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Some considerations about the constructions of the set of real numbers: is there a need for an epistemological reference model?[1]

Algunas consideraciones sobre las construcciones del conjunto de los números reales: ¿necesidad de un modelo epistemológico de referencia?

Quelques considérations sur les constructions de l'ensemble des nombres réels : est-ce une nécessité pour un modèle épistémologique de référence ?

Algumas considerações sobre as construções do conjunto dos números reais: uma necessidade para um modelo epistemológico de referência? [2]

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Abstract

The aim of this article is to provide some elements about the set of real numbers and a synthetic view on the motivation of its rigorous constructions in the 19th century. Since the seminal works of Cauchy and Weierstrass, such constructions became a requirement for the arithmetization of the mathematical analysis. We also analyze some didactic considerations regarding teaching the set of real numbers in high school and at the beginning of university. With this article, we hope to provide subsidies for the elaboration of epistemological models of reference (EMR) for studies and research on the contents of functions, limit and continuity, among others.

Keywords: Real numbers, Historical evolution, Mathematical analysis, Epistemology, Didactics, Epistemological reference model.

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Resumen

El objetivo de este artículo es presentar algunos elementos sobre el conjunto de números reales y una visión sintética de las rigurosas construcciones de este conjunto en el siglo XIX, que se convirtieron en un requisito para la aritmética del análisis matemático con los trabajos de Cauchy y Weierstrass. Analizaremos algunas consideraciones didácticas relativas a la enseñanza del conjunto de números reales en el bachillerato y en los inicios de la universidad. Con este artículo, esperamos otorgar subsidios para la elaboración de modelos epistemológicos de referencia (MER) para estudios e investigaciones sobre los contenidos de funciones, límite, continuidad, entre otros.

Palabras clave: Números reales, Evolución histórica, Análisis matemático, Epistemología, Cosas didácticas, Modelo epistemológico de referencia.

Résumé

L'objet est de donner quelques éléments sur l'ensemble des nombres réels et un aperçu synthétique sur leurs constructions rigoureuses au 19$^{\text{ième}}$ siècle. A cette époque, de telles constructions sont devenues une exigence pour l'arithmétisation de l'analyse mathématique, avec les travaux de Cauchy et Weierstrass. Quelques considérations didactiques sur leurs rapports à l'enseignant de l'analyse mathématique au lycée et au début de l'université, sont présentées. Avec cet article, nous espérons fournir des subventions pour l'élaboration de modèles épistémologiques de référence (MER) pour des études et des recherches sur le contenu des fonctions, la limite, la continuité, entre autres.

Mots-clés : Nombres réels, Évolution historique, Analyse mathématique, Épistémologie, Didactique, Modèle épistémologique de référence.

Resumo

O objetivo deste artigo é apresentar alguns elementos sobre o conjunto dos números reais e uma visão sintética das construções rigorosas desse conjunto no século XIX, que se tornaram um requisito para a aritmetização da análise matemática com os trabalhos de Cauchy e Weierstrass. Analisaremos algumas considerações didáticas concernentes ao ensino do conjunto dos números reais no ensino médio e no início da universidade. Esperamos que este artigo possa fornecer subsídios para a elaboração de modelos epistemológicos de referência (MER) para estudos e pesquisas dos conteúdos de funções, limite e continuidade, entre outros.

***Palavras-chave**:* Números reais, Evolução histórica, Epistemologia, Didática, Modelo epistemológico de referência.

Some considerations about the constructions of the set of real numbers: Is there a need for an epistemological reference model?

The study of the evolution of the concept of real number and the constructions of the set of these numbers can make an important and necessary contribution to the development of an epistemological reference model (ERM) for this topic. Thus, we can say that the teacher, aware of the historical evolution of these constructions and of the epistemological obstacles, can find ways to overcome the difficulties linked to the approach to this concept. In addition, they can enrich their culture with knowledge that is generally lacking in textbooks, as well as in undergraduate and postgraduate teaching in general. More generally, we can ask the following question: "*How can historical development and epistemological obstacles contribute to the EMR of a mathematical concept?"*

In our article, we present a theoretical approach to the constructions of the set of real numbers and their necessity for the arithmetization of the mathematical analysis, as well as epistemological obstacles that occurred in the evolution of real analysis and its teaching. On the other hand, based on the mathematical analysis of the textbooks used and the exchange with our colleagues, we were convinced of the need for a synthetic presentation of the constructions of the set of real numbers $\mathbb{R}$ and the epistemological obstacles connected with them. We also noticed that in the syllabi of calculus courses, there are no suggestions or guidelines on how to approach these constructions or their necessity for understanding the concepts of limit and continuity. Houzel (1979) draws attention to the historical evolution of mathematical concepts: *"The work of mathematicians is often dedicated to taking up old theories and reformulating them in a new framework [...]; the successive rediscoveries that lead mathematics to produce new theories, which erase a history of mathematics." (Houzel, 1979, p. 3).*

It seems for us, that this process of erasing or denying the history of the mathematical concepts is very detrimental to the development of an EMR that aims to overcome epistemological obstacles in the approach to these concepts. Over the last few decades, studies, and research have tended to show that the history of mathematics can play an important role in its teaching. In addition, the historical evolution of a concept before it reaches educational textbooks requires considerable work because of possible epistemological obstacles. So, in this context, knowledge of this historical development and the obstacles surrounding this concept will enable the teacher to take an appropriate didactic approach to teaching it. In addition, such an approach can help promote students' learning of the concept and its properties through the acquisition of knowledge based on its historical.

This study focuses on some reflections related to the construction of the set of real numbers, and the concept of limits. We begin with a brief overview of the evolution of real numbers and the epistemological motivations, as well as their constructions, resulting from the arithmetization of the mathematical analysis. In fact, the few historical considerations of the real numbers that interest us are intended to illustrate the importance of the impact of the close link between the need and the deep motivations for a rigorous construction of the set of real numbers, in order to establish solid foundations, bases for the arithmetization of modern mathematical analysis. From there, we propose some didactic considerations about the importance of the properties of the set of real numbers and the teaching of real analysis. And so, in a second moment, we will make some didactic considerations about its relationship with the practice of the teacher of mathematical analysis in high school and at the beginning of university. Finally, a conclusion is presented.

Initially, it is worth remembering that in the context of the arithmetization of analysis, the 19th century saw the emergence of different constructions of the set of real numbers, since the properties of these numbers are the basis for the study of the limit and continuity of real functions with real variables. Today, the importance of real numbers for teaching mathematical analysis is also evident in textbooks, as well as in high school curricula or university curricula. Therefore, the aim is to establish a didactic approach to the set of real numbers, more precisely to the concept of real numbers. In agreement with Artigue (1990), we believe that what interests the didactic is:

> [...] the identification of local conceptions that manifest themselves in situations and the analysis of the conditions of transition from one local conception to another, whether to reject an erroneous conception, to establish a conception that makes it possible to improve efficiency in the solution of a given class of problems, or to promote mobility between conceptions that are already available (Artigue, 1990, p. 278).

On the other hand, we think it's important to work with the properties of real numbers so that they help to deepen functional thinking. It can also help introduce students to the concepts of limit and continuity of real functions. In this sense, we find in the paper of Burigato and Rachidi (2023):

> It is also an opportunity to understand and delve into aspects of the important role of the properties of the set of real numbers. The modern concept of limit appeared with Cauchy, with his study of infinitely small and infinitely large quantities, basing his argument on the properties of the real numbers. This shows the importance of the set of real numbers for the entry into functional thinking and the work of the teacher who will have to deal with this set to make an interesting teaching proposal with the formal definition" (Burigato and Rachidi, 2023, p. 42).

Faced with the challenge of addressing content in undergraduate Calculus and Mathematical Analysis courses, such as functions, limits and continuity, among others, we believe that a more in-depth study of some aspects concerning the emergence of the set of real numbers is essential. We consider it fundamental to provide support for teachers and researchers of these disciplines for the development of *epistemological models of reference* (EMR), aimed at finding alternatives to *dominant epistemological models* (DEM) current in the institutions, according to Gascón (2014). To this end, we present some considerations on the construction of the real numbers, which we consider fundamental for approaching the content of courses in which this set is the basis for the others.

In this way, we present an approach to the different procedures used by Cauchy, Cantor and Dedekind in the theoretical construction of the set of real numbers, complemented by illustrative examples that can be used by teachers and researchers to develop epistemological reference models. We present both mathematical, didactic, and epistemological aspects of the construction of the set of real numbers, which can be useful for developing mathematical and didactic organizations of Calculus and Mathematical Analysis content.

Real Numbers: Evolution and Constructions

Since the time of Euclid, the evolution of the concept of real numbers went from the intuitive use of the notion of quantities to a rigorous mathematical construction in the 19th century. As Bronner writes:

> The construction of the real, based on the theory of ratios of magnitudes and derived from the Euclidean tradition, no longer satisfied mathematicians. The arithmetization of quantities and ratios by Descartes and Stevin is always “marked” by geometry. Moreover, geometry no longer had the legitimacy of previous eras. A new current, called formalism, is emerging, where it is a question of constructing or creating objects that we will consider to be whole numbers (Bronner, 1997, p. 43).

First, the problem of continuity of real functions was a concern of mathematicians in the 18th century. In fact, this is manifested in the work of Euler, who, in his 1748 definition of the concept of a real function with real values, writes: *"A constant quantity is a given quantity that always retains the same value*. (Euler, 1796-1797). In Euler's definition, there is one element that can help us understand his ideas on the subject: the idea of "**physical time**", which is at least implicit in the definition: "which is always preserved". The presence of **time** in Euler's work will play an important role in his study of real functions. That is, in this last notion of time, we will find the notion of continuity. In fact, the continuum will remain in a physical context, since the actual real line does not exist under the numerical form in this work (the field of real numbers $\mathbb{R}$ and the real line were developed in the 19th century).

Like Euler, Bolzano was also interested in continuity, trying to prove the intermediate value theorem, and he came up with a clear vision of the set of real numbers. In fact, in 1817 Bolzano formulated the following theorem:

> "Zwischen je zwei Werthen, die ein entgegensetztes Resultat gewähren, liege wenigstens eine reelle Wurzel der Gleichung":

Between two values that give results of opposite signs, there is at least one root of the equation (Journal: Rein analytischer Beweis).

In other words, "*there is at least one real root of the equation between every two values that give an opposite result*". In fact, this theorem that will bear his name is equivalent to the current "intermediate value theorem": "Let f: $[a, b] \longrightarrow \mathbb{R}$ (where a<b) be a continuous function such that (a). $f(b) < 0$. Then, the equation f(x) = 0 admits a solution $x_0 \in [a, b]$, namely, there exists $x_0 \in [a, b]$ such that $f(x_0) = 0$."

For the proof of his theorem, Bolzano proposed to establish it without the help of geometric intuition. This is an analytical proof. The result used by Bolzano in his proof is now known as the *least upper bound* property: "*Any non-empty upper bounded part by real numbers admits a least upper bound*." For reason of conciseness and clarity, we recall that a set $A \subset \mathbb{R}$ is *upper bounded* by a number M if, for all $x \in$ A we have $x \leq M$. For example, the set $A = \{x \in \mathbb{Q};\, x^2 < 2\}$ is *upper bounded*, by the number 2 or by other values *M* greater than 2. Therefore, for an upper bounded set $A \subset \mathbb{R}$, the least upper bound is characterized as follows: if there is a real number $m \in \mathbb{R}$ such that:

1. For every $x \in A$, we have $x \leq m$;
2. Any real number strictly smaller than *m* is not an upper bound of the set *A*.

Therefore, the real number *m is* unique, and it is called the *least upper bound of* the set *A*. Generally, the least upper bound of a set A is denoted by *sup(A).* For an *upper bounded* set, its least upper bound represents the smallest upper bound. For example, the least upper bound of the set $A = \{x \in \mathbb{Q};\, x^2 < 2\}$ is given by $\sup(A) = \sqrt{2}$. Following Vergnac and Durand-Guerrier, Boniface considers that Bolzano had already realized the limits of basing the foundations of analysis on geometric arguments:

> Bolzano (1781-1848) played a crucial role in the foundations of analysis because, unlike many of his contemporaries whose goal was the development of science, he was

> essentially concerned with legitimizing the methods used (Vergnac, Durand-Guerrier, 2014, p. 8).

Bolzano's distrust is actually towards geometry, as he writes:

> [...] But it is equally evident that this is an intolerable fault against the correct method, which consists in trying to deduce the truths of pure or universal mathematics (that is, of arithmetic, algebra, or analysis) from considerations belonging only to an applied (or special) part, namely geometry (Bolzano, Memoirs on the Theorem of Intermediate Values, 1817, p. 210). [cited in Vergnac, Durand-Guerrier, 2014].

It appears that Bolzano was not convinced by the geometric argument for studying the problems of mathematical analysis. For example, Bolzano did not accept the "geometric" explanation that if the values of a "continuous" function change sign, then there exists a real x_0 such that $f(x_0)=0$.

At the beginning of the 19th century, there was a need for rigor in analysis. This was the beginning of the arithmetization of mathematical analysis with Cauchy, about whom Cauchy writes in the introduction of his book "*A Course in Analysis*": "As for the methods, I have tried to give them all the rigor required in geometry, so as never to have recourse to arguments derived from the generality of algebra" (Cauchy, 1821, p. ij).

Cauchy clarifies his method and his vision of rigor on the first page of his Summary of the Course of Analysis (1823, p. iv). Cauchy writes:

> The methods I followed differed in many respects from those set forth in similar works. But the main thing was to reconcile the rigor, which I had never done in my course of analysis, with the simplicity that resulted from the direct consideration of infinitely small quantities (Cauchy, 1823, p. iv).

We can say that Cauchy initiated the genesis of the treatment of epsilon and delta, and thus contributed to the birth of the modern mathematical analysis.

From then on, the demand for a rigorous construction, independent of the geometric aspect, in which the character of the irrational numbers is well explained and made more explicit, becomes a necessity. As a result, constructions based on the set $\mathbb{Q}$ of rational numbers were developed using various methods, such as the method of aggregates (with Weierstrass, 1872), the method of Cauchy sequences (with Méray, 1869 and Cantor, 1872), and the method of cuts (with Dedekind, 1888). For these methods, which are equivalent (Dhombres, 1978), the mathematical construction of the set of real numbers is long and requires properties and computational techniques. However, according to Bronner (1997):

> The new set is endowed with operations and an order structure that leaves the structure of the ordered field "permanent". Moreover, Dedekind shows the "continuous" and "complete" character in terms of cut, which is equivalent to the property of nested intervals, often regarded as the Dedekind-Cantor axiom. (Bronner, 1997, p. 44).

Given the complexity of the construction of the three previous methods, this makes them out of reach for students. Therefore, the construction of the set of real numbers $\mathbb{R}$ is currently presented axiomatically in higher education textbooks. The axiomatic presentation of the set $\mathbb{R}$ of real numbers considered in textbooks will then allow for "important developments that reveal, in axioms or as a consequence, the various algebraic, ordinal, and topological properties." (Bronner, 1997, p.121). More precisely, the use of the axiomatic approach, with the axiom of order and the properties of the intervals of $\mathbb{R}$, makes it possible to study the important algebraic and topological properties of this set.

It is important to note that the various constructions of the set of real numbers are motivated by the fact that the real numbers are closely related to the concepts of limit and continuity of real functions, which constitute the basic foundations of mathematical analysis. Furthermore, the links between geometry and the set $\mathbb{R}$ are explained by the construction of a bijection between and a graded line thanks to the Dedekind-Cantor axiom (Bronner, 1997). The construction of a complete graded real line is given by Monge and Ruff (1962), who point out that there is "an analogy between certain axioms or properties of numbers and the properties of points on the real line" (Monge and Ruff, 1962). And Bronner adds:

> Thus, we derive the properties of infinity and the Archimedean order of the real line and number systems ($\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$), and then the property of density of order: "between two points or two reals, there is a third". [...] all these properties are verified by $\mathbb{Q}$ and $\mathbb{R}$, and they identify no difference between these two sets (Bronner, 1997, p. 112).

Currently, these geometric "properties" are presented in Euclidean geometry as "axioms" of incidence and order. To conclude this section, we can say that the work of 19th-century mathematicians helped to clarify the construction of real numbers and to highlight the foundations of the articulation of real numbers with the concepts of limit and continuity. It seems that this connection, already appeared in the work of Bolzano and Dedekind, is based on the famous intermediate value theorem, to which we will return later.

Historical evolution of notations for numerical sets

According to Rousselet (2021):

> *It took us 5,000 years to get a clear idea of what numbers are. In* Babylon and Egypt, whole numbers and fractions were used. For the Greeks, only integers were numbers. Negative numbers were invented by China and India. Decimals were invented by Arabian mathematicians. Numbers like √2 or √5 remained without status for a long time. The number π was not recognized as an irrational number until the 18th century (Rousselet, 2021, p. 317).

Moreover, following Rousselet (2021) it was not until the end of the 19th century that the numerical sets ℕ**,** ℤ**,** ℚ**,** ℝ and ℂ were defined and organized according to the well-known inclusions: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$. The notation of these sets is due to the mathematicians Gauss, Dedekind, Cantor, and Peano. More precisely, we have:

- The notation ℕ refers to the set of natural numbers. More precisely, the notation ℕ was introduced by Peano in 1894, and the mathematical construction of this set was proved independently by Peano and Dedekind towards the end of the 19th century.
- Historically, the origin of the notion of fraction can be found in the Egyptian papyri, in particular the Rhind papyrus, which dates back to -1650 B.C. In 1895, Peano made a mathematical construction of the set of fractions, and he chose to name this set ℚ, referring to the letter of the Italian word "quoziente", which means "quotient".
- For Dedekind: The notation ℤ refers to relative integers and is derived from the first letter of the German word "Zahlen" (meaning "count" or "numbers").
- The notation ℝ was used by Cantor to denote the set of real numbers. This is because some numbers, such as √2, √5, or π, cannot be expressed as fractions, so the set ℝ containing these numbers was defined at the end of the 19th century by Cantor and Dedekind.
- The notation ℂ was proposed by Gauss at the beginning of the 19th century to denote the set of complex numbers.

Didactic and epistemological considerations

In general, as already mentioned, the three constructions of Weierstrass, Méray-Cantor, and Dedekind of the set of real numbers cannot be taught in school or at the beginning of university. For this reason, textbooks and university programs do not pay special attention to the rigorous construction of real numbers and their properties. This sometimes leads to intuitive approaches in high school.

Students' conceptions of real numbers are based on the use and selection of the proposed set of numbers in the practice of mathematical activities in primary and secondary schools. This practice is reinforced through calculators and software, which has the effect of leading students to a conception of the real number in integer or decimal form. As a result, when they go to university, students will find it difficult to acquire concepts of mathematical analysis, such as the concept of the limit of functions or the limit of numerical sequences.

In an interesting study, Vergnac and Durand-Guerrier highlight the approach to the geometric representation of the set of real numbers, which consists of emphasizing the one-to-one correspondence of the set of real numbers and the real line:

> Although it is based on the geometric intuition of the real line, which is partly at the origin of Dedekind's construction, it seems to us, considering the study we have carried out, that this approach, although it allows us to approach the notion of order and the first notions of analysis in high school work, is not sufficient to construct the different aspects of the concept of the real number. It seems to us that, to develop this concept, it would be interesting to establish activities on the writing registers of a number (Vergnac, Durand-Guerrier 2014, p. 9).

In textbooks, the set of decimals and the set of rational numbers are studied, by using activities or value tables to introduce real functions or to construct curves representing certain functions. This type of approach makes it possible to manipulate real numbers and give them a concrete meaning. Moreover, in tables of values for introducing functions through practical activities, the values are chosen a priori to achieve the desired results (Job, 2023). In addition, the close link between real numbers and the concept of limits of real functions is expressed by Schons (1965), who introduced in his book a "**notion of variable",** which is **defined** as follows:

> Consider a variable **x** that passes successively through an infinite number of values. A variable passes successively through several values if, considering two of these values, we can say that one of them precedes the other, and again, which one precedes the other. It passes successively through an infinite number of values, none of which is the last. These values can follow each other discontinuously and be, for example, successive terms of an infinite sequence of numbers. They can also follow each other continuously, so that no value goes from one to another without passing through all intermediate values. This is what happens, for example, when you represent the x-axis of a point moving along an oriented real line (Schons, 1965, p. 156).

It seems here that Schons, out of didactic necessity, felt compelled to draw a parallel between the mathematical notion and the physical movement of the continuous displacement of points on the real line. Thus, Job will use the notion of a "**dynamic**" variable: *"[...] in the case of Schons, we are therefore faced with a definition of the limit of a function that is based on a notion of a "**dynamic**" variable, [...] " (Job, 2023, Slides p. 196).*

In fact, the "dynamic" variable aspect of the variable ***x*** reflects both the real values that this variable takes and the geometric representation of the set of real numbers, namely, the real line. We find at the same time aspects of the "intermediate value theorem", Dedekind's "continuous" character of the set of real numbers, and the real line. In fact, according to Bronner (1997):

> Dedekind shows that the separation of geometry and arithmetic is effective at the level of "theoretical creation" [...]. Dhombres quotes a reply to Kant in which Dedekind explains his position on this issue: "The concept of space is for me completely independent, completely separable from the representation of continuity, and this latter property serves only to specify from the general concept of space the special concept of continuous space (Bronner, 1997, p. 44).

According to Bronner, from a didactic perspective, this separation between geometry and arithmetic appears to be very difficult, even very complicated.

Considerations on the construction of the set of real numbers

As mentioned earlier, there are several methods to construct the set of real numbers from the set of rational numbers, from the integers, or by purely axiomatic methods. These include Weierstrass (1872), Cauchy's method of sequences (with Méray, 1869, and Cantor, 1872), and Dedekind's method of cuts (1888). Here we present a brief approach to these three notable historical methods of mathematical construction of the set of real numbers, in order to better understand their mathematical and also didactic differences.

Weierstrass aggregate method. The Weierstrass method had no significant impact. This method was derived from the work of Weierstrass' students, who evaluated it based on the collected notes of the Weierstrass' courses. The method involves the use of finite or infinite multi-sets, consisting of natural numbers and inverses of natural numbers. For example, the set F= {2, 5, 7, $\frac{1}{6}$, $\frac{1}{3}$}, is multi-set. A multi-set F is said to be **bounded** if, for any **finite** subset E of F, there exists δ in $\mathbb{Q}$ such that:

$$\sum_{a\in E} a < \delta.$$

Let us consider the set $\mathscr{F}$ of bounded multi-sets, in which he defines addition and multiplication naturally. Weierstrass then equips the set $\mathscr{F}$ of bounded multi-conjuncts with an order relation defined as follows: for any bounded multi-set E and F of $\mathscr{F}$, we say that E is less than or equal to F, and we write $E \leq F$, if for any **finite** subset C of E and every **finite** subset D of F, we have:

$$\sum_{a\in C} a \leq \sum_{b\in D} b.$$

Finally, Weierstrass introduces the following relationship:

$$E \sim F \iff E \leq F \text{ e } F \leq E$$

We can see that $\sim$ is an equivalence relation on the set $\mathscr{F}$. Therefore, thanks to this equivalence relation ~, Weierstrass was able to identify the set of strictly positive real numbers as the quotient set:

$$\mathbb{R}^{*+} = \mathcal{F}/\sim$$

We can say that the Weierstrass construction allows us to obtain the set of positive real numbers $\mathbb{R}^{*+}$**,** as a set of equivalence classes of the equivalence relation ~ .

Cauchy Sequences Method (with Méray, 1869 and Cantor, 1872). A sequence of real numbers $(\boldsymbol{u_n})_{\boldsymbol{n}}$ is said to be convergent of limit L, and we denote it by, $\boldsymbol{L} = \lim\limits_{\boldsymbol{n}\to+\infty} \boldsymbol{u_n}$ se:

For all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $|u_n - L| < \mathscr{E}$

To apply this definition we observe that knowledge of the limit L is required, which represents an obstacle. Therefore, to have a practical criterion, the idea of the epoch consists of studying the difference between any two terms starting from a given number N: A sequence (u_n) is said to be a Cauchy sequence if:

For all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m > N$ we have $|u_n - u_m| < \mathscr{E}$.

With this definition, we can see that the limit does not appear, and a Cauchy sequence in $\mathbb{Q}$ does not necessarily converge in $\mathbb{Q}$. In fact, for example, the following (u_n) defined by:

$$u_n = \sum_{j=0}^{n} \frac{1}{j!},$$

does not converge in $\mathbb{Q}$. In fact, its limit is the number *e=exp(1)* which is not an element of $\mathbb{Q}$.

Let $\mathbb{Q}^{\mathbb{N}}$ denotes the set of sequences of rational numbers. This set can naturally be provided with the operations of addition, multiplication and the relation of order <. Let (u_n) and (v_n) be two arbitrary sequences of the set $\mathbb{Q}^{\mathbb{N}}$**,** the following operations are defined:

- **Addition:** $(w_n)_n = (u_n)_n + (v_n)_n$ with $w_n = u_n + v_n$, for every $n \in \mathbb{N}$;
- **Multiplication**: $(w_n)_n = (u_n)_n \times (v_n)_n$ with $w_n = u_n \times v_n$, for every $n \in \mathbb{N}$;
- **Order:** $(u_n)_n < (v_n)_n$ com $u_n < v_n$, for every $n \in \mathbb{N}$

The set $\mathbb{Q}^{\mathbb{N}}$ equipped with these addition and multiplication operations is a commutative ring, and it is denoted as follows $(\mathbb{Q}^{\mathbb{N}}, +, \times)$. However, the relation < is not a total order relation in $\mathbb{Q}^{\mathbb{N}}$**,** namely, for two sequences (u_n) and (v_n) of $\mathbb{Q}^{\mathbb{N}}$ we cannot conclude in general that $(u_n) < (v_n)_n$ or $(v_n)_n < (u_n)_n$.

Let $\mathfrak{R}$ be the set of Cauchy sequences. We can show that $\mathfrak{R} \subset \mathbb{Q}^{\mathbb{N}}$ and also inherits the above addition and multiplication operations. Therefore, we show that $(\mathfrak{R}, +, \times)$ admits a subring structure of ring $(\mathbb{Q}^{\mathbb{N}}, +, \times)$. Let $\mathfrak{I}$ be the subset of $\mathfrak{R}$ formed by sequences converging to 0. The set $\mathfrak{I}$ is a maximal ideal of the ring $(\mathfrak{R}, +, \times)$. The set $(\mathfrak{R}/\mathfrak{I}, +, \times)$ is a commutative field denoted by $(\mathbb{R}, +, \times)$.

Let us note that constant sequences equal to a rational number are Cauchy sequences, in addition to the application of $\mathbb{Q}$ with values in $\mathfrak{R}/\mathfrak{I}$ associating a rational number of $\mathbb{Q}$ with the constant sequence equal to this rational number is injective. Therefore, the set of rational numbers $\mathbb{Q}$ is naturally injected into $\mathbb{R}$. In addition, concerning the topological level, we can show that the set $(\mathbb{R} = \mathfrak{R}/\mathfrak{I}, +, \times, <)$ is a totally ordered field, in which every Cauchy sequence is convergent.

In conclusion, Cauchy's method allows us to construct the topological field of real numbers $\mathbb{R}$, as being the set of equivalence classes of the quotient of the set of Cauchy sequences $\mathfrak{R}$ by the ideal $\mathfrak{I}$ of the sequences convergent to 0.

The method of cuts (with Dedekind, 1888). A Dedekind cut of $\mathbb{Q}$ is formed from a pair (C^-, C^+) of its parts that form a partition of $\mathbb{Q}$ with $C^- \neq \emptyset$ and $C^+ \neq \emptyset$ such that:

a) $\mathbb{Q} = C^- \cup C^+$ and $C^- \cap C^+ = \emptyset$: ($C^- \neq \emptyset$ and $C^+ \neq \emptyset$ is a partition of $\mathbb{Q}$),

b) $C^- < C^+$: For every $x \in C^-$ and every $y \in C^+$ we have $x < y$.

As an example of a cut, the two sets: $C^- = \{x \in \mathbb{Q}; x < 0 \text{ e } x^2 < 2\}$ e $C^+ = \{x \in \mathbb{Q}; x > 0$ $e\ x^2 > 2\}$ represents a cut of $\mathbb{Q}$. In fact, we can see that this pair (C^-, C^+) is a cut because:

a) Every rational number belongs to C^- or a C^+ with $C^- \cap C^+ = \emptyset$

b) Every rational number of C^- is strictly smaller than any rational number of C^+; that is, for all $a \in C^-$ and all $b \in C^+$ we have $a < b$.

Dedekind then constructs the operations of addition, subtraction and multiplication, that give the cuts a field structure: this is the field of real numbers $\mathbb{R}$. In other words, the set $\mathbb{R}$ is the set of cuts of $\mathbb{Q}$. More precisely, it can be shown that the set of real numbers $\mathbb{R}$ is itself complete, i.e., all cuts of the same type define a real.

In conclusion, given the above, we can see the complexity of constructing de set $\mathbb{R}$ of real numbers, with the aid of the three previous methods. This complexity makes them inaccessible to students. Therefore, generally the construction of the set of real numbers $\mathbb{R}$ is currently presented axiomatically in the textbooks, based on these construction methods.

Some additional mathematical observations

To complement the approach of the epistemological and the didactic aspects, on the set $\mathbb{R}$ of real numbers, we present in this section some observations involving the real numbers.

Observation 1: Insufficiency of the set of rational numbers and the least upper bound axiom.

Historically, especially since Greek times, the insufficiency of rational numbers has been noted when using them in practice for studying some known problems. For example:

Aristotle reports the first proof, due to the Pythagoreans, of the need to consider irrational numbers: no rational can represent the ratio between the lengths of the diagonal of a square and the side of the diagonal of that square (Thuizat *et al.,* 1980, p. 26).

Furthermore, according to Thuizat *et al*. (1980), the terminologies of natural, rational and real numbers can be interpreted as follows:

- **Natural**: Creating by nature,

- **Rational**: Creating by reason,

- **Real**: Actually existing.

As for the word "irrational", etymologically, it means "contrary to reason". Therefore, this adjective itself reflects the students' difficulties with these real numbers. Furthermore, this difficulty seems to be at the origin of the orientation of Greek mathematics more towards geometry. In general, the inadequacy of the set of rational numbers $\mathbb{Q}$ is justified in some textbooks by the fact that algebraic equations such as:

$x^2 = 2$ or $x^2 = 3$,

do not admit solutions belonging to $\mathbb{Q}$. However, there is another fundamental requirement, which is related to the order relation $\leq$. More precisely, this requirement concerns the existence of the least upper bound of a bounded subset of $\mathbb{Q}$.

Let us recall that a subset E of $\mathbb{R}$ is bounded if there exists a number M such that $x \leq M$, for any $x \in E$. The least upper bound *sup(E)* is defined as the smallest of the upper bound of the bounded set E. We can see that if E an upper bounded subset of $\mathbb{N}$ (or $\mathbb{Z}$), then E admits a least upper bound belonging to $\mathbb{N}$ (or $\mathbb{Z}$). And in this case, the least upper bound sup(E) of E belongs to E. However, this property of the existence of the least upper bound is not verified by all bounded subsets of $\mathbb{Q}$. Namely, the property of the existence of the least upper limit of the sets ℕ and ℤ is not preserved by the set ℚ. Indeed, for example, the following subsets of the set ℚ of rational numbers:

$E_1 = \{x \geq 0$ with $x \in \mathbb{Q} \mid x^2 \leq 2\}$ or $E_2 = \{x \geq 0$ with $x \in \mathbb{Q} \mid x^2 \leq 3\}$.

We can see that:

- The sets E_1 and E_2 are not empty (0 belongs to E_1 and also to E_2),
- The number M = 2 is an upper bound of E_1 and E_2.

However, none of the sets admits a least upper bound belonging to the set ℚ. But each of these sets E_1 and E_2 is included in the set of real numbers ℝ and admits a least upper bound that belongs to C, namely, $sup\ (E_1) = \sqrt{2}$ and $sup\ (E_2) = \sqrt{3}$, which are not rational numbers.

This question of the existence of the least upper bound was one of the key points in the different constructions of the set of real numbers in the 19th century. This leads to the axiom of the least upper bound in the definition of the set of real numbers, namely,

Upper bound axiom: Any non-empty, bounded subset of ℝ admits a least upper bound

The upper limit axiom has been used to establish many properties of real numbers, such as:

- **Archimedes' Property (or Principle):** "If a and b are positive real numbers, then there is a positive integer *n* such that: $n \times a > b$".
- **The Density of the set ℚ in the set of real numbers** ℝ: "Given any $a, b \in \mathbb{R}$, with $a < b$; there exists an $r \in \mathbb{Q}$ such that $a < r < b$".
- **The property of nested segments (or The Nested Set Theorem)**: "Let $\{I_n\}_{n\geq 1}$ be a nested (or descending), namely, $I_{n+1} \subset I_n$, countable collection of nonempty closed sets of real numbers for which I_1 bounded. Then, we have $\bigcap_{n=1}^{+\infty} I_n \neq \emptyset$".

In general, the least upper bound of a bounded subset of ℝ is not an element of that set. For example, the subset of strictly negative real numbers of ℝ:

$E = \{x \in \mathbb{R}$ with $x \neq 0 \mid x \leq 0\}$,

is majored and has an upper bound of $sup\ (E) = 0$. However, we can observe that $0 \notin E$.

- The set ℚ becomes a subset of ℝ;

The preservation of the properties of addition +, multiplication × and the order relation ≤;

Observation 2: Irrational numbers and the structure of the set of real numbers.

Any construction of the set of real numbers $\mathbb{R}$ from the set of rational numbers $\mathbb{Q}$, is characterized by:

- The complement $\mathbb{R}\backslash\mathbb{Q}$ of $\mathbb{Q}$ in $\mathbb{R}$ is called the set of irrational numbers.

In symbolic mathematical language, the ordered field ($\mathbb{Q}$, +,×,≤) is isomorphic to a sub field of the ordered field ($\mathbb{R}$, +,×, ≤).

On the other hand, any rational number of $\mathbb{Q}$ can be written under the form: $\frac{p}{q}$, where p, $q \in \mathbb{Z}$, with $q \neq 0$. However, there is no general notation for irrational numbers. The only exception is a restricted class of usual numbers, such as □, e, $\sqrt{a}$ ($a \geq 0$) or more generically $\sqrt[p]{a}$. Perhaps, this difficulty of a general notation for irrational numbers also represents an epistemological obstacle in the teaching of real numbers.

Observation 3: The construction of real numbers is carried out at the intersection of geometric, numerical and analytical aspects.

It can be said that the simultaneous introduction of the constructions, as well as the axiomatization, of the set of real numbers $\mathbb{R}$ by Cantor, Dedekind and Weierstrass represents the foundations of real analysis based on a well-constructed set $\mathbb{R}$. That is, it reflects the importance and necessity of a rigorous construction of $\mathbb{R}$ as a fundamental tool for the foundations of mathematical analysis. Moreover, according to Margolinas (1988):

> Dedekind speaks of this very clearly (Dedekind, 1872) and insists on the motivation for his construction. He places $\mathbb{R}$ at the intersection of three major mathematical fields, the geometric, the numerical, and the analytic. It is based on the need to release the notion of continuity from geometric proofs. His exposition begins with the properties of rational numbers "considered as necessary consequences of arithmetic" in order to compare the properties of $\mathbb{Q}$ with those of an intuitively continuous real line and to conclude that there is a lack of continuity of $\mathbb{Q}$ (Margolinas, 1988, p. 52).

Observation 4: Teaching real numbers and mathematical analysis.

Several questions arise about the knowledge of real numbers in mathematics and in teaching programs in primary, secondary and higher education. Indeed, we can observe that:

1- In teaching, some particular irrational numbers such as π, $\sqrt{2}$, e are considered. However, the study of the fundamentals of the set $\mathbb{R}$ of real numbers, as a structured set with an order relation, is less in-depth.

2- The set ℝ is at the intersection of several mathematical domains.

Therefore, the following fundamental question arises:

What would be the educational model of real numbers for the study of mathematical analysis and the construction of the functional thinking?

This question has always been considered seriously in Brazil and elsewhere. As mentioned by Margolinas (1988), there is a tension between the two models of teaching:

- Axiomatic model, in which **the construction of** ℝ is the basis of mathematical analysis,
- Teaching model, in which **the study of** ℝ is the basis for the study of analysis.

Again, according to Margolinas (1988), considering the historical model for a teaching model, one should have:

- Historical model, in which **the construction of** ℝ is the final step, for the study of mathematical analysis,
- Teaching model, in which **the study of** ℝ is the final step, for the study of mathematical analysis.

It seems to us that the teacher, aware of these back-and-forth between the different models of teaching mathematical analysis, can form his own reflection on the teaching of mathematical analysis at high school or university level, through the approaches proposed by the official program and the didactic books.

Discussion

In this study, we have tried to draw on some elements related to the origins and motivations behind the beginning of the arithmetization of mathematical analysis, to show the didactic importance of university students mastering the properties of the set of real numbers. This condition can help overcome some difficulties inherent in the concept of the limit of real functions with real values. In fact, the formal definition of the limit with (ε, δ) is difficult to access for many students, especially since it is not usually used in solving exercises.

What can history contribute to teaching a concept? In general, with historical knowledge, the teacher is able to take a step back to understand and identify the students' difficulties in a different way. It can also help to understand the construction of students' mathematical knowledge. For example, the teacher can propose problem situations that can facilitate the construction of mathematical knowledge by the student, which can contribute to helping him learn the concept studied. On the other hand, the historical justifications of the concept will allow a great motivation and a greater motivation on the part of the students.

What precautions should be taken when using history to teach a concept? Oversimplified and reductive representations of concepts in textbooks can obscure the real problems of the past and interpret the mathematics of another era with current knowledge: the **practice of anachronism**. In addition, teachers need to be aware of the historical misrepresentation of a concept in some textbooks. Moreover, certain studies invite us to distance ourselves to avoid bringing historical and/or epistemological difficulties closer to students' difficulties and conceptions.

Rachidi-Magalhães de Freitas-Junqueira Godinho Mongelli (2020) and Rachidi-Burigato-Junqueira Godinho Mongelli (2023) considered a preliminary study of the properties of real numbers in their books. In these two books, some chapters were dedicated to the study of the algebraic and topological properties of the set real numbers. On the other hand, considering the program of Calculus 1 (in the UFMS), Burigato and Rachidi (2023) presented activities in which the existence of limits for certain functions necessarily requires the use of the formal definition of limit. More precisely, the demonstrations of the activities proposed in the article by Burigato-Rachidi (2023) are based on the formal definition of limit, in which properties of certain sequences of real numbers play a fundamental role. The motivation for this approach is due to its relationship with functional thinking, which generally describes certain elements that characterize it, i.e., according to Blanton and Kaput:

> We broadly conceptualize functional thinking to incorporate the construction and generalization of patterns and relations using various linguistic and representational tools and treating generalized relations of functions as useful mathematical objects in their own right (2004, p. 8, according to Burigato-Rachidi's translation).

Considering this fact, we find in Georges the following reflection:

> In view of the preeminence of functional thinking and the availability of the various mathematical methods for the interpretation, representation, generalization, and

application of functional relations to make possible the acquisition of correct habits of functional thinking, we are led to believe that this is the principal aim of the teaching of mathematics (1929, p. 608, according to Burigato-Rachidi's translation*)*.

Consequently, we can conclude that:

> Functional thinking is linked to the concept of function so that we can find it in various branches of mathematics in which the concept of function is present. Functional thinking thus goes beyond mathematics, which makes it possible to enrich the student's education in different areas (Burigato-Rachidi, 2023, p. 29).

Conclusion

Regarding the content of this paper, we would like to highlight some aspects that we consider important and that we have covered in this article. We have highlighted and presented general elements about important constructions of the set of real numbers and their role in the arithmetic of analysis. On the other hand, the history of the real numbers and their constructions are important tools for teaching and for the scientific culture of teachers. We also highlight various didactic and epistemological observations related to the history of real numbers and their constructions. We have also highlighted the importance of the history of real numbers and their construction for the development of an EMR.

Thus, we can say that, in general, the rigorous mathematical constructions of the set of real numbers during the second half of the 19th century were motivated by the importance of this set for the study of limits and continuity of real functions with real values. In particular, mathematicians of that time realized that the foundation of modern mathematical analysis necessarily requires a rigorous mathematical construction of the real numbers. Furthermore, based on this observation, we present in this study some elements and approaches to illustrate the didactic importance of an in-depth study of the set of real numbers and their properties by students. In fact, this can enable them to study continuity and limits more adequately and, even more, to approach mathematical analysis more rigorously.

Finally, we believe that this study can contribute to improving the approach to limits and continuity at the end of high school and in the topics of Calculus and Mathematical Analysis in undergraduate courses in the area of Exact Sciences. We hope that we have been able to provide support for the development of *Epistemological Reference Models (*EMR), aimed at developing studies and research on mathematical and didactic organizations concerning the contents of functions, limits and continuity, among others.

References

Artigue M. (1990). Épistémologie et didactique, *Recherche en Didactique des Mathematiques*, vol 2.3, pp. 241-286.

Blanton, M. L.; Kaput, J. (2004). *Elementary Grades Students' Capacity for Functional Thinking*. International Group for The Psychology of Mathematics Education, v. 2, 2004, pp. 135-142.

Bronner, A. (1997). *Étude didactique des nombres réels : idécimalité et racine carrée*. Education. Joseph Fourier. Université Joseph Fourier (Grenoble 1) (UJF), Grenoble, FRA. Français. NNT: tel-03797170.

Burigato, S.M., Rachidi, M. (2023). O uso da definição formal de limite finito para funções reais: uma proposta para o ensino, *Revista Sergipana de Matemática e Educação Matemática*, v. 8 n. 3. https://doi.org/10.34179/revisem.v8i3.18348

Dhombres, J. (1978). *Nombre, mesure, et histoire*. Cedic - Nathan.

Euler, L. (1796). *Introduction à l'Analyse Infinitésimale*. Tome 1 / Léonard Euler; trad. du latin en français avec des notes et des éclaircissements par J. B. Labey (1752-1825). https://gallica.bnf.fr/ark:/12148/bpt6k3884z/f4.item#

Gascón, J. (2014). Los modelos epistemológicos de referencia como instrumentos de emancipación**.** *EDUCACIÓN MATEMÁTICA*, 25 años, Marzo de 2014. https://www.redalyc.org/pdf/405/40540854006.pdf

Georges, J. S. (1929). Functional thinking as an objective of mathematical Education. *School Science and Mathematics*, v. 29, n. 5, p. 601-608, 1929.

Houzel, C. (1979). « Histoire des mathématiques et enseignement des Mathématiques », *Histoire des mathématiques et épistémologie*, Bulletin inter IREM, n° 18, p. 3-6, 1979.

Job, P. (2023). Didactique et notion de limite. Vers un modèle épistémologique de référence (MER) partagé ? *Conférence Didactique sur l'Analyse*. INMA – UFMS, Campo Grande. Journée 3, Septembre 2023.

Margolinas, C. (1988). Une étude sur les difficultés d'enseignement des nombres réels. *Petit x* n.16, p. 51-66.

Monge, M. et Ruff, P. (1962), *Ensembles et nombres*, Librairie Belin, 1962.

Oudot, X., Le *Théorème de Bolzano-Weierstrass*, Culture Math. https://culturemath.ens.fr/

Rachidi, M.; Freitas, J. L. M. & Mongelli, M. C. G. J. (2020). *Limite de funções de uma variável real com valores reais e generalizações*.1ª edição. Campo Grande: Editora UFMS. Disponível em: https://repositorio.ufms.br/handle/123456789/3548.

Rachidi, M.; Burigato, S. M. M. S. & Mongelli, J. G. M. C. (2023). *Conceitos Básicos para a Introdução ao Cálculo Diferencial e Integral*, Editora UFMS. Disponível em: https://editora.ufms.br/livros-digitais-edital-ufms/

Rousselet, M. (2021), *La belle histoire des Maths*, ADAPT-Snes. De Boeck-Supérieur.

Schons, N-J. (1965). *Éléments d'algèbre*. 8e éd.; *Procure.*

Vergnac, M. et Durand-Guerrier, V. (2014), Le concept de *nombre réel* au lycée et en début de l'université: un objet problématique. *Petit x,* n, 96. pp. 7-28.

Weierstraß, K. *Einleitung in die Theorie der analytischen Funktionen*. Springer, 1988.