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Reflections about modeling processes: a case of translational motion in a horizontal direction

Reflexiones sobre los procesos de modelado: el caso del movimiento traslacional en la dirección horizontal

Réflexions sur les processus de modélisation : le cas du mouvement translationnel dans le sens horizontal

Reflexões sobre o processo de modelagem: o caso do movimento translacional na direção horizontal

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Abstract

Motion is not usually included in basic education, and only a static view of isometries is studied. The main purpose of this paper is to present some results of research on the analysis of modeling processes of isometric translational motion, carried out by first-year high school students from a public school in the state of Goiás, Brazil, from the perspective of school mathematical modeling. The data were collected from solutions provided by participants in face-to-face and online workshops, which were not compared. The analyses were conducted qualitatively, according to the claim-making/participatory conception, where participatory action is dialectic and focuses on producing changes in practice. This research is exploratory, descriptive and explanatory. The data were categorized, and we observed that the modeling processes encompassed: graphic- geometric approach; arithmetic approach; algebraic approach; written native language; and written mathematical language. This research provided students with an opportunity to raise their awareness of thinking and processing motion modeling in the Euclidean plane.

***Keywords*:** Mathematical modeling, High school, Translational isometric motion.

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Resumen

Generalmente los estudios sobre movimientos isométricos no se incluyen en la educación básica de forma dinámica y sólo se estudia una visión estática de las isometrías. El principal objetivo de este artículo es presentar algunas reflexiones sobre modelación, modelos y algunos resultados de una investigación sobre el análisis de procesos de modelación del movimiento traslacional isométrico, realizada con estudiantes de primer año de secundaria de una escuela pública de Goiás-Brasil, bajo la perspectiva de la Modelación Matemática de la Escuela. Los datos se recolectaron a partir de las resoluciones de los participantes en talleres presenciales y en línea. Los análisis se realizaron de forma cualitativa, no comparativa, según la concepción participativa, donde la acción participativa es dialéctica y se centra en producir cambios en las prácticas. Esta investigación se caracteriza por ser exploratoria, descriptiva y explicativa. Se categorizaron los datos y se observó que los procesos de modelación involucraron: enfoque gráfico geométrico; enfoque aritmético; enfoque algebraico; lengua materna escrita; y lenguaje matemático escrito. La investigación brindó un momento para que los participantes tomaran conciencia de su pensamiento matemático y también de un proceso para modelar el movimiento traslacional en el plano euclidiano.

Palabras clave: Modelación matemática, Enseñanza secundaria, Movimiento traslacional.

Résumé

Généralement, les études sur les mouvements isométriques ne sont pas incluses dans l'éducation de base de manière dynamique et seule une vision statique des isométries est étudiée. L'objectif principal de cet article est de présenter quelques réflexions sur la modélisation, les modèles et quelques résultats d'une recherche sur l'analyse des processus de modélisation du mouvement translationnel isométrique, réalisée avec des élèves de première année du secondaire d'une école publique de Goiás-Brésil. Du point de vue de la modélisation mathématique scolaire. Les données ont été collectées à partir des résolutions des participants lors d'ateliers en personne et en ligne. Les analyses ont été réalisées de manière qualitative et non comparative, selon la conception participative, où l'action participative est dialectique et vise à produire des changements de pratiques. Cette recherche se caractérise par être exploratoire, descriptive et explicative. Les données ont été catégorisées et il a été observé que les processus de modélisation impliquaient : une approche graphique géométrique ; approche arithmétique ; approche algébrique ; langue maternelle écrite ; et langage mathématique écrit. La recherche a permis aux participants de prendre conscience de leur pensée mathématique et également d'un processus de modélisation du mouvement de translation dans le plan euclidien.

Mots-clés **:** Modélisation mathématique, Lycée, Mouvement de translation

Resumo

Geralmente, os estudos sobre os movimentos isométricos não são incluídos na educação básica de forma dinâmica e apenas uma visão estática das isometrias é estudada. O objetivo principal deste artigo é apresentar algumas reflexões sobre modelagem, modelos e alguns resultados de uma pesquisa sobre análise de processos de modelagem de movimento de translação isométrica, realizada com alunos do primeiro ano do ensino médio de uma escola pública de Goiás-Brasil, sob uma perspectiva de Modelagem Matemática Escolar. Os dados foram coletados a partir das resoluções dos participantes em oficinas presenciais e on-line. As análises foram realizadas de forma qualitativa não comparativa, segundo a concepção participativa, onde a ação participativa é dialética e tem como foco a produção de mudanças nas práticas. Esta pesquisa caracteriza-se como exploratória, descritiva e explicativa. Os dados foram categorizados e observamos que os processos de modelagem envolveram: abordagem gráfica geométrica; abordagem aritmética; abordagem algébrica; língua materna escrita; e linguagem matemática escrita. A investigação proporcionou um momento para que os participantes tivessem consciência de seu pensamento matemático e de um processo para modelar o movimento translacional no plano euclidiano.

Palavras-chave: Modelagem matemática, Ensino médio, Movimento translacional isométrico.

Reflections about modeling processes: a case of translational motion in a horizontal direction

Mathematical modeling is used by many teachers and researchers in the classroom to help mathematical content make sense to students (Dunn & Marshman, 2020; Villa-Ochoa, 2015; Malheiros & Honorato, 2016; Hernandez-Martinez & Vos, 2018; Almeida & Lara, 2021). Thus, we analyzed some students' processes of modelling translational motion by manipulating a concrete figure, not by software. Geometry is an important content to develop thinking, as stated by Jablonski (2024):

> […] geometry seems to be a relevant aspect for mathematical modelling in relation to reality and vice versa. On the one hand, the solution of modelling problems related to reality requires geometry knowledge. On the other hand, geometry can be discovered in reality and allows interesting questions in the scope of mathematical modelling. The relation between modelling and geometry seems to contribute to both the general aims of geometry teaching and learning and the enhancement of mathematical modelling. Therefore, the relevance to involve geometry modelling tasks in mathematics classes is evident. (Jablonski, 2024, p.3)

Due to the complexity and the diverse character of a reflection on School Mathematical Modeling (SMM) (Cifuentes &Gabardo, 2012; Levy & Espírito Santo, 2010; Araújo, 2019; Brown & Ikeda, 2019), here we make an initial reflection through a historical context of human development with the art of modeling, but there are other topics of reflection that are also important.

We start by providing a basis with human experience situations that enable individuals to interpret and influence the environment in which they live in order to, above all, facilitate their existence. The historical timeline linked to the progress of sciences, and aided by the scientific modeling of mathematics, leads us to the main goal of this paper, which is to present a piece of research based on the theoretical principles that locate School Mathematical Modeling (SMM) in contextualized situations beyond those usually found. This means that we also approach SMM through aspects that discuss mathematics by mathematics itself, mathematics as a deductive and logical system, with its own language (Tarski, 2001). In Machado's words (2001, p.53): "The possibility that mathematical abstractions can be viewed as a thing in itself, detached from the empirical substratum that generated them, cannot be denied".

More than 10,000 years B.C., in the prehistoric period, records already existed, and through them we know what Paleolithic civilizations were like: drawings on cave walls, sculptures, hunting and cooking utensils, and others. Cave paintings of people and animals in motion allow us to imagine what hunting was like. But it was only around 6,000 B.C., when writing began to exist, that mathematics had a boost, but it was far from formalization. Evidence shows (Contador, 2006) that writing and recording began with people who settled in the area known as Iraq today. However, another growing civilization was also established in Egypt. The first ones used clay tablets and, in Egypt, ink on papyrus, but there is evidence that writing was also used in China and India at the same time. Records, representations of ideas, problem-situations, and daily views…it seems that this is where we had the first marks of modeling – representing and mathematizing ideas to analyze and understand phenomena. At that time, and in those regions, the first representations of how they conceived the world began to appear (Contador, 2006; Katz, 2009) through drawings or writing.

Now we should discuss records and representations. Schematics and drawings were (and still are) part of a common language for everyone, regardless of culture, and they certainly accompanied geometric and even arithmetic ideas. They provided interpretations, abstractions, connections, locations, etc. Maps are examples of them.

Drawings were common ways to communicate, represent, and metaphorize ideas, like Babylonian tablets and papyri. Euclid was an expert at this, and so were astronomers, navigators, and traders. To translate, interpret and even communicate the phenomena, they used mainly non-verbal language, which includes visual signs such as symbols, drawings, paintings, and others. However, interpretations were mostly outlined by models and prototypes that aimed to explain the characteristics of a phenomenon. It was symbolic mathematical language, often accompanied by drawings that described it (see Figure 1). For Levy and Espírito Santo (2006), the history of mathematical knowledge is abundant when it comes to the construction of models based on practice, on the concrete, in line with philosophical and epistemological currents that favor the importance of empiricism. For example, Eratosthenes, who “is especially remembered for his measurement of the Earth – not the first nor the last of such assessments in antiquity, but, in all things, the most successful" (Boyer, 1974, p.117), and Thales, who, at the request of the pharaoh, measured the height of a pyramid, an important landmark for Trigonometry. Another example is Napier, who used two straight lines to represent particle motion and, stemming from this, the initial idea of logarithms emerged (Fig.1).

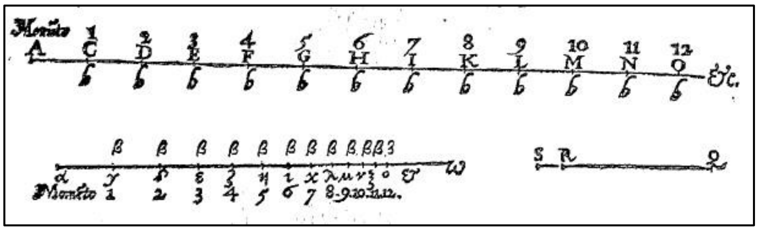


Figure 1.

Napier's draft for representing the velocities of two particles: one following a geometric progression and the other an arithmetic progression. (Clark & Montelle, 2011, n. p.)

From this historical digression, we observe that representing and modeling are inherent to man as a way of understanding the world and that it has been happening for thousands of years. Civilizations have used models beyond the scientific way to solve daily problems, as a real-world necessity. For some years now, the mathematics education teaching and research community has been using models for teaching and learning mathematics.

Biembegut and Marrow (2008) point out that, in a collection of American texts written between 1958 and 1965 on the work done by the School Mathematics Study Group, there was an overview of mathematical applications to teaching and the process of model building. This may be an initial indication of the SMM, which has advanced and solidified in the past decades (Nunes, Nascimento & Souza, 2020).

The practice of creating and elaborating models uses material and intellectual resources. In the case of using mathematical tools, we refer to this practice as Mathematical Modeling, whose objectives are multiple. Observations on the behavior of a model can serve as a guide for experimental tests. In addition, models can help identify and generate new questions (D’Ambrósio, 2009).

Kepler's laws, which define the motion of the Earth around the Sun, and Newton's laws, used to determine the dynamics of the motion of bodies, are real-world models; they are optimal due to their predictive power. Biologists and mathematicians derive equations (create models) of population growth and epidemic spread. Economists create models that describe some aspects and help study real-world economics.

Although mathematics has its genesis in social practice, its study as a pure science – almost always without immediate concern for application – has often proved useful. This was the case with Boole’s studies of conics and algebra, Kepler’s astronomical discoveries, and the development of computers.

In this context, a model of the proposed problem-situation is essential for understanding it, regardless of its origin. Let us consider a more current aspect: Newton's theory of planetary motion was one of the first modern models. According to Davis and Hersh (1985), the art of modeling is the art of adopting the appropriate strategy. The authors exemplify it by presenting a problem-situation with six different main models – unlike Newton's single model at the time, which was unique –, so we can have several of them, depending on the situation.

According to Aris (1978), a mathematical model is a complete and consistent set of mathematical equations designed to correspond to some other entity, its prototype. The prototype can be a physical, biological, social, psychological or conceptual entity, perhaps even another mathematical model. Nevertheless, for Davis and Hersh, in Aris' (1978) sentence "one can substitute the word equations for structures because one does not always work with a numerical model" (Davis & Hersh, 1985, p. 107). What about a design, a scheme? Are they mathematical models? Not necessarily, but we can use one to start the process of creating a model.

An interesting point discussed by the same authors is about the goals of building a model, defined by them as five: to get answers about what will happen in the physical world; to influence further experimentation or observations; to promote conceptual progress and understanding; to aid the axiomatization of the physical situation; and to encourage mathematics and the art of producing mathematical models.

School Mathematical Modeling (SMM) as a teaching strategy not only serves the development of practical mathematical thinking, of the real world, of the student's daily life, of the proximity of the problem-situation to the school universe, but also develops creativity, language and observation. Numerous researchers debate the meaning of SMM and its use, such as Bassanezi (2009), Weyns et al. (2017); Dunn and Marshman, (2020); Villa-Ochoa (2015); Rehfeldt, Dente and Neide (2017), and others.

To consolidate our theoretical framework, we point out that it is not only students' routine situations that can be addressed by SMM. Gabardo and Cifuentes (2013) state that:

> The mathematical literacy of the citizen, the study of the reality in which this citizen is inserted, and the construction of a critical and reflective attitude towards using mathematics as a tool with political power are placed as reasons for the development of modeling activities in school. However, we wonder if only motivations external to the process of mathematical modeling would justify its approach in mathematics education. Wouldn't there be reasons related to the nature of this process for mathematical modeling to be approached for teaching and learning mathematics? We assume and will show that yes, there is. (Gabardo & Cifuentes, 2013, p. 2)

We are based on the aspect that SMM does not need to emerge only from students' daily problems; that its nature comes from human activities aimed at interpreting various phenomena;

that this approach promotes the emergence of concepts and theories; that the Didactic Transposition of models could be important; and that the action of modeling stimulates the development of mathematical thinking, as pointed out by Dreyfus (1991). The idea of modernizing mathematics teaching in schools by inserting content from new scientific contexts was introduced by what we call New Math. In Brazil, it was discussed and implemented in the 1960s and one of its goals was to teach students more practical and contextualized mathematics in order to eliminate the high level of abstraction and complexity of the "old mathematics", influenced by Euclid's systematized ideas. However, what happened was the opposite: we taught more Mathematics as a Pure Science than as an Applied Science.

Dreyfus (1991) aligns MM with the process of representing. He said:

> The representation process is analogous to the modeling process to some extent but on another level. In modeling, the situation or system is physical, and the model is mathematical; in representation, the object to be represented is the mathematical structure, and the model is a mental structure. Thus, the mental representation is related to the mathematical model as the mathematical model is related to the physical system. Each one is a partial representation of the other. Each of them reflects some (but not all) properties of the other. And it improves the individual's ability to mentally manipulate the system under consideration. (Dreyfus, 1991, p. 31)

Nevertheless, representation does not remain only in the mind, in thought. It is also recorded, as stated by Duval (2011). In this sense, the action of modeling involves several other mental processes (Alvarenga, 2021), depending on the problem-situation, and that is fundamental for the advancement of mathematical thinking.

In terms of scientificity and aiming at obtaining a model that explains the phenomenon, Paty (2001) highlighted Poincaré’s and Einstein’s contributions to physical and mathematical theories. Although there were significant differences between their respective philosophies of scientific knowledge (the former mixed elements of empiricism and conventionalism in his own way, while the latter professed critical realism and rationalism), they shared the conviction that scientific ideas, in the elaboration of physical and mathematical theories, are 'free constructions of thought'. In this sense, they understood that they are not induced logically and univocally, necessarily and compulsorily from the data of experience, and, also, that they are not inscribed in an innate or *a priori* structure of thought. It is in this space of freedom that the idea of scientific creation that leads to discovery comes in (Paty, 2001).

Paty's words refer us directly to the French theory of Didactic Transposition, according to which:

> Some knowledge content that has been defined as knowledge to be transmitted subsequently undergoes a set of adaptive transformations that will make it suitable to occupy a place among the objects of teaching. The 'work' that makes an object of knowledge to be taught into an object of teaching is called didactic transposition. (Chevallard, 1991, p. 39, our translation)

However, what scientificity do we want to transpose? In this research, we focused on transposing the idea of motion in the Euclidean plane, which includes several other human practices: the creation of mosaics; symmetries like translation, reflection, rotation and glide reflection; animation in movies and electronic games, etc. Working with motions of plane or spatial figures has been lost in the curriculum, both in basic and higher education, and here we have evidenced this type of motion.

In this text, we consider Model the result of the modeling process; Modeling is the process of translating a problem-situation into mathematical language; School Mathematical Modeling is the process of translating a situation-problem into mathematical language for teaching and learning.

With this in mind, we present here a part of a piece of research conducted with first-year high school students in a public school in the state of Goiás, Brazil. The investigation focused on analyzing their processes of creating a mathematical model for horizontal translational motion in the plane.

We also point out that there are models that do not meet scientific systematization, but that meet cultural needs. For example, the Sona geometry of the Angolans and Congolese; the Peruvian Quipus; the Mayan Calendars; the Aztec Abacuses; the measuring systems of farmers from various cultures and world regions, and others.

Thus, based on these theoretical ideas and aiming to bring to school an approach to the motion of plane figures in the Euclidean plane, we developed two workshops to observe participants' modeling processes. To explore the idea of motion, they used a type of image, created by them, which we call Auxiliary Manipulative Material (AMM). We emphasize translational displacement among the four isometries: translation, rotation, reflection and glide reflection. They were chosen because they are the simplest to model and appropriate for this school level, besides allowing a broader approach to the content of functions. Isometries are present at various moments in students' lives, although they are not aware of it, and they are part of computer animation, so they are present in the technological and social world of children, teenagers and young people. At the time the research was being conducted, the teacher-researcher was developing the content of functions, so together we decided to approach it differently from traditional textbooks.

In this investigation, we started from the statement that it is possible to develop basic education students' reasoning, especially those between 13 and 14 years old, about the translational movement through SMM. Thus, our guiding questions were: how is this possible? What are the mathematical modeling processes that appear in this development?

In light of this, when preparing the workshops, our focus was to provide a problem-situation to address translational motion for them to model, bearing in mind that translating a dynamic motion into mathematical language deviates from the standards normally found when school mathematics teaching and learning are harnessed. With this focus, the data analysis was directed at students' modeling processes, whether they were correct or wrong, not just at the models they could find.

It is important to mention that the BNCC refers to this content as follows: "Recognizing and constructing figures through translational, rotational and reflectional symmetries", (Brazil, w.d, p. 309; 271). For high school, although it mentions motion and position, the related skill is: (EM13MAT105) "Using notions of isometric transformations (translation, reflection, rotation and their combinations) and homothetic transformations to analyze different human productions such as civil construction, works of art, among others".

Only in higher education, for those taking a degree program in exact sciences, is this content studied in a dynamic way. Thus, mathematics in basic education always seems to be static, and it seems that the phenomena related to motion are only seen in physics, as if this were an attribute belonging only to this scientific field. Visualizing the motion of figures through software can even cause confusion, considering that they are point-to-point and not exactly a figure (Brazil, w. d, p. 525).

Isometries

It is possible to contextualize and study isometries through mathematical tools, and there are several everyday situations where we can observe them: in art, in artistic productions, in harmonious patterns, in nature and in chemical and physical compounds.

The ideas about symmetrical patterns have three characteristics: the unit to be repeated using motion in the plane, the repetition of this unit, and the organization of this system, which standardizes such motions and repetitions. These three features form a geometrically patterned figure with a combination of colors. These motions are widely used in computer graphics, where, for example, an image of one or several polygons is shifted harmonically, generating the visualization of the motion of the figure.

Isometry is a transformation in the plane, that is, transformations in the points belonging to the plane, where the distances and angles between points are preserved. There are 4 types of isometries: translation, reflection, rotation, and glide reflection.

When preparing the workshop, it was fundamental for us to create opportunities to highlight the importance of motion, specifically the translational one, because it is the simplest of all and can be observed and studied from elementary school to university courses, as stated in *Groups and Symmetry – A guide to discovering mathematics* (Farmer, 1996).

With a specific approach to basic education, by avoiding the use of extremely formal language and making the language more accessible and easier to understand, we have prepared the following presentation for the content of isometries according to Veloso (2012).

Translation is the displacement of points in the plane following the direction of any $\overrightarrow{AB}$ line (Fig. 2). When we apply several translations to the object, in the same direction, the result is still the translation of the object, which can be obtained from a single motion. This isometry maintains the orientation of the plane but has no fixed point. Therefore, as displacement occurs, no point remains in the same place.

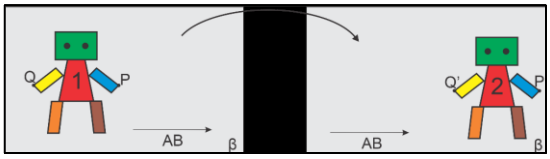


Figure 2.

Example of the translational motion of the line $\overrightarrow{AB}$ in the horizontal direction (prepared by the authors using CorelDraw software,2021)

About the mathematical language, we have the following considerations: the geometric transformation T corresponds to every point P in the plane, the point P', and to every point Q in the plane, the point Q'. Therefore, there is an isometry T, such that $P' = T(P)$ and $Q' = T(Q)$. Also, the distance $d(Q,P) = d(Q',P')$, or $\overline{PQ} \equiv \overline{P'Q'}$. Consequently, the transformation that links puppet 1 to puppet 2 is a translation, a type of isometry which preserves distances.

Materials and Methods

To approach School Mathematical Modeling (SMM) and deal with translational isometric motion, we developed two workshops. We employed the methodological approach of teaching through SMM, not only to contextualize the content for teaching purposes but also to observe the models, model constructions/representations, actions and the modelling processes used by the participants.

The main objective of the workshops was to build a mathematical model to represent horizontal translational motion in mathematical language, which also allowed for data collection.

Workshops 1 and 2 were developed in a public state school in Goiânia, the capital of Goiás state, in an outlying neighborhood in the northwestern part of the city, with a mostly low-income population, but with different participants and contexts. Such workshops were planned with the goal of encouraging students to model motion in the Euclidean plane. In general, they understand mathematics in a static way. Translation in the plane is a simpler form of symmetry and motion that involves knowledge of ordered pairs, location, rectilinear motion, functions and others.

Our main goal was to observe how a group of first-year high school students from a public school modeled motion in the Euclidean plane with support from SMM. This was where our research emerged from. The workshops were planned to take place in two stages in the same school, but with two different student groups and at different times, 3 months apart. One of them happened face to face, and the other took place online.

First data collection (workshop 1)

The design of Workshop 1 was based on pre-defined activities planned using PowerPoint, with figures that moved rectilinearly in various directions (see Fig. 3). After the students observed the motions, they repeated the same motions through a paper figure created by them. Then, the teacher defined translational dynamic motions and, once again using a paper figure, which was moved by the students. Later, they described the motion by using mathematical language or not, with or without help from the teacher-researcher.

Workshop 1 was designed to take place face-to-face and was done this way, so we evaluated its potentialities and weaknesses to make changes, if necessary, before a subsequent data collection, which occurred in Workshop 2. Fourteen teenagers participated, and we analyzed all the answers collected (first stage of the research). Next, it was possible to list the initial categories related to the process of building the mathematical models and identify some necessary changes to improve the activity for Workshop 2.

In this stage, the contents of relationships, variables, symbology and mathematical operations to represent displacement were discussed, that is, the use of mathematical language to represent a dynamic phenomenon – translation. To do so, the sequence of activities encompassed three situations related to motion: vertical, diagonal, and horizontal (this one to be performed outside the classroom, individually), each containing five questions to help in the development of mathematical thinking and to direct students' actions. The same happened in workshop 2. The questions were proposed by the teacher-researcher orally. Finally, we applied a questionnaire on Google Forms.

Final data collection (Workshop 2)

The approach and materials were reworked, and the focus changed to an online presentation, which made use of Power Point, too, just like Workshop 1, to represent motion in the Cartesian plane (see Fig.4). To do it, the lessons were guided by a sequence of previously designed in-class activities and questions based on the experience we had with Workshop 1 and sent to the students. The main difference was the availability of written material, to serve as a guide.

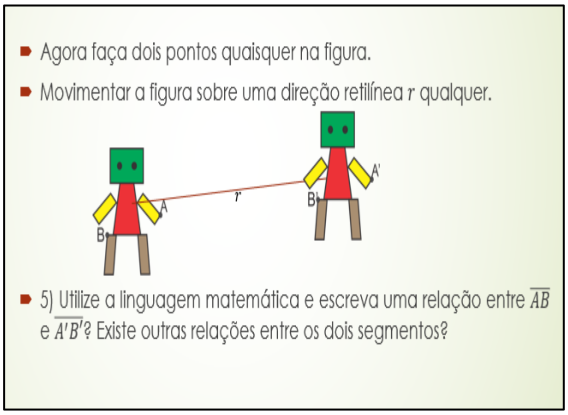
A sequence of written activities was necessary because the first experience showed that they did not answer some of the activities; probably they did not remember the questions that were asked only orally, so we did it in writing this time. We think this may have happened for several reasons: too much information; the activity was very different from the ones usually presented to them, which are generally based on a traditional teaching and learning sequence (definition, examples, and exercises, often with the same ideas and, therefore, repetitive); the idea of motion was characterized by most as a non-mathematical situation, when we asked them whether the activity was a type of mathematics situation or not. Our hypotheses were based on the answers to the questionnaire applied after Workshop 1, which was about their experiences with mathematics: what mathematics was, how they learned mathematics, where mathematics could be seen in daily life, and so on.

The implementation happened in stages. First, we contextualized the theme. Isometric patterns found in nature, architecture and works of art were presented, and we used a YouTube video.

In the next stage, the activities were related to producing displacements in the Euclidean plane with the support of the AMM, and students did them at home. As in Workshop 1, they used an AMM, and each student made the paper figure they wanted: a puppet, a circle, a polygon, a piece of paper, or others.

They should mark two points in the figure and move it randomly, at first. Then we debated this motion, raising questions about some properties referring to translation (Cf. Fig.

4) and how to write, to record this motion. Next, we asked them to record this motion using mathematical language; they could create it.



-Now make any two points in the figure

-Move the figure in any straight direction

-Use mathematical language to write a relationship between $\overline{AB}$ and $\overline{A'B'}$.

Are there other relationships between the two segments?

Figure 3.

Example of the original slide used in this step (our own production - PowerPoint slide used in workshops 1 and 2, 2021)

In the last stage, the teacher-researcher approached vertical motion with the information needed to build a model with the students as a team. Finally, a homework activity about horizontal motion was proposed, and it should be done individually. They received a script (Fig. 4) with a sequence of activities to be performed.

The first situation of the script, which should be performed individually and outside the classroom, involved analyzing the motion of the AMM, centered at the point $(0, 0)$ with a horizontal displacement of 4 units along the *c*-axis in the positive direction (Fig. 4, the original script) in the Euclidean plane. In the proposed situations, the axes were named ***h*** and ***c*** to show that we can use different mathematical language.

Have the following materials on hand:
-a picture you like (it could even be a small piece of paper)
-a colored pen

Horizontal movement

-1st stage

Draw the Cartesian plane on your sheet of paper

-2nd stage

Mark a point in the center of your figure.

-3rd stage

Place the figure on the Cartesian plane with its center (0,0) and move it 4 units horizontally to the right

	Componente Curricular:	Professor (a):	Edson Ferreira	
	Matemática	Aluno (a):		
		Série/ turma:		Valor: 10,0
		Data:	___de ___________de 2020	Nota:

Tenha em mãos os seguintes materiais:

- uma figura de sua escolha (pode até ser um pedaço pequeno de papel);
- uma folha de papel para registar as suas respostas;
- caneta com cor de acordo com a sua preferência.

Movimento na Horizontal

1° passo

Desenhar o plano cartesiano $\mathbb{R}^2$ na sua folha de papel.

2° passo

Pegar a sua figura e marcar um ponto no centro dela.

3° passo

Colocar a figura sobre o plano cartesiano com o seu centro no ponto (0, 0) e realizar um deslocamento de 4 unidades na horizontal para a direita.

Figure 4.

Part of the script that the students used in the out-of-the-classroom activity. (created by the authors, 2021).

There were 15 questions, separated into five groups according to their characteristics: group 1 – questions concerning the line on which the horizontal motion was performed; group 2 – questions concerning the new position of the center of the AMM after the motion; group 3 – description of the motion in mathematical language; group 4 – generalization to any point and displacement in any direction; and group 5 – description of the motion in students’ mother tongue.

The questions were the same, but the situations were different: (*i*) place the figure on the Cartesian plane, with its center at (0,0), and move it 4 units to the right horizontally; (*ii*) place the figure on the Cartesian plane, with its center at (4,0), and move it k units to the right horizontally; and (*iii*) place the figure on the Cartesian plane, center it at any point (a, b) belonging to the figure and move it k units in the horizontal direction, to the right. Here we will present some results from group 3: describing motion in mathematical language, with answers analyzed for situations *i*), *ii*) and *iii*), as mentioned in this paragraph.

At the end, a questionnaire was applied using Google Forms in order to evaluate students’ participation and the development of the activity, to build each participant’s profile and to understand the context in which the investigation was carried out. The entire methodological process of the research is represented in figure 5.

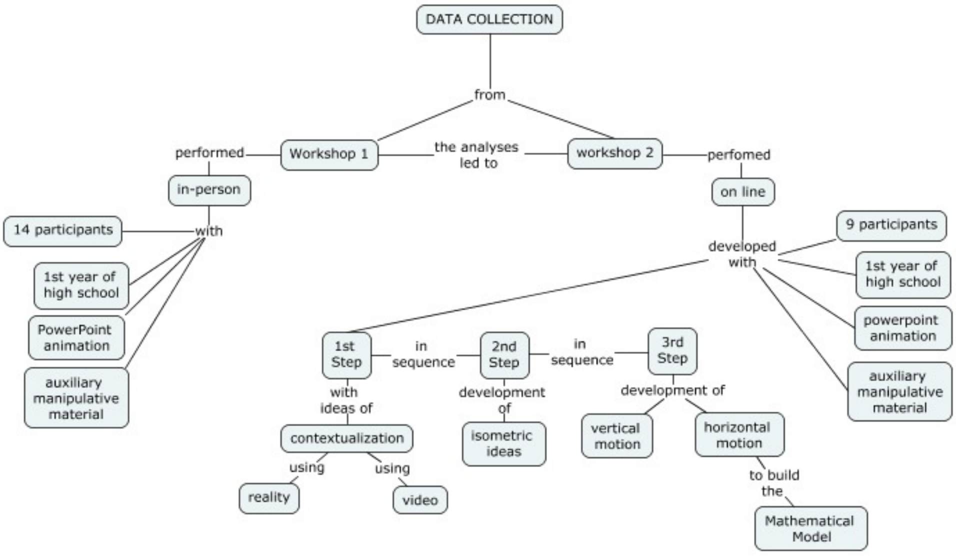


Figure 5

Data collection routines (created by the authors with CmapTools, 2023)

Data analysis methodology

We conducted the data collection with first-year high school students in order to analyze model construction processes (see Fig. 5). The results were separated into five groups, but for the discussion in this paper, we selected the third one: description of motion in mathematical language for the three previously presented situations: i), ii) and iii). We consider that group 3 (Fig. 6) represents well the reflections presented in the introduction of this paper: what a mathematical model can be, the mathematical records and representations. It is important to remember that the script encompassed questions referring to the mathematical description of horizontal motion with values different from those discussed in the classroom, where we also discussed and worked with a mathematical model for vertical translational motion.

Groups	Question about horizontal displacement
Group 3	How can we write this motion mathematically, and how can we represent this motion using mathematical language?

Figure 6.

Group 3 (created by the authors, research results)

Based on the initial analyses of the data, we organized the sample related to Group 3 into five categories that emerged according to solutions patterns: geometric-graphic approach; arithmetic approach; algebraic approach; written native language; and written mathematical language. We have only presented two of them in this article. The geometric-graphic approach is characterized by the spatiality of the displacement of the AMM along a line, without and with the Cartesian plane; the arithmetic one only involves numerical and spatially defined numbers and operations; the algebraic one involves generalization; and the other two are the ways through which students express themselves by mathematical or native language. Somehow these approaches can overlap.

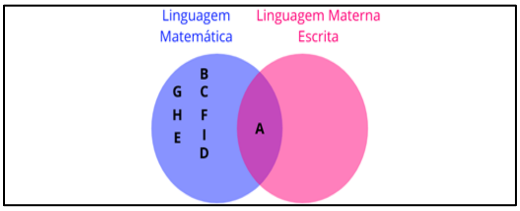
The analyses were conducted qualitatively, according to the participatory conception, where participatory action is dialectical and focuses on producing changes in practice, suggesting an action agenda (Creswell, 2010) for changing the conception of mathematics teaching and learning, not as mere knowledge reproduction, but as a mathematical creation. As for the objectives, the research is also characterized as exploratory, descriptive and explanatory (Vilelas, 2017; Sampieri et al. 2013).

Some results

We observed that most of the students had difficulties interpreting, solving problems and doing simple arithmetic calculations. In response to the questionnaires administered at the end of the workshop, all of them said that they go to school to learn, and most of them also go to school to get a good job. The participants believed that they never used mathematics to describe any situation and associated it mainly with numerical calculations, numbers, a set of rules, letters and symbols. We emphasize that they did not realize, or did not remember, the use of these mathematical tools in building the model for horizontal motion. It is assumed that they did not realize that they were doing and creating mathematics all the time.

These subdivisions emerged as we studied the recorded data and the ideas proposed by Sierpinka (2000), Lorenzato (2008) and Lins and Gimenez (1997).

Most of them used mathematical language, and in only one case did we notice the use of written first language as well (Fig.7).



Blue: Mathematical Language

Figure 7.

Data from group 3's answer. The letters indicate the respondents (own elaboration in Canva software based on students' solutions, 2022)

In this set of answers, participant A used written native language where he was asked to describe horizontal translational motion mathematically. He has possibly decided to use his native language because it presents the register closest to his understanding, as stated by Sierpinska (2000).

In most cases, the participants used the addition of ordered pairs (center of the figure + displacement): B, C, D, E, F, H and I. The same fact can be observed in participant G's answer (Fig.8). He modeled the motions horizontally in an expected way.

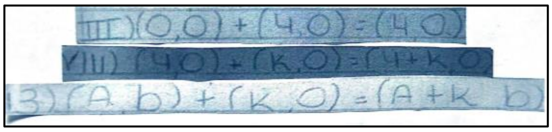


Figure 8.

G's answer to the three situations in group 3: a), b) and c)

(participant's solution, 2022)

However, participant I may have made a new construction to represent the ordered pairs or reproduced the idea of vertical motion once again (Fig 9). We noticed that the unit of displacement used by him refers to vertical motion, which is different from what the activity proposed (Fig.6). This can be an oversight, lack of attention or because he did not understand the directions of the motions. He also used some nonsensical equalities, like (0,0) = (0,4), and additions without following the logical rule: the first element of one pair with the first element of another. His understanding appeared confused.

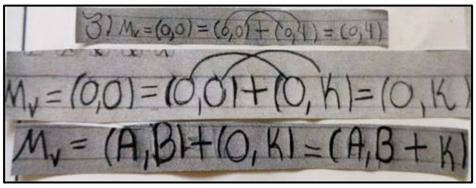


Figure 9.

Participant I's answer to the proposed questions (Fig. 6) in group 3

(participant's solution, 2022)

We also detected the need to improve written records. Let us observe respondent C. He incorrectly performed the addition of the ordered pairs. In fact, in the 2^{nd} line of Figure 11, he used a comma instead of the addition sign, because it should be (4,0) + (k,0) = (4+k, 0), instead of (4, k,0) (Fig. 10).

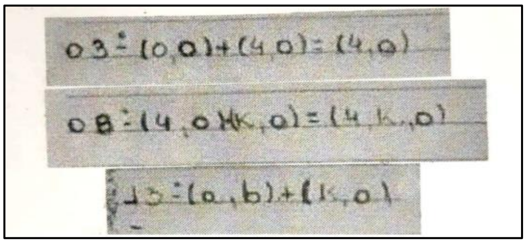


Figure 10.

Participant C's answer to the question in group 3 (Fig.6) (participant's solution),2022.

This participant also stated, in the questionnaire, that he likes math, but that he has difficulty doing calculations, a fact that was evidenced in the activity. In this case, it was related to the variable *k*. He claims not having understood very well the representation of the addition of two points in the plane with variables.

Respondent H also had difficulties with his record, as he used the equality between pairs for different pairs (Fig. 11). The concept of the solution was to add ordered pairs, but he used the equal sign.

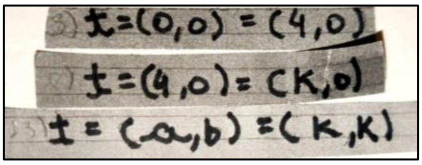


Figure 11.

Participant H's answer to group 3 (participant's solution), 2022.

In this investigation, we recognize that introducing the addition of points, as ordered pairs, may be a novelty – especially for representing motions – even though in middle school students had already been exposed to points and ordered pairs in the Cartesian system.

In the proposed problem-situation, it was important to use the three types of thinking: synthetic-geometric, analytic-arithmetic and analytic-structural (Sierpinska, 2000) to analyze the situation in graphic and geometric terms and to describe it algebraically and numerically. Thus, we consider that participants B, C, D, E, F, G, H and I used algebraic, arithmetic and geometric approaches to answer the questions.

Participant A even tried to use the coordinates of the Euclidean plane to answer the questions in the problem-situation (see Figure 12), but he still used written native language in most of his answers.

-3rd The motion brings zero to the center

-8th a motion to the right

-13th because this motion is being A and b (0,1)

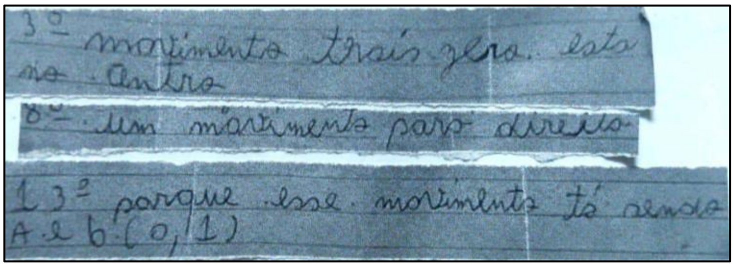


Figure 12.

Participant A's answer to the questions in group 3 (participant's solution), 2022.

Participant A used written native language as a resource to answer the questions, and this occurred in most situations. In addition, in some cases, we also noticed the need to improve written records, both arithmetic and algebraic. This may be related to the fact that, in school tradition, the teaching of algebra usually starts right after arithmetic, often in a disconnected way. Perhaps through a joint study of these two fields, students may understand the relationships between them, such as the use of letters to express generalization. Otherwise, mathematical confusion will appear in both contexts (Castro, 2012).

In this sense, we emphasize the need to stimulate mathematical skills to work with models, to develop algebraic, arithmetic and geometric thinking, and to describe situations and phenomena, as these are skills prescribed for high school by Base Nacional Comum Curricular (BNCC) (Brazil, 2017).

The prevailing mathematics language in this case was geometric (see fig. 13) for one student. **Overall,** the 3 types of languages were used equally. Arithmetic was considered because as some displacement was requested numerically, it was necessary to properly use the units in the Euclidean space, in addition to the numerical operations involved.

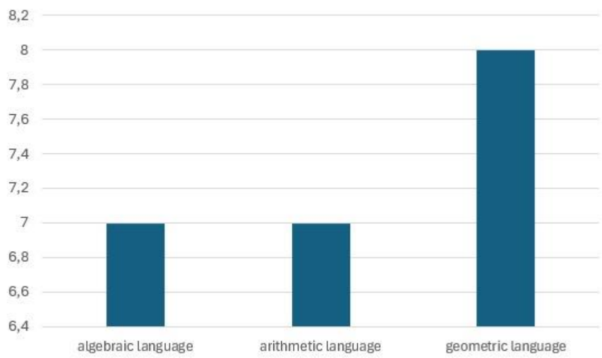


Figure 13.

Types and quantitative use of mathematical language employed by participants, 2022.

We further highlight the importance of implementing motion in mathematical ideas instead of just approaching it statically, as evidenced by Veloso (2012). Teaching based on repetition, in a mechanized way, does not provide an understanding of the usefulness of mathematics (Oliveira & Laudares, 2015). This is observed when most of the participants stated, in the questionnaires applied after the workshop, that the use of mathematics is mainly related to numerical calculations. However, it seems that many did not realize that they built a mathematical model during the activity, and that they are able to create their own mathematical languages and representations to express that motion.

Thus, the participants in these workshops did much more than just performing some numerical and algebraic calculations. They observed, reasoned and modeled a situation that is part of the world around them, the translational motions and symmetries.

Conclusion

Our objective was to propose a situation completely different from what participants are usually faced with, so that they could exercise their ability to experience mathematical modeling in a context where translational motion was studied and formalized. We were curious to know what the modeling process used by them would be like. We understood that the students had many difficulties in developing a mathematical model to express motions in the plane by using parameters and numbers to show displacement and to understand the notion of direction and sense through a straight line. We noticed that experiencing the possibility to do mathematics, to represent and mathematically record translational motion made them aware of their abilities to mathematize phenomena. For us, this was the most important thing.

We are convinced that it is necessary to provide modeling situations beyond the ordinary ones usually found in books and on the internet, although the latter are useful, especially when it comes to a critical mathematical modeling approach. We must believe in our students' ability to mathematize "scientific" situations proposed by us and, consequently, they may feel they are co-participants in a mathematical culture, in a way of doing and creating mathematics. Motion phenomena are overlooked by SMM, and it is important to study them.

In this research, we started from the statement that it is possible to develop basic education students’ reasoning, specifically those between 13 and 14 years old, about translational motion through SMM. After the two workshops and the data analyzed, we observed that despite teenagers’ difficulties with mathematically modeling dynamic situations, it was possible to engage them in this type of situation. Nevertheless, most of the models produced had flaws, the main one being the lack of rigor in mathematical language. Moreover, even though they knew Cartesian Systems and ordered pairs, they had difficulties in interpreting motion in this system. The manipulation of a concrete figure is essential for them to feel motion and directions.

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