas (2015). In what follows we propose a more detailed and precise reformulation of this notion, i.e. a characterisation of what we mean by FM. This characterisation will clarify the meaning of the conjecture of Ruiz-Munzón (2010) according to which the raison d'être (or a possible raison d'être) of the EDC is located in the field of FM.

Activity diagram of functional modelling as an outline of the REM

In order to reformulate the notion of FM, we will start by making a detailed scheme of the types of tasks that we propose as components of the *four stages* of the mathematical modelling process and, in particular, of the FM processes (Chevallard, 1989; Gascón, 2001). We will materialise this REM scheme by means of an activity diagram (Figure 1) which we will call *activity diagram of functional modelling*. In order to clarify the content of this diagram, we will now describe each of its components, as well as the possible relationships between them. It may happen that at the end of a functional modelling process, new problematic issues arise that refer to the model itself. In this case we will say that 'the model has become independent of the initial system' and has come to play the role of a new system, thus demonstrating the *recursive character of the mathematical modelling process*.

The diagram contains the four stages of any mathematical modelling process, without prejudging a linear temporal succession between them.

- *First stage*: Delimitation or construction of the system to be modelled in which problematic questions and conjectures are formulated.
- *Second stage*: Construction of the mathematical model and reformulation of the initial questions.
- *Third stage*: Technical work within the model and interpretation of this work and the results in terms of the system.
- *Fourth stage*: Need for a new modelling process to answer new questions.

Furthermore, the diagram is divided into two major domains: the discrete and the continuous. Thus, when a certain activity (or type of task) is located on the equatorial line, it means that it can help to move from one field to the other, or that the activity can be carried out in both the discrete and the continuous field.

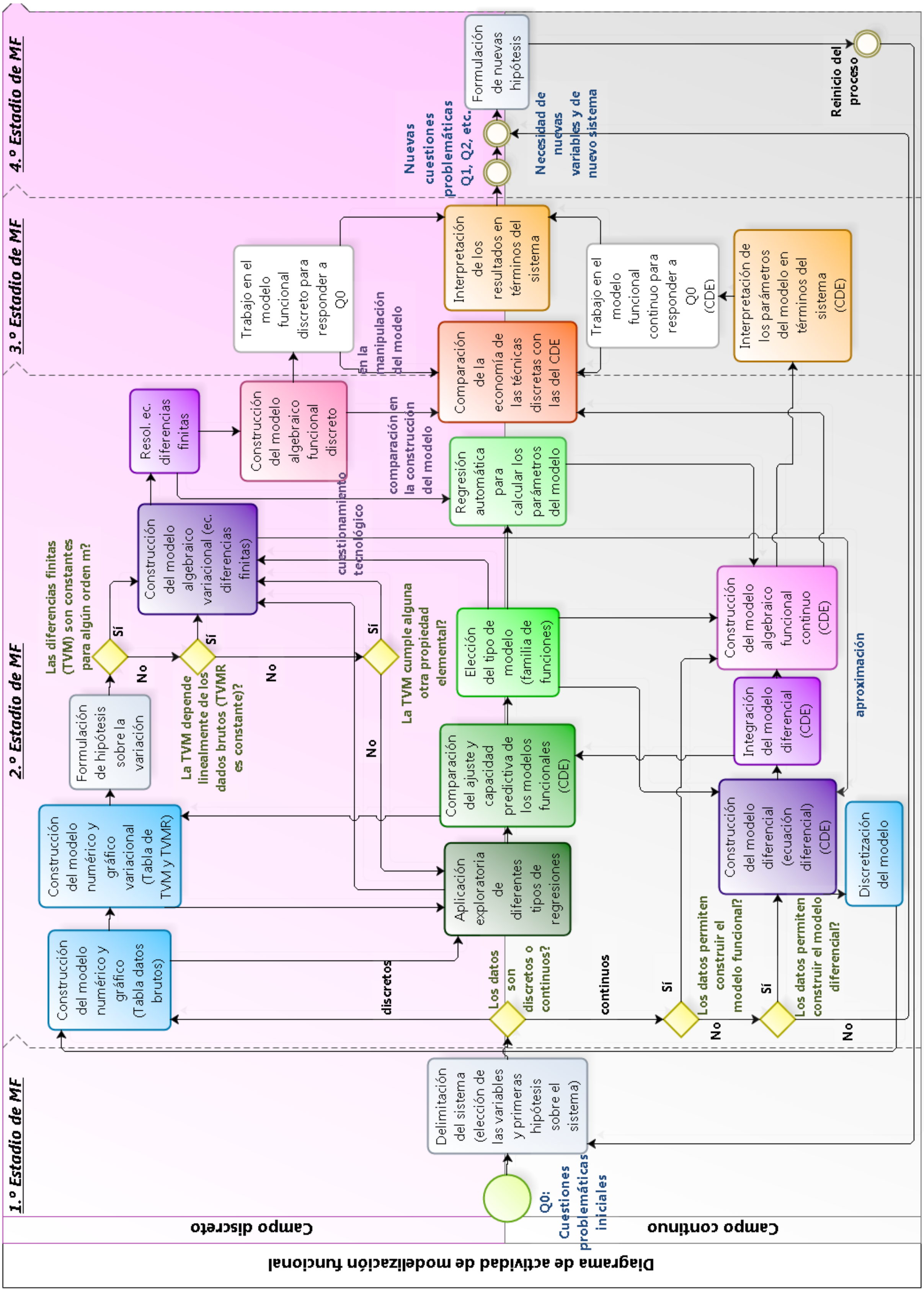

Figure 1.
Activity diagram of functional modelling

Description of the components of the activity diagram of functional modelling

In this section we will briefly describe the components of the activity diagram, specifying in more detail the tasks and issues that are part of the second stage of the FM process, i.e. the stage we call *construction of the model*. The third stage of the process, that refers to the *work within the model*, is much less detailed, because the tasks that are part of it are much more present in school mathematics.

First stage: Construction of the system to be modelled

Initial problematic questions: Any study process necessarily starts with a set of problematic questions, usually not very precise, but sufficiently rich to generate derived questions capable of guiding the study process along different paths. For example: *How can we predict the development of an epidemic? How can we study the behaviour of the mass of a yarn*[3] *from its mass density function? With how many friends should I share an advertising 'spot' so that, at the end of a certain period of time, it has been viewed by a certain 'target' number of users of a given social network? How does the length of a curved arc vary?*

Delimitation or construction of the system: The initial system is a field of 'reality' (mathematical or extra-mathematical) in which the initial problematic questions appear, which constitute what we call the 'generating question' that we want to answer. The delimitation or 'construction of the system' consists of the choice of certain aspects of it, which are symbolised by *variables* and which, we postulate, are the relevant ones for constructing a functional model useful for answering the questions posed. In the case of FM, *a dependent variable and one or more independent variables* are usually considered, among which the variable 'time' can be found. At this point, the first hypotheses about the system are formulated (often implicit) in terms of relationships between the variables chosen to build the functional model. It is true that, in some cases, it may seem that no initial hypothesis is made (for example, when a few discrete data are given on the number of infected people in a certain epidemic) but, in reality, the knowledge that it is an epidemic of a certain type (such as, for example, influenza) already determines some hypotheses (even if they are qualitative) on the possible relationships between the *time* variable and the *number of infected* people.

[3] The linear mass density of the yarn can be measured by weighing a known amount of yarn length. The density is the mass of the yarn per unit length.

Second stage: Construction of the model

Continuous data: These are relationships between variables that may be functional or non-functional, and whose information is represented by a condition on the data or by a verbal description of that same relationship.

Discrete data: These are given either by certain conditions on the variation of the data (knowing, for example, in advance that the ARC of certain empirical data is constant) or by a finite number of values of the dependent variable corresponding to pre-specified values of the independent variable(s).

Construction of the 'exact' continuous algebraic-functional model: If the data are continuous, i.e. if they are given in terms of functions or relationships between variables, it may happen that they allow the exact functional model to be constructed directly. This would be the simplest case. However, it is also possible to construct this model in the case where only the variation of the continuous data is known in terms of a differential equation (algebraic-differential model).

Construction of the 'exact' algebraic-differential model: If the data are continuous, it is quite usual that, although it is not possible to construct the exact functional model directly, it is possible to construct an exact differential algebraic model, i.e. an exact differential equation.

Integration of the 'exact' differential model: By means of the integration of an elementary differential equation which, in the institution in which we are situated, will be reduced to the calculation of an almost immediate primitive, the 'exact' continuous algebraic-functional model is constructed.

Discretisation of the continuous differential model: In the case where it is possible to construct the differential model, but an unsolvable differential equation is obtained by means of elementary techniques, the model can be *discretised* by transforming the continuous data into discrete data, going on to work with numerical (tabular) and graphical models.

Numerical and graphical model building: If we start from *discrete data* (coming, for example, from an empirical system), that is, if the data are a few values of the dependent variable (corresponding to a pre-fixed set of values of the independent variable(s)) we will start building different types of tabular and graphical models, both from 'raw' data and from the average rate of change (ARC) and relative average rate of change (RARC) built from the raw data. We will call the latter *numerical tabular variational models*.

Formulation of hypotheses about ARC and RARC: From the tabular models it will be possible to prove (or conjecture) that ARC or RARC fulfils a certain elementary property. In the transition from secondary school to university, we will restrict ourselves to a set of properties that characterise a *certain universe of elementary discrete models*.

Construction of the algebraic-variational model: From the different tabular variational models, and according to the hypotheses that we formulate with respect to the properties that ARC or RARC fulfil, different algebraic-variational models can be constructed in terms of *equations in finite differences*.

Solving the equation in finite differences: using, for example, *recurrence techniques* or the technique of determining the *indeterminate coefficients* to calculate the parameters and constructing a particular solution (a functional model) from the *general* solution (a family of functional models) of the finite difference equation, the *discrete algebraic-functional model* can be constructed from the algebraic-variational model. In particular, an interpolating polynomial model (passing through all points corresponding to the discrete data) can be constructed algebraically. To calculate the parameters of that model, the technique of calculating the indeterminate coefficients by solving a compatible and determinate system of equations can be used. Other algebraic techniques can be used to construct the interpolating polynomial model, such as: *Lagrange's formula* or *Newton's divided difference technique*.

Automatic regression to calculate the parameters of the model: As algebraic techniques for solving finite difference equations can usually be very laborious and costly, the need may arise for a *technological questioning* of these techniques and, consequently, the search for a more economical technique that produces equally reliable results. Thus, starting from the general solution of the finite difference equation, the parameters of the model can be calculated by the technique of direct application of a specific automatic regression (of the type of the general solution) on the discrete data and the *'approximate' continuous algebraic-functional model* can be constructed. For example, in the case of the construction of the interpolating polynomial, the algebraic technique of determining its coefficients 'with pencil and paper' can be very costly, especially in cases where the degree of the polynomial is very high, i.e. when we have a large amount of discrete data. In such cases, the automatic regression technique can be used to calculate the parameters of the model with a graphing calculator, Excel, GeoGebra or other suitable software. This automatic regression technique on the discrete data to calculate the model parameters can also be used after the choice of the family of functional models that best describe the system. For example, when integrating a differential model describing a given variation of discrete data, a parameter arises that can be automatically fitted by approximating the integral of the regression to the raw data.

Construction of the discrete algebraic-functional model: this model corresponds to a particular solution of the equation in finite difference obtained from the data describing the system.

Construction of algebraic-differential models by approximation to an equation in finite differences: Each discrete model expressed in terms of an equation in finite differences can be approximated by means of a differential equation which we shall call an 'approximate' algebraic-differential model.

Integration of the 'approximate' algebraic-differential models: By means of the integration of an elementary differential equation, which here is reduced to the calculation of an almost immediate primitive, 'approximate' algebraic-functional models are constructed.

Construction of the 'approximate' continuous algebraic-functional model: this model corresponds to the particular solution of the 'approximate' algebraic-differential model (the differential equation).

Discretisation of the continuous differential model: In the case where the differential equation cannot be solved with the techniques available in the institution, the continuous algebraic-differential model (differential equation) can be approximated by means of a discrete numerical-variational model (equation in finite differences). We are not going to study these cases because they are not part of the institutional environment in which we are working.

Exploratory' application of different types of regressions: From tabular numerical models (both from the raw data table and from the ARC or RARC tables) regressions can be applied to: (i) Go from a discrete numerical/graphical model to a continuous algebraic-functional model; (ii) Know a possible algebraic law that can describe a functional relationship between discrete variables, whose domain is the natural numbers (discrete algebraic-functional model). GeoGebra can be used to make some types of automatic regressions such as: polynomial regressions of different degrees, exponential, logarithmic, logistic, potential, sinusoidal and regressions resulting from the composition of some of these. This work will be carried out in an exploratory way and without using specific criteria to decide which type of regression should be selected to approximate a given set of discrete data. Of the various resulting functions, for the sake of economy of the subsequent analysis, it is of interest to choose the best ones according to some predetermined criteria:

1. *Technique of direct observation* of the approximation results and curve fitting to the points;

2. *Automatic technique for calculating the R^2 value* (coefficient of determination);
3. *Technique for calculating the 'smallest absolute error'* to determine the curve fit, i.e. calculation of the distance between the approximate image and the real image: $|y_{aprox} - y_{real}|$;

In Lucas (2015) we chose to select for each discrete data set the best regressions according to these criteria and then choose the 'best regression' according to its *ability to fit* the points, but not forgetting its *ability to predict* the trend of the 'future' data, i.e. the consistency of the model with the evolution of the modelled system and also with the data not used to build it.

Comparison of the fit and predictive ability of functional models

From the models that best fit the data, we will test and evaluate different techniques in terms of their effectiveness in choosing the model that best predicts future data in the *short, medium and long term*:

$\boldsymbol{\tau_1}$: Comparison of the prediction errors of the 'approximate' raw values obtained by the different regressions relative to the corresponding 'real' raw data;

$\boldsymbol{\tau_2}$: Comparison of the prediction errors of the ARC (or RARC) of the 'approximate' models, i.e. of the 'approximate' variational values obtained by the different regressions in relation to the corresponding 'real' variational data;

Choice of model type (family of functions): The choice of the 'model type' that best fits a given data set arises as a result of comparing the fit and predictive ability of various 'candidate' functional models to describe the system.

Construction of 'approximate' differential algebraic models by regression: Depending on the nature of the system to be modelled, the regression will be applied on the ARC or RARC (and, very rarely, on the raw data). Depending on the type of regression used and the type of discrete data used to perform the regression, we will obtain a function that either approximates the functional model sought (if we have started from the raw data), or approximates the derivative of the model sought (if the regression has been applied on the ARC) or, finally, approximates the quotient between the derivative and the said model (if the regression has been applied on the RARC). In these last two cases, which, as we shall see, are the most commonly used, we will obtain different approximate differential algebraic models (in terms of differential equations) which, by integration, will provide us with the approximate functional model.

Comparison of discrete techniques and EDC techniques in terms of economy: Working with discrete models (which we have called variational algebraic and which are materialised in finite difference equations) and with continuous functional models allows us to compare the economy of the techniques used in each case to respond to the initial problematic questions and other derived questions. On the one hand, it will be possible to compare the economy of the discrete techniques with those of the EDC in the construction of the algebraic-functional model, that is, to compare the cost of the techniques for solving finite difference equations with the cost of the techniques for solving differential equations. On the other hand, it will be possible to compare the economy of the two types of techniques in the manipulation of the previously constructed model, that is, to compare the cost of the techniques for calculating the average rate of variation in an interval with the cost of the derivative techniques. For example, if the model is simple (linear or quadratic), the economics of discrete and EDC techniques for building and manipulating the model will be very similar. However, if you intend to build and work within a slightly more complex model, e.g. logistic or rational model, EDC techniques will be much more economical than discrete techniques.

Third stage: Working within the model and interpreting in terms of the system

Work on the functional model (continuous or discrete) to answer the initial problem questions: the development of this work within the model previously constructed to answer the initial problematic questions (Q_0) presents different aspects such as, for example, the study of the evolution of the model, in particular, the study and interpretation of monotonicity and possible extremes, concavity/convexity intervals and inflection points, zeros and signs, asymptotes, limits, periodicity, parity, etc. While in the case of work within continuous functional models, EDC tools can be used to answer Q_0, in the case of work within discrete functional models, only finite difference techniques can be used. After carrying out the work and manipulation of the functional model, it will be possible to extract information about the behaviour of the system and to predict the short, medium and long term evolution of the system. In addition, the (continuous) functional model itself is a good tool to discover, by means of extrapolation techniques, intermediate data that are unknown or whose information has been 'lost' and whose recovery would be of great use.

Interpretation of the model parameters in terms of the system: In the study of the 'long-term' behaviour of the model, the influence of the parameters on the shape of the model graph and, in particular, on the extreme values and long-term behaviour of the system (study of the asymptotic behaviour) can be interpreted. In particular, when working with continuous algebraic-functional models, EDC tools can be used to interpret the parameters of the model in terms of the system. Thus, according to the type of model constructed, the parameters can be identified with, for example: the initial velocity or acceleration, with the elasticity, with the percentage variation of the dependent variable when the independent variable increases by 1 unit, etc.

Interpretation of results in terms of the system: Sometimes the results of the work within the functional model (whether continuous or discrete) needed to answer the initial problematic questions Q_0 do not correspond to valid values in the system. For example, the results may not be part of the domain or may not be interpreted in the context of the problem at hand. Another aspect of this interpretation is to perceive the relevance of the results obtained to advance the knowledge of the system. It should be noted that the same results of the model work may have different interpretations according to the field of study, for example: the geometrical interpretation of the results in terms of the system may be different from the algebraic interpretation of the results; the physical interpretation of the results may also differ from their mathematical interpretation, and so on.

Fourth stage: Need for a new functional modelling process

Emergence of new problematic issues: The system interpretation of the work within the constructed functional model will generate new problematic questions (Q_1, Q_2, Q_{21}, Q_{211}, etc.) different from the initial ones (Q_0) that may require the use of new mathematical techniques (and even a new technological-theoretical discourse) and, consequently, the widening of the scope of the developed mathematical activity. For example, the need to work with families of functions with one or more parameters, which will entail the need to study new functional models (Fonseca, Gascón & Lucas, 2014).

Need for new variables and a new system: New problem questions may require the consideration of new variables to delimit (construct) the system, which will lead to the emergence of a new system. For example, if in the initial system we had assumed that the effect of a medicine was independent of the weight, age and sex of the patient and it turns out that the analysis of the results suggests that this hypothesis is questionable, then we will have to introduce these new variables and we will have a more complex system in which finer questions can be raised. It may also be the case that the model constructed becomes independent of the system modelled by it, so that problematic questions emerge concerning the model itself. In this case it is the model that takes on the role of the new system and the modelling process that is initiated will be a process of intra-mathematical modelling, thus highlighting the reflexivity of the mathematical modelling activity and the impossibility of separating the modelling of extra-mathematical systems from that of intra-mathematical systems.

Formulation of new hypotheses: In the aforementioned case where new variables need to be taken into account to delimit the new system (such as, for example, the variables weight, age and sex of the patient), it will also be necessary to formulate hypotheses on possible relationships between these variables and the effect of the medicinal product. These hypotheses will necessarily be 'new' since they involve variables that were not considered in the first model. The formulation and testing of these new hypotheses is one of the main objectives of the FM process.

A reference didactic paradigm for the study of differential calculus in the transition from secondary school to university

The need to establish a reference didactic paradigm: As we said in the introduction, in order to carry out a didactic research related to the study of a certain field of knowledge in a school institution, it is not only necessary to make explicit a REM of this field, but it is also necessary to make explicit a *didactic paradigm* (DP), that is, a *modality of study* of the field in question which, more or less explicitly, is taken as a reference from the research and, therefore, it is called a *reference didactic paradigm* (RDP).

Each didactic paradigm around a certain domain of knowledge specifies a possible way of organising the study of this domain in a given institution. It is characterised by a complex system consisting of four subsystems: an *epistemological model* (EM) of the knowledge at stake; the *educational ends* or, more generally, the *didactic ends* (DE) pursued by such a study; the *didactic means* (DM) used by the study community to achieve these purposes; and the *didactic phenomena* (Dφ), surprising and usually 'undesirable' from the perspective of the institution responsible for constituting the DP. Taking these phenomena into account is the starting point (the *initiator*) of the DP *construction process* (Gascón & Nicolás, 2021a, 2021b; Gascón, 2023, 2024). Schematically we will put:

$$DP = [EM, DE, DM, D\varphi]$$

To make things a little more precise, let us say that each DP is supported by a EM that constitutes a particular *representation*, among other possible ones, of the knowledge at stake. The EM provides the notions and terms necessary to formulate the DE that are pursued, while at the same time it conditions the type of DM that could be useful to achieve these ends, among which is the avoidance of didactic phenomena Dφ (Gascón, 2001). This highlights the *functional unity* of the DP, which must therefore be considered as *complex systems* whose determining characteristic is the *interdefinability* and mutual dependence of the functions fulfilled by their components or subsystems within the total system (Rolando García, 2006).

We distinguish between the *current* didactic paradigm (CDP) in an educational institution around a certain field, such as, for example, the current didactic paradigm in the transition from secondary school to university, around the study of functional modelling:

$$CDP_{SU}(FM) = [CEM, CDE, CDM, CD\varphi]$$

and the reference didactic paradigms (RDP), which are theoretical artefacts constructed by the research to analyse and represent the CDP in an institution and, where appropriate, to analyse the conditions required to promote a change of it in a given direction.

$$RDP_{SU}(FM) = [REM, RDE, RDM, RD\varphi]$$

The same arguments that served to justify the convenience and even the need to make explicit the *reference epistemological model* (REM) used in research to carry out a *praxeological analysis* of a certain field of knowledge (Gascón, 2014) now serve, with even more reason, to justify the need to make explicit the DP that is taken as a reference. The explicitness of a RDP will make it possible to clarify and specify, by contrast, the didactic analysis of the current modality of study, and of the possible modes of study, in an institution. Just as the main features of the CEM and, above all, its 'shortcomings' and 'limitations' are described in contrast and comparison with a REM, a RDP will be used (as a contrast) to analyse the CDP.

Given the importance of the chosen RDP, it should be stressed that this choice (i.e. the scientific community's construction of a RDP instead of other possible ones) is never arbitrary, is guided by convictions, preferences or assumptions of the researcher or the scientific community in question (even if, in some cases, it is motivated solely by methodological issues) and, ultimately, is underpinned by a value system. As we have said, a RDP embodies a scientific hypothesis according to which the study processes governed by such a DP would, under certain conditions, make it possible to avoid the 'undesirable' didactic phenomena that emerge in the study processes governed by the CDP.

Didactic paradigm based on the reference epistemological model constructed:

Based on and in coherence with the $REM_{SU}(FM)$ constructed, the $RDP_{SU}(FM)$ based on the said REM advocates specific *didactic ends* associated with the study of EDC in the transition from Secondary to University, at the same time as it proposes *didactic means* to achieve these ends.

The *didactic ends* constitute the answer to the following question: *why study EDC in the transition from secondary to university?*

Before explaining the answer provided by the RDP_{SU}(FM) to this question, it is necessary to make a small circumlocution.

In ATD we assume that all knowledge ultimately refers to a system or a type of system. Knowledge is obtained through a process of modelling such a system and is embodied in the answers to certain questions posed with the elements and notions of the system. Therefore, what is modelled is, in reality, a system provided with questions, and these answers are constructed by working on the model, which has a praxeological structure.

Returning to the previous question, we shall say first of all that, in general, the didactic ends of the study of a field of knowledge, i.e. the answer to why study a certain field in a certain institution, can simply be formulated as follows: to increase knowledge (both in extension and in precision and rigour) about a system or a type of system. This (quantitative and qualitative) increase in knowledge is materialised, as we have said, in the answers to questions about the system and in the relationships that are established between these answers.

In the particular case we are dealing with here, to the question: *why study EDC in the transition from secondary to university?*, the RDP_{SU}(FM) answers: to construct functional models (from discrete or continuous data) of all types of systems (mathematical or extra-mathematical); to carry out mathematical work on these models; to interpret the results of this work; and, in short, to respond to the problematic questions initially formulated in the different types of systems (physical, biological, economic, geological, social, numerical, geometric, etc.) and thus extend knowledge about these systems. This synthetic response can be made more precise and detailed by specifying the role played by the EDC in the construction and study of the different types of functional models, as detailed in the construction of the REM_{SU}(FM).

The *didactic means* proposed by the RDP_{SU}(FM) to achieve these ends constitute the answer to the following question: *how to study the EDC in the transition from Secondary School to University in order to achieve the above-mentioned didactic ends?*

The response proposed by the RDP_{SU}(FM) begins by placing the study of the EDC within the scope of the FM redefined in the terms proposed by the REM_{SU}(FM). Consequently, the RDP_{SU}(FM) proposes to study the EDC as a *key instrument in the functional modelling of all types of systems*. More specifically, it proposes to study the role played by the EDC: in the comparison of the fit and predictive ability of functional models; in the comparison between the economics of discrete techniques and those of the EDC; in the construction of differential models and continuous algebraic-functional models; in the analysis of the properties of continuous functional models and in the interpretation of the model parameters in terms of the system. To this end, the RDP_{SU}(FM) proposes to carry out different functional modelling processes in the continuous field and in the transition between the discrete and continuous fields, as suggested in the activity diagram (Figure 1). The didactic means to manage the aforementioned functional modelling processes are materialised in different study and research paths whose implementation helps to connect different mathematical praxeologies that usually arise in an atomised way (for example, the resolution of differential equations, the calculation of primitives and the graphical representation of functions) by integrating them in functional modelling processes.

Didactic analysis of the current modality of study in the transition from secondary school to university in differential calculus: From the perspective of the RDP_{SU}(FM), the empirical analysis of curricular documents in various countries reveals the didactic phenomenon of the lack of school visibility of the activity of functional modelling and the consequent school absence of the activities of construction, comparison and interpretation of functional models. This is an important manifestation, at the subdisciplinary level, of the disciplinary phenomenon of rigidity and disarticulation of school mathematical organisations (Fonseca, 2004) in the particular case of EDC and FM in the transition from secondary school to university (Lucas, 2015). This disarticulation causes difficulties for the school system (and, therefore, for teachers and students) in making sense of the school study of EDC.

The role that EDC might play in building models from discrete data, in comparing the fit of functional models to empirical data, and in interpreting model parameters in terms of the variation of one system variable relative to another is not explored in school mathematical activity. Consequently, what are the *current didactic ends* associated with the study of EDC at the transition from secondary school to university?, i.e. what is the CDP's answer to the question: *What is the purpose of studying EDC at the transition from secondary school to university?*

The response of the curricular documents shows that the official raison d'être of EDC in the transition from Secondary School to University, i.e. the type of tasks assigned to it in the curricular documents is focused on the analysis of the properties of certain types of functions and on the solution of some optimisation problems. Consequently, the *didactic ends* formally pursued with the school study of EDC are limited to increasing knowledge of the properties of certain types of functions, and are materialised in the answers to questions concerning the domain, monotonicity, continuity, derivability and other characteristics of these functions.

In coherence with the school disarticulation between EDC and FM (whose presence is, moreover, very fragile), the *current didactic means* in the transition from Secondary School to

University for the study of EDC, that is, the didactic means that the educational system uses to achieve the ends it advocates, are reduced to the teaching proposal of the use of rather stereotyped techniques to determine the properties of certain types of functions and to use some of these properties to solve optimisation problems.

Up to this point we have carried out a brief analysis of the *economics of the current modality of study* in the transition from secondary school to university around the EDC from the perspective of the RDP_{SU}(FM). The didactic analysis should be completed with an *ecological analysis*, which would require, in particular, answering questions such as: *what conditions are required and, in particular, what constraints hinder the evolution of the current modality of study in the direction set by the* RDP_{SU}(FM), *what role could the EDC play in the establishment of these conditions, and what constraints, stemming from what levels of didactic codetermination, hinder the implementation of such a modality of study?*

Didactic paradigms complete the functions of epistemological models

The reason why we construct reference epistemological models (REM) is linked, in the first place, to the need to support the praxeological analysis of a certain field of mathematics. We thus obtain a representation of some features of the way mathematics is conceptualised as living in an institution that is ultimately linked to the didactic phenomena, usually 'undesirable' (from a certain perspective) that these features bring to light. The incipient *phenomenotechnical function*[4] of REMs is thus revealed (Gascón, 2014; Lucas et al., 2019).

The praxeological analysis of the mathematics involved in a certain field brings with it the questioning of the school conceptualisation of mathematics and, consequently, makes it possible to distance oneself from the conditioning factors involved in this conceptualisation. Consequently, the REM on which the praxeological analysis is based can be considered as an *instrument of epistemological emancipation* of the didactician and didactic science. Only in this way can didactics emancipate itself from the current epistemological model in the institutions concerned and autonomously construct its own object of study (Gascón, 2014).

The usefulness of REMs is not limited to their phenomenotechnical and emancipatory functions. It is assumed that a REM, in addition to clarifying and specifying certain praxeological features of institutionalised mathematics, can *guide the design and management*

4 REM are heuristic tools to make certain *didactic phenomena visible*. This is their phenomenotechnical function.

of study processes that will (presumably) meet certain conditions and allow certain ‘desirable’ (from a certain perspective) educational goals to be achieved. In fact, *empirically testing* a REM means, in essence, checking whether it actually fulfils all these functions.

To complete the praxeological analysis supported by a REM, didactics constructs *reference didactic paradigms* (RDP) as instruments to support the didactic analysis of institutionalised study processes. A RDP is a *scientific hypothesis* that develops and specifies the hypothesis embodied in the REM. It can be formulated as follows: if a study community (fulfilling certain conditions) were to carry out a study process *governed* by the RDP, i.e. based on the REM and using RDM means, then this community would avoid certain phenomena that occur when the study of this field is *governed* by the CDP, and would achieve the didactic ends advocated by the RDP. Like any scientific hypothesis, it must be empirically tested.

In addition to specifying the hypothesis embodied by the REM, a RDP extends and completes the *emancipatory function* of the REM since, by providing a representation of the CDP, in addition to making epistemological emancipation possible, it allows distancing itself from the influences coming from the CDM and the CE, making visible didactic phenomena that had remained invisible. Consequently, the explicit assumption of a RDP constitutes a first step in institutional emancipation, which could be defined, in general, as the liberation of didactic science from subjection to the dominant ideology of the institutions that form part of its object of study. Thus, on the basis of a RDP, didactics can, in a relatively autonomous manner, construct the phenomena and formulate the didactic problems that constitute its object of study.

In short, didactic paradigms, in addition to completing and specifying the hypothesis that embodies the REM on which they are based, complete the phenomenotechnical and emancipatory functions of the latter. It thus becomes clear that in order to tackle a didactic problem involving a field of mathematics, it is not only necessary to make explicit a REM of that field, but also to establish (even if only provisionally) a didactic paradigm based on that REM.

Acknowledgements

This work has been funded by projects PID2021-126717NB-C33 and UIDB/05422/2020 (Portuguese funds: FCT - Fundação para a Ciência e Tecnologia).

References

Bosch, M. & Gascón, J. (2005). La praxéologie comme unité d'analyse des processus didactiques. In Mercier, A. et Margolinas, C. (Coord.) *Balises en Didactique des Mathématiques* (pp. 107-122). Grenoble: La Pensée Sauvage.

Chevallard, Y. (1989): *Arithmétique, Algèbre, Modélisation. Étapes d'une recherche*. Publications nº 16 de l'IREM Aix-Marseille.

Fonseca, C. (2004). *Discontinuidades matemáticas y didácticas entre la Secundaria y la Universidad* [Tesis doctoral, Departamento de Matemática Aplicada, Universidad de Vigo].

Fonseca Bon, C., Gascón Pérez, J. & Oliveira Lucas, C. (2014). Desarrollo de un modelo epistemológico de referencia en torno a la modelización funcional. *Revista Latinoamericana de Investigación en Matemática Educativa, 17*(3), 289-318. http://www.redalyc.org/articulo.oa?id=33532494003.

Gascón, J. & Nicolás, P. (2021a). Incidencia de los paradigmas didácticos sobre la investigación didáctica y la práctica docente. *Educación Matemática, 33*(1), 7-40. https://doi.org/10.24844/EM3301.01

Gascón, J. & Nicolás, P. (2021b). Relaciones entre la investigación y la acción en didáctica de las matemáticas, *Avances de Investigación en Educación Matemática (AIEM)*, *20*, 23-39. https://aiem.es/article/view/v20-aascon-nicolas/4033-pdf-es

Gascón, J. (2001). Incidencia del modelo epistemológico de las matemáticas sobre las prácticas docentes. *Revista Latinoamericana de Investigación en Matemática Educativa, 4* (2), 129-159.

Gascón, J. (2014). Los modelos epistemológicos de referencia como instrumentos de emancipación de la didáctica y la historia de las matemáticas. *Educación Matemática*, número especial, XXV aniversario, marzo de 2014, p. 143-167.

Gascón, J. (2023). La formación del profesorado en tiempos de crisis paradigmática, *Educação Matemática Pesquisa*, *25*(2), 211-237, (25 años de la revista EMP). https://revistas.pucsp.br/index.php/emp/article/view/62075

Gascón, J. (2024). Problematic issues regarding didactic paradigms, Extended Abstracts 2022 -*7th International Conference on the Anthropological Theory of the Didactic (*CITAD 7). Springer. (in press)

Lucas, C. (2015). *Una posible «razón de ser» del cálculo diferencial elemental en el ámbito de la modelización funcional* [Tesis doctoral, Departamento de Matemática Aplicada, Universidad de Vigo]. http://www.investigo.biblioteca.uvigo.es/xmlui/handle/11093/542

Lucas, C., Fonseca, C., Gascón, J. & Schneider, M. (2019). The potential phenomenotechnical of reference epistemological models. The case of elementary differential calculus, In M. Bosch et al. (Eds.) *Working with the Anthropological Theory of the Didactic in Mathematics Education: A comprehensive Casebook*, 77-97. Routledge. DOI: 10.4324/9780429198168-6

García, R. (2006). *Sistemas complejos. Conceptos, método y fundamentación epistemológica de la investigación interdisciplinaria.* Barcelona: Gedisa Editorial.

Ruiz-Munzón, N. (2010). *La introducción del álgebra elemental y su desarrollo hacia la modelización funcional* [Tesis doctoral, Universitat Autònoma de Barcelona]. https://dialnet.unirioja.es/servlet/tesis?codigo=22189