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An epistemological reference model for the limit[1]

Un modelo epistemológico de referencia para el límite

Um modelo epistemológico de referência para o conceito de limite

Un modèle épistémologique de référence pour la limite

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Résumé

Dans ce travail, nous présentons de façon détaillée un modèle épistémologique de référence (MER) sur la notion de limite, en s'appuyant sur la théorie anthropologique du didactique et la théorie des champs conceptuels. La construction d´un MER (en s´appuyant sur la dimension épistémologique de la notion de limite), nous a permis de mettre cet objet mathématique en relation avec les notions qui permettent de lui donner du sens et de diriger nos analyses économico-institutionnelle et écologique. Ces analyses nous ont permis de mettre en évidence le modèle épistémologique de référence dominant (MED) dans les différentes institutions (les programmes et les manuels scolaires, etc.) analysées. Celui-ci a permis de montrer quelle est la signification donnée à la notion de limite dans ces différentes institutions. Pour la phase expérimentale (que nous ne décrivons pas ici), um modèle épistémologique alternatif de référence (MEAR) a été construit à partir de la confrontation entre le MED et le MER, pour

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réduire l’écart entre le savoir savant et le savoir enseigné. Ce travail nous a permis d’actualiser les connaissances connexes à la définition formelle de la limite telles que les activités graphiques, la résolution des inéquations avec valeur absolue, les intervalles (notion de voisinage), les quantificateurs universel et existentiel, l’introduction à la logique des prédicats.

Mots-clés : Concept de limite, Modèle épistémologique de référence, Organisation mathématique.

Abstract

In this work, we present in detail an epistemological reference model on the notion of limit, based on the anthropological theory of didactics and the theory of conceptual fields. The construction of a ERM (based on the epistemological dimension of the notion of limit, allowed us to put this mathematical object in relation to the notions which make it possible to give it meaning and to direct our economic-institutional and ecological analyzes. These analyzes allowed us to highlight the dominant epistemological reference model (DERM) in the different institutions (curricula and textbooks, etc.) analyzed. This made it possible to show what meaning is given to the notion of limit in these different institutions. For the experimental phase (which we do not describe here), an Alternative Epistemological Reference Model (AERM) was built from the confrontation between the (DERM and the ERM, to reduce the gap between scholarly knowledge and taught knowledge. This work allowed us to update knowledge related to the formal definition of the limit such as graphic activities, the resolution of inequalities with absolute value, intervals (notion of neighborhood), universal and existential quantifiers, introduction to predicate logic.

Keywords: Concept of limit, Epistemological reference model, Mathematical organization.

Resumen

En este trabajo presentamos en detalle un modelo epistemológico de referencia sobre la noción de límite, basado en la teoría antropológica de lo didáctico y la teoría de los campos conceptuales. La construcción de un MER (basado en la dimensión epistemológica de la noción de límite), nos permitió poner este objeto matemático en relación con las nociones que permiten darle significado y orientar nuestros análisis económico-institucionales y ecológicos. Estos análisis permitieron nos permitió resaltar el modelo de referencia epistemológico dominante (MED) en las diferentes instituciones (currículos y libros de texto, etc.) analizados, lo que permitió mostrar qué significado se le da a la noción de límite en estas diferentes instituciones para la fase experimental. (que no describimos aquí), se construyó un modelo de referencia epistemológico alternativo (MEAR) a partir de la confrontación entre el MED y el MER, para reducir la brecha entre el conocimiento académico y el conocimiento enseñado. Este trabajo permitió actualizar conocimientos relacionados con el conocimiento formal. definición del límite como actividades gráficas, resolución de desigualdades con valor absoluto, intervalos (noción de vecindad), cuantificadores universales y existenciales, introducción a la lógica de predicados.

Palabras clave: Concepto de límite, Modelo epistemológico de referencia, Organización matemática.

Resumo

Neste trabalho apresentamos detalhadamente um modelo de referência epistemológico sobre a noção de limite, baseado na teoria antropológica do didático e na teoria dos campos conceituais. A construção de um MER (baseado na dimensão epistemológica da noção de limite, permitiu-nos colocar este objeto matemático em relação com as noções que permitem dar-lhe sentido e orientar as nossas análises económico-institucionais e ecológicas. Essas análises nos permitiram destacar o modelo epistemológico de referência dominante (MED) nas diferentes instituições (currículos e livros didáticos etc.) analisadas. Isso permitiu mostrar qual o significado que é dado à noção de limite nessas diferentes instituições. Para a fase experimental (que não descrevemos aqui), um modelo epistemológico alternativo de referência (MEAR) foi construído a partir do confronto entre o MED e o MER, para reduzir a lacuna entre o conhecimento acadêmico e o conhecimento ensinado. Esse trabalho nos permitiu atualizar o conhecimento relacionado à definição formal do limite, tais como as atividades gráficas, resolução de desigualdades com valor absoluto, intervalos (noção de vizinhança), quantificadores universais e existenciais, introdução à lógica de predicados.

Palavras-chaves: Conceito de limite, Modelo epistemológico de referência, Organização matemática.

A reference epistemological model for the concept of limit

The notion of limit has been the subject of several works in the didactics of mathematics, including those by Cornu (1983), Sierpinska (1985), Artigue (1996a, 1996b, 1989), Bkouche (1996, 1997), Job (2011), Hitt (2006) and Lecorre (2016), among others. These works highlighted the epistemological obstacles and the need for a formal definition in teaching the notion of limit.

The polysemic nature of the notion of function does not help students understand the notion of limit. One of these meanings is that the limit is the end; there is nothing on the other fside. It is a static notion for students, which Cornu reinforces when he states that:

> The notion of a limit is, above all, a static notion, that is, a prohibition or an impossibility of crossing something. This limit can be in time or space (we stopped...). It may also include the idea that it is difficult to approach the limit, much less reach it. The expression "tend to" generally has a more dynamic meaning. (Cornu, 1983, p. 122)

According to Cornu (1983), the limit appears as an insurmountable border that cannot be reached; an insurmountable border that can be reached by others; a point that we approach without reaching it; a point that we approach until reaching it; an upper limit, a lower limit; a minimum, a maximum, extremities, borders.

For our purposes, we are interested in the epistemological dimension of the notion of limit. We rely on Boch and Gascón (2005), who specify that a mathematical organization (MO) is a practical praxeological method of the mathematics curriculum found in textbooks and programs. The identification of these MOs, whose use involves the characterization of types of institutional tasks, is a (re)construction based on the analysis of textbooks and curricula. From the point of view of didactic transposition, two postulates are proposed:

1. We cannot understand or explain a learned MO without understanding and explaining the MO of the previous stages;
2. The unit of analysis of the didactic process must contain a didactic organization that allows the application (establishment) of at least one local MO.

We must add an epistemological model of "praxeological reference," which Chaachoua and Bittar (2019) call the praxeological reference model (PRM), to characterize and analyze the praxeologies to be employed (Figure 1).

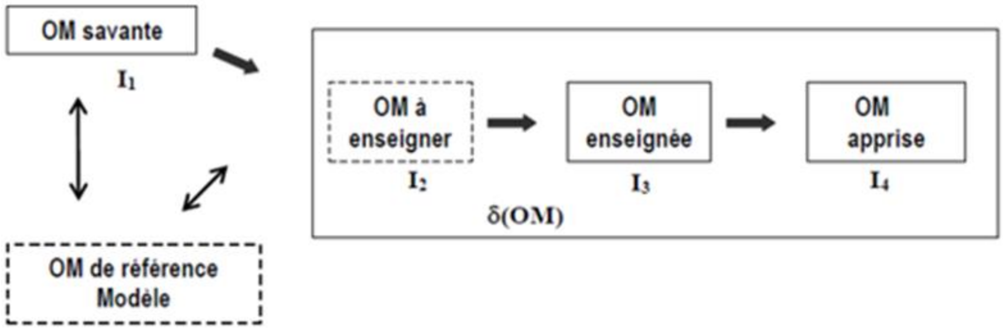

Figure 1[5].

Units of analysis of didactic process (Boch & Gascón, 2005, p. 11)

The MO to teach constitutes a praxeological model of the mathematics curriculum. The empirical basis for the development of this model is found in curriculum documents (official programs) and textbooks. Its influence on the didactic organization associated with the observed mathematical organization δ (MO) is central, although neither the teacher nor the institution has this model explicitly; only praxeological materials are roughly articulated with each other. Boch and Gascón state that:

> [...] This influence cannot be correctly interpreted if we do not have an epistemological point of view. This point of view is provided by an MO of reference whose description generally relies on the δ (MO) that legitimizes the teaching process. The MO of reference is the one the researcher considers in his/her analysis. It does not necessarily coincide with the academic MOs from which it derives (because it includes them in the analysis), but it is formulated in very similar terms. The MO of reference is the one that the researcher puts to the test of contingency and that, for this reason, undergoes constant reformulation. (Boch & Gascón, 2005, p. 11)

The development of a praxeological reference model has become an essential step in most of the work developed within the framework of the ATD (anthropological theory of the didactic). The PRM is, in itself, a didactic result and a tool for conducting didactic analyses.

Chevallard (2007) states that:

> What the theory of didactic transposition says, in other words, is that there is no "privileged point of reference" from which one can observe, analyze, and judge the world of knowledge and, more broadly, of praxeologies. "Academic knowledge" is a function, not a substance, from which the didactician must expressly distance himself/herself. It follows that the work of the didactician is, each time, to construct a never definitive point of reference from which to analyze the praxeologies whose dissemination he/she is studying. (Chevallard, 2007, p. 12)

[5] **Translation of the figure**: Academic MO; MO of reference Model; MO to teach; MO taught; MO learned]

Finally, since scientific concepts are never alone and cannot be completely isolated, it is necessary to take into account the relationships between the different concepts involved in any situation linked to the conceptual field at stake. Vergnaud (1990) defines a "conceptual field" as an area of problems or problem situations whose treatment involves concepts and procedures of various kinds in close connection. This notion of makes it possible to:

- replace a concept in a set of neighboring concepts;
- specify the classes of problems in which these concepts are used as solution tools (i.e., specify their meanings).

Therefore, let us relate the notion of a limit to its conceptual field and then specify the task types. To do this, we built a conceptual map highlighting some of the mathematical objects that feed the concept of a limit and the mathematical notions constructed with the help of the limit.

To construct this conceptual map, we based ourselves on the results of the study of the historical-epistemological dimension carried out by Doumbia (2020).

Historical study of the notion of a limit

Based on works by Cornu (1983), Sierpinska (1985), Artigue (1996a, 1996b), Bkouche (1997), Job (2011), and, mainly, Doumbia (2020), we will foreground the obstacles related to the notion of limit, the problems that led to the emergence of the notion of a limit, and some contradictory debates on the definition of the notion of a limit. We will also present an analysis of the formal definition of ℓ, which is the limit of a numerical function of a real variable at a finite point a.

Among the historical examples, we can mention the paradoxes of Zeno of Elea, the notion of infinity (potential infinity and real infinity, and infinitely large and infinitely small), the method of double reduction by absurdity (method of exhaustion), sequences and series, and the geometric problems: the calculation of area and volume, indivisibles, Newton's method of fluxions (calculation of derivatives).

In antiquity, Zeno of Elea's paradoxes brought the question of whether an infinite sum could be finite. One of the paradoxes is: "It is not possible to traverse an infinite number of points in a finite time." We can also see the connection with time. Back then, no one could explain those paradoxes. Aristotle considered them as lines divided into points, in other words, in the nature of geometric objects (a line cannot be considered a juxtaposition of points).

The notion of infinity is also present but in a potential form. According to Aristotle, a quantity can be halved indefinitely: the unlimited exists potentially but is never achieved.

Around 400 BC, the leading mathematical problems revolved around geometry: calculating lengths and measuring areas. For example, Hippocrates of Chios (430 BC) wanted to demonstrate that the ratio between the measures of the areas of two circles was equal to the ratio between the squares of the measures of their diameters.

To do this, he inscribes similar regular polygons in the two circles and, by indefinitely increasing the number of sides, superimposes the two circles. At each stage, the ratio of the areas of the inscribed polygons is equal to the ratio of the squares of the radii of the circles: the result is that "in the limit," the same applies to the areas of the circles. This passage to the limit, not very clearly explained, was clarified a century later by Eudoxus of Cnidus (408-355 BC) method of exhaustion, whose principle is described as follows: "If two unequal magnitudes are proposed; if we subtract from the largest a part greater than its half; if we subtract from the remainder a part greater than its half; and if we do this again and again, a certain magnitude that is smaller than the smallest of the magnitudes proposed will remain." In other words, by successive divisions by 2, we can make a quantity as small as we want, which is the principle of exhaustion. That means that for any $\varepsilon > 0$, there is a regular polygon inscribed in the circle whose area approximates the area of the circle over an interval of ε.

Although the method of exhaustion is quite close to the notion of a limit in certain aspects, we cannot say that the Greeks possessed the concept of a limit. It is, above all, a geometric method that allows us to demonstrate results without resorting to infinity. We should also note that the method of exhaustion applies to geometric quantities, not numbers.

> The unifying concept of a limit of numbers is not present, there are no general results, and the method is applied to each example without having actually identified the tool that would allow the problems to be solved more generally. (Ovaert *apud* Cornu, 1983, p. 42)

The problem of dividing geometric figures into elementary ("indivisible") figures, the calculation of sums of series, and the notion of function were very uncertain: Fermat solved problems of minimum and maximum, Newton's method of fluxions, with the notions of "first and last ratios." The notions of "first ratio" and "ultimate ratio" are paramount in the history of the notion of limit. They highlight the meaning of epsilon and delta and mark the difference between limit and continuity, limit and image. They gave rise to heated debates.

Definition of the notion of a limit

Euler (1707–1783) did much to free calculus from its geometrical support. He did not work with quantities but with functions. The study of functions was one of the factors in the development of the notion of a limit. In Euler's time, the notion of a function was not yet evident; it was essentially a matter of algebraic expressions. Euler developed functions in series by introducing several orders of infinitesimals *dx, (dx)*^{2}, which allowed him to achieve several results about numerical series. It is important to note that Euler was not only interested in the qualitative aspect (convergence or divergence) but, above all, in the quantitative aspect (speed of convergence or even divergence).

D'Alembert (1717-1783) considered the problem of the infinitely small and the infinitely large to be very sensitive. For him, the infinitely small was a matter of "metaphysics" and had no place in mathematical reasoning: it resulted from reasoning involving quantities that "vanish into thin air." He stated that "a quantity is either something or nothing; if it is something, it has not yet disappeared; if it is nothing, it has disappeared completely. The assumption of an intermediate state between these two is a chimera" (D'Alembert, 1805, p. 353).

D'Alembert, therefore, set out to identify the notion of a limit in this "metaphysics" of the infinitely small and to give it a precise definition:

> A quantity is said to be the limit of another quantity when the latter can approach the former more than it can approach a given quantity, however small it may be, without the quantity it approaches being able to exceed the quantity it is approaching; in such a way that the difference between that quantity and its limit is absolutely unattributable. (La Chapelle, D'Alembert, 1751, p. 542)

He insisted that a quantity never becomes equal to its limit: he took the circle, the limit of inscribed polygons, and the sum of a geometric progression as examples. He defined the "sum of a sequence" (that is, of a series) as "the limit of its various terms, that is, a quantity that can be approximated as much as desired, always taking into account an ever-increasing number of terms." He was very reticent about divergent series.

The notion of a limit introduced by D'Alembert contrasts it with that of infinitely small for the sake of mathematical rigor. Sierpinska (1985) points out that the notion of a function does not appear in this definition; it is not about numbers, but about magnitudes and, finally, the expressions approximation, difference between magnitudes, and unattributable are also not defined.

Lagrange (1736-1813) was reticent about the notions of limits and infinitesimals. About the limit, he wrote:

> The type of metaphysics that one is obliged to employ is, if not contrary, at least foreign to the spirit of analysis, which must have no other metaphysics than that which consists of the first principles and the first fundamental operations of calculus." (Cornu, 1982, p. 52)

He wanted to reduce all analysis to algebraic calculation, and, to this end, he worked on developing series functions. Lagrange, who rejected the notion of limit, was one of the most important architects of the transition to the numerical domain, which led to the unification of the concept of limit. This transition also led to the establishment of a static definition of the limit. For Lagrange, the limit did not involve infinity; he developed the practice of majorizing and minorizing, especially to control the remainder of a series. For him, series were, above all, formal algebraic objects. When the numbers are replaced by indeterminates, the problem of convergence arises and hence the need to increase the remainder: after establishing the formula: $f(x) = f(x - xz) + xzf'(x - xz) + \frac{x^2z^2}{2}f''(x - xz) + etc.$, he takes care to calculate the remainder for the case where "you want to stop at your first, second, third, etc. term," and obtains, for example, "$fx = f + xf' + \frac{x^2}{2}f'' + x^3R,\ R$ being a function of z that vanishes when z=0". After working in the numerical domain, he applied his results to geometry and mechanics.

Cauchy (1789-1857) gave the notion of a limit its definitive place, reorganizing the analysis around it. He considered the concept of a limit fundamental to analysis and, at the beginning of his course at the École Rurale Polytechnique, he defined the limit and introduced operations on limits:

> When the values successively assigned to the same variable approach indefinitely a fixed value so as to differ from it as little as possible, the fixed value is called the limit of all the others. Thus, for example, an irrational number is the limit of several fractions that provide increasingly approximate values. In geometry, the area of a circle is the limit to which the areas of the inscribed polygons converge, while the number of its sides increases more and more. (Cornu, 1983, p. 53)

He based all analysis on limits: continuity, derivatives, and integrals. However, Cauchy was still influenced by the infinitely small numbers and used them. The notion of a limit had not yet been definitively refined; it was still confused with the notion of accumulation point: lim(sen($\frac{1}{x}$) has, for Cauchy, an infinite number of values between -1 and +1. There is also a misperception between the study of continuity and the study of the limit at a point and between the limit and the image.

The functional relationship already appears when numerical values are assigned. However, these are consecutive values, which led Sierpinska (1985) to say that this definition referred only to numerical sequences. She points out that quantifiers are implicit in this definition, as are indefinite expressions such as "approaching indefinitely." She also points out that Cauchy did not indicate the dependence between the neighborhood of the point at which the limit is calculated and the neighborhood of the point that is the limit. She draws attention to the fact that, in natural language, we do not pay much attention to word order and the resulting subtle differences.

Although Cauchy's ideas were not immediately adopted in France, the notion of limit developed rapidly in Germany, and Weierstrass (1816-1897) gave a "static" formulation to it in the form we know today: If given any positive real number ε, there is a number η_0 such that for $0<\eta<\eta_0$, the difference $f\,(x_0 \pm \eta) - \mathcal{L}$ is smaller than ε in absolute value, while $\mathcal{L}$ is the limit of *f(x)* for $x = x_0$. According to Weierstrass, there is nothing to add, nothing to remove, or still $(\forall \varepsilon > 0\ \exists \delta > 0, \forall x \in Df\ (0 < |x - a| < \delta \Rightarrow |f(x) - \ell| < \varepsilon)$.

Epistemological study of the notion of a limit

In this study, we present the epistemological obstacles to the notion of a limit, an analysis of the formal definition of □ as the limit of a numerical function of one variable at a finite point a, and the difficulties in teaching and learning the notion of limits.

We have the limit as a "metaphysical notion," like the notion of the infinitely small and the infinitely large. These mysterious notions have been one of the main obstacles. Is there an intermediate state between what is zero and what is not? Is there a number greater than all the others? The debates surrounding evanescent quantities have shown how difficult it is to make a quantity tend to zero. Is it possible to reach a limit? And numerical transposition: this is an obstacle linked to the "difficulty of detaching oneself from the geometric and kinematic context, to work not with quantities, but with numbers," in short, a difficulty with the arithmetization of the notion of a limit.

The research carried out by Sierpinska (1985) aimed to highlight the epistemological obstacles[6] related to the notion of a limit, identifying two aspects of the notion of epistemological obstacle: the inevitability of the emergence of obstacles and the repetition of their emergence in the phylogenesis and ontogenesis of concepts.

[6] The author defines the epistemological obstacle related to a concept as all the causes of slowness in the acquisition of a concept that are specific to that concept, and only to it, and are such that awareness of them is essential for the development of that concept.

Sierpinska (1985) presents the following epistemological obstacles linked to the notion of limit: *horror infiniti*[7], the obstacles linked to the notion of function, the "geometric" obstacles, the "logical" obstacles, and the obstacle of the symbol.

Michèle Artigue (1996a) also identified a series of epistemological obstacles, which we present below:

> - The ordinary meaning conveyed by the term limit, which favors a conception of the limit as an insurmountable and even unattainable barrier, as a frontier, or even as the ultimate end of a process, which also tends to reinforce strict monotonic conceptions of convergence;
> - The principle of "continuity" (so-called in reference to Leibniz), which consists of treating the limit process as a "finite" algebraic process, passing to the limit the properties common to the elements of the process and, in general, not paying attention to what differentiates this specific operation from the usual algebraic operations;
> - Conceptions that rely too much on a "geometry of form," which do not force us to clearly identify the exact objects involved in the process of limit and the underlying topology. This makes it difficult to perceive the subtle interaction between the numerical structure and the geometric structure underlying the process of limit and induces or reinforces erroneous convictions, such as the constant belief that if an object tends "geometrically" toward an object, all quantities associated with it will have as their limit the corresponding values of the quantities for the limit object. (Doumbia, 2020, p. 74)

Artigue (1996a) also emphasizes the difficulties associated with the dual operational and structural status of the limit, which is reflected in the difficulty of disengaging from a view of the limit as a process composed of terms that prevent the object of the limit from being dissociated from its construction process in order to endow it with an identity of its own.

Finally, the author highlights the difficulties of the standard formalization of the notion of limit, which functions as an indivisible whole,

> when, spontaneously, the student is more likely to consider two distinct processes: one related to the variable, the other to the values of the function; moreover, the intertwining occurs in a way that is not at all natural: to write that the limit of the function is ℓ at infinity, for example, we do not write that for a big *x*, *f(x)* is close to l, but rather, there is a neighborhood of ℓ and one tries to ensure that if *x* is big enough, *f(x)* will be in this neighborhood! In all standard definitions, this imbrication p leads to an alternation of quantifiers, which we know is logically poorly mastered at this level of education. Based on this observation, we can say that mastery of quantifiers is crucial for understanding and implementing the formal definition of the limit. (Doumbia, 2020, p. 75)

[7] This expression refers to Georg Cantor: "The horror of infinity is a form of myopia that prevents us from seeing the real infinity, although in its highest form this infinity created and maintained us, and in all its transformed secondary forms, it manifests itself around us and reaches our minds" (Cantor, 1932).

Michèle Artigue highlights the great qualitative leap between the intuitive definition and the formal definition of the limit:

> Indeed, there is a great qualitative leap between the relatively intuitive handling of the notion of limit and the standard formalized notion, a leap that is attested by the history of the concept itself. In the sense of Imre Lakatos (1976), the formalized concept appears as a concept made to "prove," in a partial rupture with the forms of knowledge that precede it. And its function as a unifying concept of the field of analysis is, at this point, as fundamental as its function of mathematical production. (Artigue, 1996b, p. 10)

One of the problems identified by Artigue (1996b) is the algebra/analysis divide, a problem that has received very little attention in research on analysis learning.

Obstacles of an epistemological nature can also be found at the formal definition level. The practices related to the notion of a limit are diverse, and this diversity extends to the definition of the limit of a function at a point a of $\mathbb{R}$. For some, the formulation of "ℓ is the limit of *f* when *x* tends to a" is:

Definition 1: We say that $\lim_{x\to a} f(x) = \ell$ if, for the entire sequence x_n that tends to a, the sequence $f(x_n)$ tends to ℓ ;

Definition 2: $\forall \varepsilon > 0 \ \exists \ \delta > 0, \quad \forall x \in Df \quad 0 < |x - x_0| < \delta \ \Rightarrow \ |f(x) - \ell| < \varepsilon$, while for others, it is:

Definition 3: $\forall \varepsilon > 0 \ \exists \ \delta > 0, \quad \forall x \in Df \quad 0 < |x - x_0| < \delta \ \Rightarrow \ |f(x) - \ell| < \varepsilon$ and yet in each case, the same notation is used: $\underset{x\to a}{lim} f(x) = \ell$.

Doumbia (2020) notes that Definition 1 brings to light the covariant aspect of the limit, which respects the "natural" order between the initial and final sets of the function under consideration. It takes into account the dynamic connotations of the expression "tends to":

> […] Definition 1 deals with the elements of the initial set and the final set in the natural order: first, the *x*s are considered, and then, only the *f(x)*s. This is close to the familiar formulation of the idea of the limit of a function *f* at a point a: "When the *x*s approach *a* the *f(x)*s approach *b*," which can be found in the canonical expression "when *x* tends to *a*, *f(x)* tends to *b*". (Hauchart, & Rouche, 1987, p. 330 *apud* Job, 2011)

Doumbia (2020), based on these authors, states that this definition explicitly contains a single quantifier (for any sequence), while Definitions 2 and 3 contain two each. However, these are apparent quantifiers because there are hidden quantifiers in the expression "x_n tending to a." Therefore:

> [...] The main difficulty with Definition 1 is that it forces us to imagine all series of limit a. This is particularly difficult due to the double infinity of objects that it involves: considering only one infinite series is not trivial; then, what happens when we must

consider all those that tend to *a*? The imagination suddenly becomes disordered. (Hauchart & Rouche, 1987 *apud* Job, 2011, p. 520)

Job (2011) points out that Definition 1 tends to mask the aspect of approximation underlying the notion of a limit. It better explains the dynamical intuitions underlying expressions like “tend to” than the definition in epsilon and delta. However, “tending to” refers to a continuous movement, while Definition 1 deals with the discrete by relying on sequences.

According to Job (2011, citing Hauchart & Rouche, 1987), Definition 2 captures the aspect of approximation underlying the concept of limit. The kind of universal quantification that occurs in Definition 2 is somehow easier than the universal quantification that appears in Definition 1 because it does not require us to consider a double infinity of objects; its objects (the real ones) are “simpler” than the sequences. Furthermore, we do not need to consider all of them, only the smallest one.

On the other hand, Definition 2 does not consider the dynamic aspects of the concept of limit. It does not respect the “natural” order between the initial and final sets of the function under consideration. Universal quantification “for all ε” masks the idea that we are only interested in “small” values of ε. Sometimes, it establishes a dialectic between covariance and contravariance.

The conceptual field of the notion of a limit

Doumbia (2020) emphasizes the need to use topological concepts when studying the local behavior of a numerical function of a real variable in the neighborhood of a point. These notions are:

> […] neighborhood, interval, adherent point, accumulation point, distance, successor and predecessor of a real number (order in $\mathbb{R}$), inequalities with or without absolute value, concepts of logic such as: universal and existential quantifiers, the sufficient condition, the implication of predicate logic, the concept of function, the composition and decomposition of functions, the direct and reciprocal image of an interval, infinitely small and infinitely large. (Doumbia, 2020, p. 85)

For our purposes, we will focus on numerical functions of real variables that are affine or affine at intervals.

The notion of a limit is fundamental for the derivative, the integral, the convergence of a sequence, and the intermediate value theorem.

The intuitive conception of the expression “x tends to a” is dynamic. The notion of a left or right limit refers explicitly to the order relation. When considered in isolation, the expression “x tends to a” has no mathematical meaning. Intervals play an essential role in studying the notion of a limit of a numerical function of a real variable. Describe the topology of $\mathbb{R}$ is equivalent to characterizing the set of neighborhoods of a point *a*[8] in $\mathbb{R}$, which we will call V*a*.

According to R. Bkouche (1997), the notion of a limit involves two issues that, although linked, have contradictory aspects. The first problem is kinematic, in the sense that it is based on the notion of movement; the canonical formulation (in terms of a function, for example): "*f(x)* tends to *b* when *x* tends to *a*" should then be understood as follows: when the independent variable *x* approaches value *a* indefinitely, then *f(x)* approaches value *b* indefinitely. The quoted assertion: "*f(x)* tends to *b* when *x* tends to *a*," is composed of two propositions: a main proposition," *f(x)* tends to *b*," and a subordinate proposition, "when *x* tends to *a*," thus indicating that the variable *x*, in its movement toward *a*, drives the variable *f(x)* toward value *b*; in other words, the variable controls the function.

The second problem is that of approximation, which can be formulated as follows:

> Be a numerical sequence x_n, say that the sequence x_n tends to a limit ℓ is to say that "the bigger the *n*, the closer x_n gets to ℓ," which is still part of the movement, but the following problem is added: how far the sequence must go for the difference between x_n and ℓ be smaller than a number given in advance or, if we prefer, so that the error we make in substituting ℓ by x_n is less than a value given in advance. (Bkouche, 1997 *apud* Doumbia, 2020, p. 90)

For Doumbia (2020), an elementary example of approximation is the approximate decimal calculation of a number: How many digits after the decimal point so that the calculated number differs from the sought number by less than a value given in advance? The advantage of this example is that it highlights the connection between the notion of approximation and the calculus itself, as in the case of the practice of Euclidean division when "it does not work."

In the approximation problem, the kinematic aspect becomes secondary because what matters is no longer the movement of the independent variable, but the order number that allows the desired approximation.

> In other words, it is the dependent variable that imposes itself, and the grammatical structure of the ritual assertion "*f(x)* tends to *b* when *x* tends to *a*" is different; now there is only one proposition that indicates both what the limit is and the principle of an approximate calculation. In this second formulation, we recognize the Weirstrassian definition of limit. (Bkouche, 1997 *apud* Doumbia, 2020, p. 91)

[8] It is a part V of $\mathbb{R}$ containing an open interval that contains *a*.

This author emphasizes that when the difference $x - s$ becomes infinitely small, then the difference $f(x) - b$ becomes infinitely small, a formulation that can be compared with Cauchy's in his *Résumé des leçons données à l'École Polytechnique* [Summary of the lessons given at the École Polytechnique] (1823).

There is a contradiction between the two questions; the first emphasizes the movement of the independent variable and the ripple effect on the dependent variable, while the second emphasizes the dependent variable and the way it forces the values of the first variable, which is what Bkouche (1997) indicates when he states that it is this contradiction that constitutes one of the difficulties of the notion of limit, a difficulty that is of mathematical order and that is why it is a pedagogical difficulty.

> A utilitarian conception of teaching (guaranteeing success, which implies sparing students from this type of difficulty) would lead us to choose a single problem and select the exercises according to this problem, or else to invent the appropriate pedagogical artifact that would allow students to succeed in the ad hoc exercises proposed to them. But what would they have understood and learned? (Bkouche, 1997, p. 16)

According to Bkouche (1997), if we look again at the notion of limit and the two issues mentioned above, we will see that several elements come into play, including the intuitive and operational aspects. We should emphasize that the intuitive aspect plays a role in each of the two questions, that of movement and that of approach, insofar as it is around this intuitive aspect that both questions under consideration are defined. However, the operational aspect has brought the problem of approximation to the fore inasmuch as the Weirstrassian formulation has been used to give a rigorous definition of the notion of limit and to deduce the conditions for calculating limits. R. Bkouche (1997) argues that ignoring in teaching the two issues that make up the notion of a limit and the two aspects, the intuitive and the operational, that allow this notion to be grasped can only contribute to mutilating the notion and, in the same way, mutilating students' mathematical thinking.

Bkouche (1997) argues that the mathematical activity itself is at stake, the way of thinking about the limit, calculating it when possible, or inventing new methods of determination when necessary. He notes that reducing the operative aspect to the problem of approximation through Weierstrass's definition is less a logical necessity than a historical choice in response to the difficulties presented by the notion of the infinitely small still used by Cauchy.

Regarding the covariant and contravariant nature of the notion of limit, Lecorre (2016), inspired by Lutz, Makhlouf, and Meyer (1996), writes:

> First, let us observe the contravariant nature of this definition, which requires us to determine, for a neighborhood of f(x), the corresponding neighborhood of x. A covariant character, on the other hand, requires us to determine, for a neighborhood of x, the corresponding neighborhood of f(x). (Lecorre, 2016, pp. 30-31)

He continues:

> If we examine this definition, we will see the decisive role played by certain elements of the formalism: universal and existential quantification (there are also three implicit universal quantifications for f, m, and k), the notions of variable, interval, neighborhood of a real number, real number, function, and implication. From this straightforward analysis of the constituent elements of the definition, we can see that it concentrates a large number of notions necessary to give it meaning. (Lecorre, 2016, pp. 30-31)

We can conclude from these analyses that, for the notion of a limit to be taught successfully, it is necessary to return to students' intuitive conceptions, update them, develop situations that challenge erroneous conceptions, and, finally, formalize the notion of a limit and make it operational—in other words, finding a balance between intuition and rigor in teaching mathematics. This highlights the importance of the dialectic between intuition and rigor in teaching mathematics. Berrou (2011, p. 1) states, "There is no choice between rigor and intuition. Both are a precious asset for the study of mathematics, and the ideal way to make mathematics an ally, and not an obstacle, is to cultivate both attitudes, especially if one knows the dominant one."

This study also allowed us to construct a conceptual map (Figure 2) of mathematical objects that feed the notion of the limit of a function and those whose existence depends on this notion.

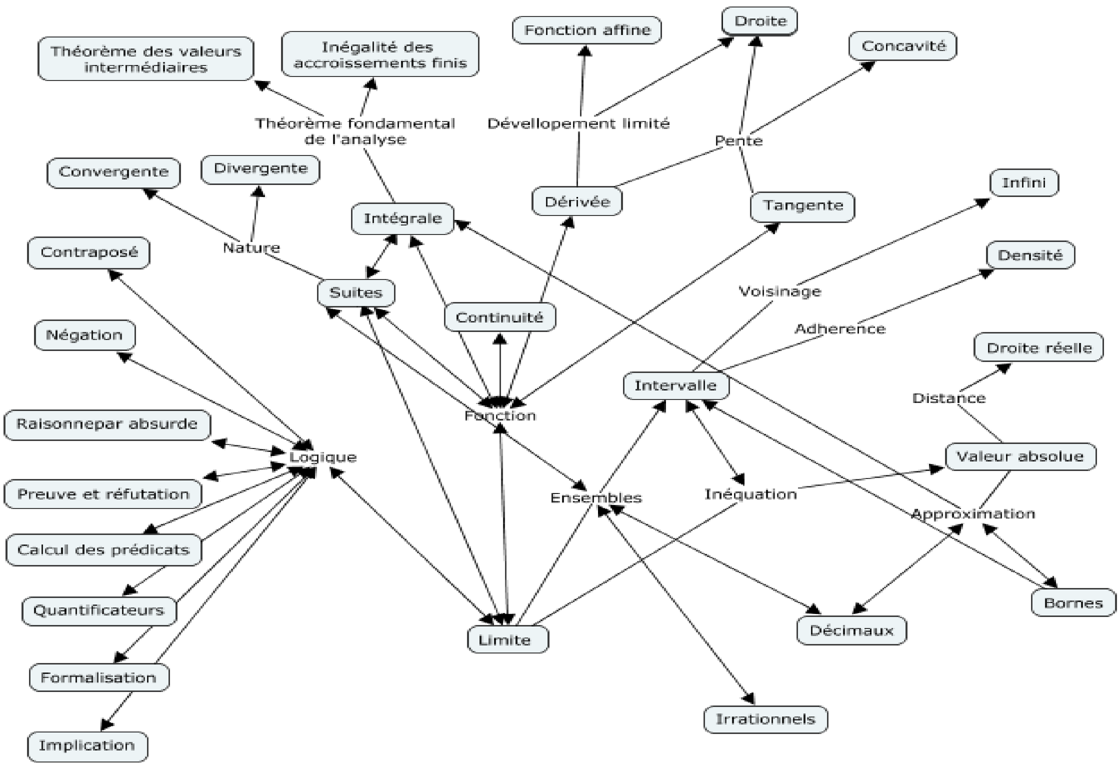

Figure 2[9].

Concept map of the concept of limit (Doumbia, 2020, p. 115)

The analysis of this conceptual map shows that the notion of a limit is closely linked to the concepts of function, variable, integral, derivative, continuity, set, inequalities with absolute value, logic, topology, notion of infinity, etc.

Current teaching of the notion of a limit in Mali does not address these important epistemological aspects. Acquiring knowledge about the notion of a limit means becoming aware of these epistemological aspects, confronting them, and overcoming them. The teaching of the notion of a limit must be organized around the epistemological obstacles (that we have identified) so that students come across them to provoke their limit crisis and, finally, provide them with the means to overcome them. Under these conditions, we would say that students learned the notion of a limit. This is only possible by constructing and applying the formal definition of a limit. For epistemological aspects to be addressed, teachers must have a clear conception of them.

[9] **Image translation**: Intermediate value theorem; Inequality of finite increments; Affine function; Right; Concavity; Fundamental theorem of analysis; Limited development; Slope [*Comb*], Convergent. Divergent, Integral; Derivative; Tangent; Infinite; Contraposited; Nature; Sequences; Continuity; Neighborhood; Adhesion; Density; Negation; *Reductio ad absurdum* [Reason by absurdity]; Proof and refutation, Predicate calculus; Quantifiers; Formalization; Implication; Logic; Function; Sets; Interval; Inequality; Distance; Real right; Absolute value; Approximation, Limits (*Bornes*). Limit (*Limit*); Decimals; Irrationals.

Using this conceptual map as a starting point, the following section analyzes the various tasks associated with the notion of a limit.

Study of the different types of tasks linked to the notion of a limit

We will present the different types of tasks related to logic, derivatives, and sets. We identified 22 task types in logic, as shown in Table 1.

Table 1

Types of tasks involving logic (Doumbia, 2020, pp. 115-116)

1. Giving the formal definition of the limit of a function *f* at a certain point *a* using quantifiers. 2. Demonstrating that a real number is the limit of *f* at one point *a*. 3. Proving that a real number is not the limit of a function *f* at one point *a*. 4. Showing that if *f* has a limit on *a*, this limit will always be reached. 5. Showing that the limit at a point *a* of the sum, product, and quotient of two functions is the sum, product, and quotient of the non-zero finite limits at that point. 6. Showing that if *f* is derivable in *a*, then it is continuous in *a*. 7. Studying the continuity of *f* in *a* 8. Studying the differentiability of *f* in *a* 9. Showing the existence of an integral in a given interval 10. Showing that, if *f* is continuous in *a*, then *f* is limited in the vicinity of *f(a)*. 11. Demonstrating the theorem for continuous monotone functions on an interval [*a*, *b*] (existence of the limit)	12. Proving the upper and lower limit theorem. 13. Demonstrating that a sequence is convergent. 14. Demonstrating that a sequence is divergent 15. Determining the fractional form of a number that can be written as an unlimited repeating decimal number 16. Demonstrating that a rational number is the limit of an unlimited periodic decimal number. 17. Demonstrating that an irrational number is the limit of an unlimited non-periodic decimal number. 18. Demonstrating that any increasing and magnifying sequence converges to a finite limit; 19. Demonstrating that any increasing non-major sequence converges to a finite limit. 20. Demonstrating that any decreasing and non-minorized sequence converges to a finite limit; 21. Demonstrating that two adjacent sequences always converge and that their limits are finite and equal; 22. Demonstrating that every decreasing and minorized sequence converges to a finite limit.

Regarding derivatives, we highlight five types of tasks, which we present in Table 2.

Table 2.

Types of tasks involving derivatives (Doumbia, 2020, p. 116)

1.	Determining the derivative of a function at a point;
2.	Determining the bounded expansion of order n of a function in the neighborhood of a point;
3.	Showing that the bounded expansion of order n of a function is unique;
4.	Demonstrating that a function has an extremum at a point;
5.	Studying the concavity of a curve at a point;

Regarding sets, 16 types of tasks are highlighted (Table 3):

Table 3.

Types of tasks involving sets (Doumbia, 2020, pp. 116-117)

1. Demonstrating that if a function has a limit at a point, that limit is unique; 2. Demonstrating that "being as close as possible" means equality in ". 3. Demonstrating that it is complete; 4. Demonstrating that any numerical function of a real variable can be approximated by an affine line in an interval of reduced radius (that is, in the neighborhood of a point belonging to its set of definitions); 5. Calculating the limit of a function at a point *a*; 6. Studying the limit of a function at a point; 7. Demonstrating that the curve of the function *f* has a vertical asymptote at *a*, a horizontal asymptote at infinity or an oblique asymptote at infinity;	8. Graphically determining the limit of a function at a; 9. Demonstrating that the limit of *gof* is equal to *g(limf)*; 10. Demonstrating the intermediate value theorem; 11. Demonstrating the fundamental theorem of analysis; 12. Demonstrating the finite increment theorem; 13. Calculating whether the limit of a sequence at infinity exists; 14. Calculating whether the limit of a function at a point of exists 15. Determining the exact value of the solution of an equation; 16. Showing that a function has an inflection point at a point *a*.

From the tasks related to the notion of a limit, we can see that its *raison d'être* is to rigorously prove properties and establish axioms about functions, in other words, to find the analysis on solid grounds in Cauchy's sense.

Le Thai Bao's (2007) thesis, inspired by Bosch et al. (2003), identifies and describes three local mathematical organizations related to the notion of a limit. The first MO concerns the algebra of limits (Bosch et al., 2003). Based on the existing problem of computing the limit, the algebra of limits results from modeling the passage to the limit in operations on functions in algebraic rules. The algebra of limits avoids the problem of infinity of the notion of limit and the problems of approximation of this notion. It also allows us to associate writing with a real number or infinity.

The second MO is characterized by the topology of the limits. It is based on the problem of the existence of the limit. Bosch et al. (2003, p. 99) present an example where they discuss the mathematical techniques associated with the types of tasks in this MO:

> Those tasks are associated with valuable mathematical techniques for producing the necessary proofs. Let us see, for example, how to approach a task of the first type: proving that the function *f(x) = sin(x)* has no limit in infinity. Consider the sequence of general terms $x_n = (\pi/2 + 2\pi n)$. We know that $\lim_{n\to+\infty} x_n = +\infty$: Let us study the behavior of the sequence of images: $f(x_n) = \sin(\pi/2 + 2\pi n) = 1$ for all n. Then $\lim_{n\to+\infty} f(x_n) = 1$. Now, let us consider the sequence with the general term $x_{n'} = (-\pi/2+ 2\pi n)$. We know that $\lim_{n\to+\infty} x'_n = +\infty$. Let us analyze the behavior of the sequence of images: $f(x'_n) = \sin(-\pi/2 + 2\pi n) = -1$ for all n. Therefore $\lim_{n\to+\infty} f(x'_n) = -1$. We have two sequences that

tend to infinity and whose images by f converge at different points. Therefore, there is no limit of f(x) = sin(x) at infinity. (Le Thai Bao, 2007, p. 65)

For those kinds of tasks, we have associated mathematical techniques that are useful for the necessary demonstrations. For example, let us look at a task of the first type: demonstrating that the function *f(x) = sin(x)* has no limit in infinity.

Consider the sequence with the general term $x_n = (\pi/2 + 2\pi n)$. We know that $\lim_{x \to +\infty} x_n = +\infty$.

Let us analyze the behavior at infinity of the sequence of images: *f(xn)* = sin(-π/2 + 2π*n*) = -1 for all *n*. We have two sequences that tend to infinity and whose images by *f* converge at different points. Therefore, the limit of *f(x) = sin(x)* does not exist (*cf*. Le Thai Bao, 2007).

In the previous example, the non-existence of the limit of the function *f(x) = sin(x)* at infinity is based on the production of numerical sequences and the calculation of the limits of these sequences. These calculations belong to the OM_1 and indirectly address the relationship between numbers and limits.

Based on the results of his epistemological investigation, he models a restriction of Bosch's OM 2 et al. (2003) in a local mathematical organization $OM_{2'}$ that is more explicitly concerned with the dialectical relationship between the notion of a limit and the notion of a real number, which emphasizes "the existence of the limit of a numerical sequence." Some of the constituent tasks of the $OM_{2'}$ are as follows:

- $T_{2'1}$: Demonstrate the convergence of a numerical sequence;
- $T_{2'2}$: Demonstrate the existence of a solution to an equation.

To carry out these demonstrations, the mathematical technique below focuses explicitly on the dialectical relationship between a number and a limit, that is, producing and verifying whether the numerical sequence satisfies one of the following convergence criteria: the Cauchy criterion, the "limited monotone" criterion, the nested intervals criterion (or adjacent sequences), the definition in terms of (e, N).

In some cases, we can also use techniques based on previously proven theorems, such as the algebraic limit theorem or the intermediate value theorem. In this case, the dialectical relationship between number and limit arises indirectly. Le Thai reinforces this idea by stating that:

> The theoretical block (technology and theory) is described by Bosch et al. (2002) as OM2. The OM2 technological discourse is centered on the use of properties of the limits of a sequence and the classical definition of "limit" in terms of *e* and *d*. This technology is based, in turn, on the theory of real numbers, which considers its structure as a metric

space and its many properties: density, completeness, characterization of the topology of R induced by the Euclidean metric, the existence of the supremum (and infimum) of any non-empty bounded subset of R, Cauchy sequences, etc. (Le Thai Bao, 2007, p. 66)

The technological discourse of the OM_2 is centered on the use of the properties of the limits of a sequence and the classical definition of a "limit" in terms of epsilon and delta. This technology, in turn, is based on the theory of real numbers, which considers its structure as a metric space and its many properties: density, completeness, characterization of the topology induced by the Euclidean metric, the existence of a maximum (and a minimum) in any non-empty closed interval of Cauchy sequences, etc. (*cf*. Le Thai Bao, 2007).

In other words, the theoretical block is based on the topology of ℝ the notion of a limit. To illustrate the $OM_{2'}$, he uses as an example an activity from the manual *Mathématiques 1re S. E*, from the *Dimathème* collection (1982), which paves the way for the introduction of the notions of a limit and continuity of a function: We call D_n ($n \in \mathbb{N}$) the set of decimals d such that the number $10^n \times d$ is a relative integer, for example, 2,427 ∈ D3.

a) How many elements does the ⊂ Dn[0;1] set have?
b) Let be $f(x) = x^3 + 2x - 1$. Show that for any natural number n, there are two real numbers a_n and b_n:
 $a_n = \text{Max}\ \{x \in Dn, f(x) < 0\}$
 $b_n = \text{Min}\ \{x \in Dn, f(x) > 0\}$
 What is the relationship between a_n and b_n?
c) Show that the sequences (a_n) and (b_n) are convergent and that $\lim(a_n) = \lim(b_n)$. Be x_0 this limit.
d) Show that $\exists k > 0$ such that "$\forall x \in [0;1]$ and " $\forall x' \in [0;1]. |f(x) - f(x')| < k|x - x'|$.
e) Deduce that "$\forall n \in \mathbb{N}, |f(b_n) - f(a_n)| < kx10\ \text{-n}$, since $-k \times 10^{-n} < f(x_0) < k \times 10^{-n}$.
 Conclude that $f(x_0) \neq 0$ leads to a contradiction (Le Thai Bao, 2007, p. 67).

The activity above is a technique to demonstrate the existence of a root of the equation $f(x) = 0$ in the range [0;1], producing two adjacent decimal sequences (a_n and (b_n) such that the sequences of its images $f(a_n)$ and $f(b_n)$ are also adjacent and satisfy $f(a_n) < 0 < f(b_n)$ for all n. This leads to the conclusion that there is a real number x_0 such that $f(x_0) = 0$ and that a_n and b_n are successively decimal values approximated to 10^{-n} close to x_0 by the relationship $b_n - a_n = 10^{-n}$.

The local OM_3 "decimal development of a real number" is the restriction of a more general OM – "usual development of real numbers" –present, for example, in Bourbaki (1960). It answers the question of the representation of real numbers according to a basic sequence that forms a dense part of ℝ, for example, development in continued fraction, dyadic developments.

We identify two types of OM_3 tasks:

- T_{31}: Decimal approximation to 10^{-n} closest to a real number.

- T32: Approximate decimal resolution to 10^{-n} close to an equation (given the existence of the root). The general technique associated with these types of tasks is to square the real number (or the root of an equation) x in a decimal pair (dn, dn') in which $dn, dn' \in Dn$ ($n \in \mathbb{N}$) such that $dn' - dn = 10^{-n}$. This technique is justified by the minimum technology required to guarantee the existence of this decimal pair (dn, dn'), because:

- D is a dense part of $\mathbb{R}$. You can always choose a decimal d close to x such that $0 < x - d < 10^{-n}$ for all $n \in \mathbb{N}$. If pn is the projection of d onto Dn (Margolinas, 1985), then $0 < d - pn < 10^{-n}$. Therefore, we have $0 < x - pn < 2.10^{-n}$. Or $-10^{-n} < x - (pn + 10^{-n}) < 10^{-n}$. Let $dn = pn + 10^{-n}$ $n \in Dn$, we have $x - dn < 10^{-n}$.
- We then have $dn - 10^{-n} < x < dn + 10^{-n}$. If $dn < x$, we choose $dn' = dn + 10^{-n}$. Otherwise, we choose $dn' = dn - 10^{-n}$.

This technology is based, as in OM_1 and OM_2, on real number theory, in particular as a metric space and its many properties (*cf.* OM_2 by Bosch et al., 2003), adding the minimal properties of D: a dense additive subgroup of $\mathbb{R}$ (*cf.* Bourbaki).

We can mention the problems of calculating the approximate value for the 10^{-n} close to the quotient or square root of *a* (in which *a* is a positive integer and *b* is a non-zero integer) as examples of tasks of type T_{31}. *Dimathème*'s (1982) activity cited above gives an example of an approximate decimal resolution technique to 10^{-n} close to an equation as a T_{32} task.

The last two MOs – $MO_{2'}$ and MO_3 – deal with the approximation of numbers by numerical sequences, especially decimal sequences (decimal approximation of a real number).

The three local MOs –MO_1, $MO_{2'}$ and MO_3 – revolve around the construction of real numbers. Therefore, they can be integrated into a regional MO (Le Thai Bao, 2007, pp. 65-68).

Maggy Schneider (2015), inspired by Hardy (2009) and Boch et al. (2003), evokes a duality between the first two MOs: a first OM_1 organized around the algebra of limits and whose tasks are calculations of limits, all cases mixed, based on a minimum technology that is the axiomatic of this algebra and a second focused on the topology of limits and whose tasks consist of demonstrating the properties of this algebra, as well as the existence of particular limits. In the first, the theoretical block is missing, and in the second, the practical block is missing. She claims that at the secondary school level in Belgium, this transposition results in a relatively systematic study of all cases of the limits of a function of $\mathbb{R}$ in $\mathbb{R}$, in one order or another, depending on the education network. However, all these cases are "mixed" in the sense that no tasks allow them to be hierarchized a priori according to an objective or a level of study. The author reports the case of a course she attended on limits, in which the first case studied was the limit of a function at a point in its continuity domain. For her, this could be justified in a more formal approach, with the concept of a continuous function playing an important role in formalized analysis. Nevertheless, she notes that in the classes in question, the atmosphere quickly becomes surreal and electric, with students surprised that the values of x are considered increasingly close to this real, while refraining from having a direct image of it. These classes usually continue with the limits of rational functions whose numerator and denominator have a common factor. The students are no longer convinced, wondering why the teacher does not directly study the "simplified" function that he identifies with the previous one without thinking that it differs from it as a continuous extension. She says we can assume that teachers present this limiting case with the didactic intention of getting them to factor and simplify a rational fraction.

Based on this observation, we can say that it is better to introduce the limit before continuity at the first moment and continuity at the second one and, finally, establish a link between the two concepts. It seems necessary to link these two MOs to allow students to acquire a mathematical understanding of the concept of limit. In order to leave a central part to intuition and logic, we believe it would be interesting to add a local mathematical organization whose task types would revolve around the following tasks:

T_{i1}: Graphically conjecture the limit of a function defined by its graphical representation;

T_{i2}: Determine the reciprocal image of an interval by a function defined by its graphical representation;

T_{i3}: Complete a table of values for a function defined by its explicit algebraic expression;

T_{i4}: Interpret the table of values geometrically.

The techniques will be founded on orthogonal projection on the axes of the reference frame, substitution of *x* for its value in *f(x)*, and comparison of real numbers. The theory-technology complex consists of a graphing calculator, the properties of orthogonal projection, elementary algebra, and plane geometry.

This MO of intuition allows us to bring to life students' conceptions of the notion of limit, the MO of the algebra of limits allows us to manipulate the notion of limit, and finally, the MO of the topology of limits allows us to update the conceptions and make them evolve. With these three local MOs, we are establishing a regional mathematical organization (Figure 3) that will allow students to acquire a mathematical sense of the notion of a limit.

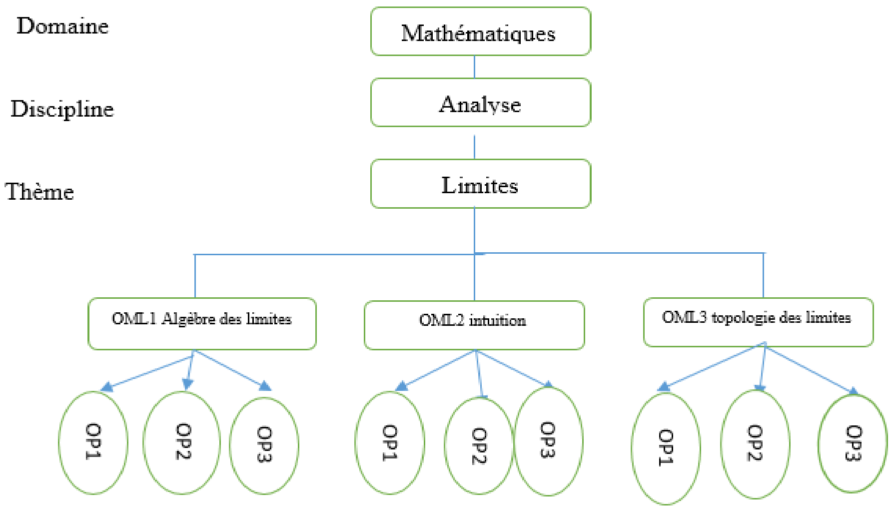

Figure 3[10].

Regional mathematical organization (Doumbia, 2020, p. 123)

The GT Task Generator: calculate, prove, find, determine, conjecture, etc. (Calculate the limit of *f(x)* when *x* tends to *a*. V1, V2, V3, V4). V1, V2, V3, V4 are task type variables. Action verbs are epistemological variables here because they trigger changes in solution techniques. They show the limits of the techniques used to answer them. Table 4 shows the different variables.

Table 4: *List of different variables* (Doumbia, 2020, pp. 123-124)

V1 Nature of *a*: V1.1 *a* can be finite or infinite; V1.2 Position of *a* in relation to the definition set of *f*: *a* belongs or does not belong to D*f* with levels of granularity (*a* is an isolated point adherent to D*f* or not; or *a* is one of the limits of D*f*, *a* is an interior point of D*f*); V2 Nature of *f*. V2.1 *f* is continuous in *a*; V2.2 *f* is not continuous in *a* (*f* can be extended by continuity at a or not). V3 Representation register of f. V3.1 *f* is defined by an explicit algebraic expression; V3.2 *f* is defined by its graphical representation; V3.3 *f* is defined by a table of values; V3.4 *f* is defined by a program.	V4 Function type of f V4.1 f is an affine function; V4.2 f is a polynomial function of degree greater than or equal to 2; V4.3 f is a rational function V4.4 f is an irrational function; V4.5 f is a circular function; V4.6 f is an exponential function; V4.7 f is a logarithmic function; V4.8 f is a combination of functions

[10] **Image translation**: Domain Mathematics; Subject Analysis; Theme Limits; OML1 Algebra of limits; OML2 Intuition; OML3 Topology of limits.

We summarize the different status of *a* and the function *f* in the following diagrams (Figures 4, 5, 6, and 7):

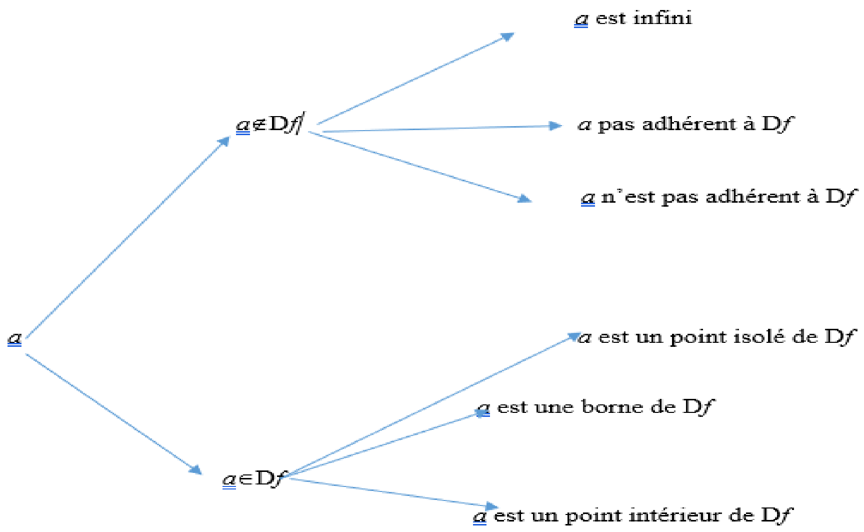

Figure 4.

Limit of f according to the nature of a (Doumbia, 2020, p. 124)

The case where a is an isolated point not adherent to D*f* should be avoided.

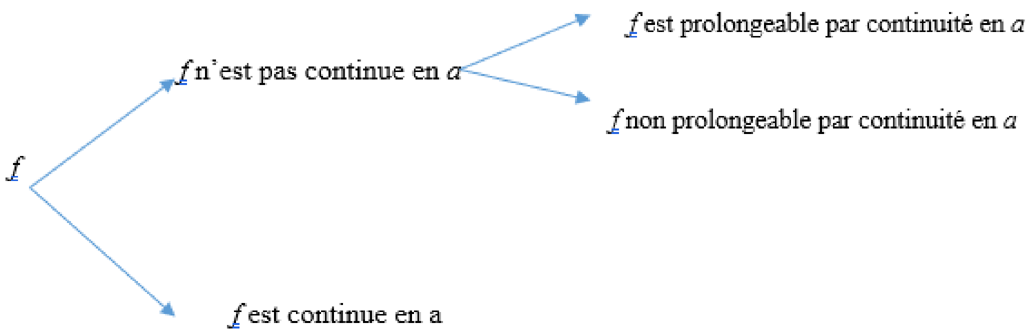

Figure 5: *Continuity or non-continuity of function f* (Doumbia, 2020, p. 125)

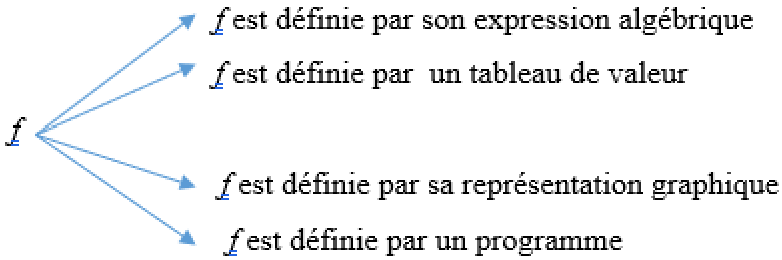

Figure 6.

The function represented in different registers of semiotic representation (Doumbia, 2020, p. 125)

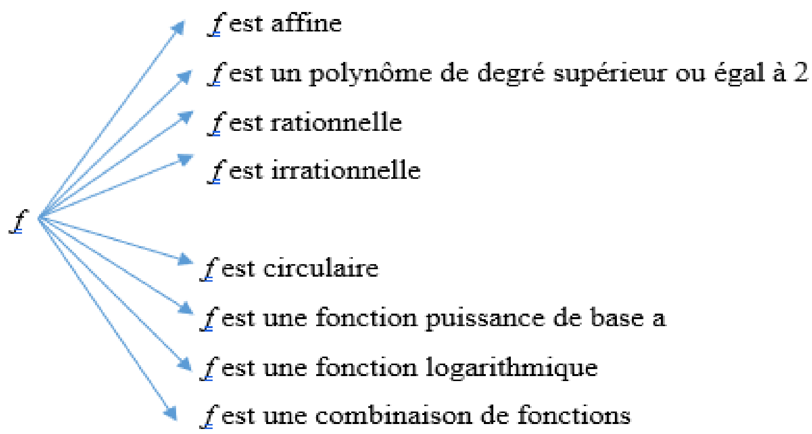

Figure 7.

Nature of the function f (Doumbia, 2020, p. 125)

We highlight the types of tasks linked to these different characterizations of the function *f*:

Table 5

Types of tasks, techniques, and associated technologies/theories (Doumbia, 2020, p. 127)

Types of tasks	**Technique**	**Technology/Theory**
T_1: Calculate the limit of *f* in *a* knowing that *f* is continuous in *a*	Substitution	Property of continuous functions, literal calculation
T_2: Demonstrate that it is the limit of *f* in *a*	Use of the formal definition of limit, increase, and decrease	Formal definition, comparison of functions, order in $\mathbb{R}$
T_3: Calculate the limit of *f* in *a* knowing that *f* is not continuous in *a*	$\tau 1$: substitution comparison (if *f* is defined by parts); $\tau 2$: factorization simplification substitution (if *f* is a rational function); $\tau 3$: rationalization factorization simplification substitution (if *f* is an irrational function); $\tau 4$: change of variable reduction substitution	Definition of limit, properties of continuous functions, and literal calculation.
T_4: Demonstrate that it is the limit of *f* in *a*	The negation of the formal definition of the limit (we show the existence of an epsilon such that, for any positive delta, the implication is false).	Formal logic, order in, weak topology of $\mathbb{R}$.
T_5: determine the sufficient condition for $\|f(x) - \ell\| < \varepsilon$	Transform the inequality with the unknown *f(x) :* $\|f(x) - \ell\| <$	

	ε in an inequality with the unknown x, $\lvert x-a\rvert<\delta$ so that delta is a function of epsilon	Properties of absolute value, inequality in ℝ.
T_6: Demonstrate that f admits a limit in a	Application of the formal definition of the limit. Transform the inequality $\lvert f(x)-\ell\rvert<\varepsilon$ into an inequality with x unknown, $\lvert x-a\rvert<\delta$ so that delta is a function of epsilon	Definition of limit and order in ℝ.
T_7: Demonstrate that if it is the limit of f in a and' that of g in a, then +' is the limit of f+g in the	Formal definition of the limit: epsilon is chosen so that	Definition of limit and order in Absolute value properties, the field of reals and their properties
T_8: Demonstrate that if it is the limit of f in a and ℓ that of g in a, then' is the limit of fg in a	Application of the formal definition of the limit	Definition of limit and order in ℝ
T_9: Demonstrate that if it is the limit of f in a and $\frac{\ell}{\ell'}$ 'that of g in a, then' is the limit of $\frac{f}{g}$in a	Application of the formal definition of the limit	Definition of limit and order in ℝ

Table 6 shows the types of tasks involving graphical representations and the case where the value of a is infinity.

Table 6.

Types of tasks involving graphical representations and the case where the value of a is infinity (Doumbia, 2020, p. 127)

Types of tasks	**Technique**	**Technology/Theory**
Graphical representation		
T_{10}: conjecture the limit of f defined by its graphical representation when x tends to a	Graphic reading, graphic determination of the reciprocal image of an interval. Show that the reciprocal image of any interval with center and radius is an interval with center a possibly deprived of {a}.	Orthogonal projection, plane geometry. If a is infinity
If *a* is infinity		
T_{11}: Demonstrate that it is the limit of f in infinity	Application of the formal definition of the limit: To do this, consider a real e strictly positive. Write that $\lvert f(x) - l\rvert < e$ and calculate this inequality to obtain an interval for $x - a$; deduce that the delta value can be considered the maximum of the absolute values of the limits of the table of $x - a$ obtained. Next, consider that delta is the maximum of the absolute values of the limits of the interval $x - a$ previously obtained. Then show that $\lvert f(x) - l\rvert < e$.	Formal definition and order in ., field of reals and their properties
T_{12}: Demonstrate that f admits an infinite limit at infinity	Application of the formal definition of the limit	Definition of limit and order in

Task types can be grouped according to the technique used to perform them. In this case, we would have the following groupings:

For techniques involving substitution, factoring, simplification-substitution, rationalization and/or change of variable, we have task types T_1 and T_2.

For the technique based on formal definition, we have the following task types: $T_{1'}$, $T_{2'}$, T_4, T_5, T_6, T_7, T_9, T_{10}.

T_3 and T_8 use the "transformation" and "graphic reading" techniques, respectively.

In the case of *f* not continuous at a, we can choose f so that the limit is in one of the indeterminate forms 0/0 or 0×∞, as in the following examples:

$f(x)=\frac{x-2}{x^2-4}$ and a=2; $f(x)=\frac{\sqrt{x+3}-2}{x-1}$ and a=1; $f(x) = \begin{cases} -x-2, x<0 \\ 3-x, x \geq 0 \end{cases}$ and a=0; $f(x)=\frac{x-2}{(2-x)^2}$ and a=2.

Summary analysis of the economic-institutional and ecological dimensions

Based on the constructed REM, Doumbia (2020) analyzed the study programs, skills, and manuals from secondary education (11th and 12th grades) and from the beginning of the university course that addressed the notion of a limit in Mali. Without going into details (given the space reserved for this text), we present some of the results of this analysis.

This analysis allowed us to identify the following types of tasks regarding the mathematical organization of the notion of limit in textbooks: "Calculate, when it exists, the limit (finite or not) in x_0 (x_0 finite or not) of a function that is a combination of reference functions" and "show that the limit when x tends to x_0 of $g(x)$ is equal to ℓ establishing a markup of the type: $|g(x_0+h)-\ell| \leq k^*|j(h)|$ with k an element of $\mathbb{R}^+$ and *j* is one of the reference functions and ℓ is a real number."

Most of the techniques used to perform these types of tasks are based on the following procedures:

a) Replace in $f(x)$ x by x_0;
b) If $f(x_0)$ is not an indeterminate form "0/0 ; $+\infty - \infty$; $0 \times \infty$; $\frac{\infty}{\infty}$" , then $f(x_0)$ is the limit;
c) Otherwise, the indeterminacy needs to be resolved using calculus techniques for algebraic expressions (factorization, finding the conjugate expression, simplification);

d) If, despite these operations, the indeterminacy is not removed, then f has no limit at x_0.

e) Establish a markup of the type $|g(x_0+h)$- $\ell| \leq$ k*| j(h) | with the element k of $\mathbb{R}^+$ and j is one of the reference functions and ℓ is a real.

The following technological-theoretical complex justifies these techniques:

a) Algebraic operations on numerical functions of a real variable (algebraic structures in R^R) (addition, multiplication, inverse, root, elevation, or squaring);

b) Rules for calculating infinitely small and infinitely large numbers.

If k $\in \mathbb{R}$ $alors \begin{cases} k+\infty=+\infty \\ k-\infty=-\infty \end{cases}$; $\begin{cases} k\times\infty=\infty \\ \frac{k}{\infty}=0 \end{cases}$ $(+\infty)(-\infty)=-\infty;\ \infty+\infty=+\infty;\ -\infty-\infty=-\infty$

In terms of didactic organization, we identified the following options:

a) Choose reference functions with zero limit at 0.

b) Bring out the intuitive idea of the limit by combining numerical, graphical, and algebraic structures.

c) Recall the properties of absolute value, minorizing, majorizing (squaring a function), and provide, without demonstrating, the properties and techniques for calculating limits.

The dominant epistemological model in textbooks is the algebra of limits.

Regarding the economic dimension, the praxeologies built around the concept of limit at a point make it possible to calculate limits at a point. However, they distance us from the mathematical meaning of the notion and are ineffective when faced with tasks such as: "Showing that ℓ is the limit of f at point a" and "Show that ℓ is not the limit of f at point a." Furthermore, we hypothesize that the knowledge mobilized in these praxeologies does not evolve students' conceptions. This MO lacks the Technology-Theory block (the justification of the properties of the limits). It does not allow the mathematical meaning of limits to be acquired, but it does allow techniques related to the concept of limits to be manipulated.

It is necessary to review these praxeologies to allow students to construct new knowledge and develop their conceptions about the notion of limit.

From the point of view of the *ecological dimension*, we observed in this study that the teaching of the notion of a limit underwent changes in curriculum texts and handbooks. Its habitat is the eleventh grade (in Mali) in the study of numerical functions of a real variable and numerical sequences; its niches are the definition of the derivative, the determination of the exact value of the solution of an equation, the study of the direction of variation of a function, the condition of convergence of a sequence, and the fundamental theorem of analysis.

The rupture between algebra and analysis, mathematical rigor, the topology of limits, the valorization of the acquisition of logic and the entry into the logic of predicates, the updating and reinvestment of notions such as absolute value, distance, inequalities with absolute value, the convergence of a sequence, proof and refutation (showing that ℓ is or is not the limit of f in a), the treatment of epistemological obstacles, it is formalizing, unifying, generalizing, and simplifying, etc., constitute the *reason for being of the concept of limit*.

This analysis shows that the teaching of the notion of a limit in Mali has changed. Initially, the intuitive and formal definitions were taught in the 11th and 12th grades with application exercises, then it was the intuitive definition of the notion that was taught with a pseudo-formalization with applications, followed by the intuitive definition of the notion, which came after its formalization without any example of application. Finally, it was simply the intuitive definition that was taught without formalization. Initially, continuity was taught before the notion of a limit, and the latter was defined using continuity. Currently, the order is reversed, and the types of tasks are limited to calculating limits, applying the rules and properties on limits established in the course without demonstration.

The formal definition and its application are gradually being abandoned in secondary education, which means that it will re-emerge at another level, that is, in higher education. In higher education, we observe that the formal definition resurfaces but is not constructed, and they show some simple application cases to make it work (the case of affine functions). These cases do not illustrate the formal definition in all dimensions (contravariance, covariance, and the dialectic between the two). It is the contravariant aspect that is emphasized in university courses. Obstacles such as "the limit can or cannot be reached" and logical obstacles are not addressed; in other words, there is no meticulous study of the limit to help students' conceptions evolve. The limit is taught as a dynamic notion and an approximation, while it is a static notion in its precise mathematical definition.

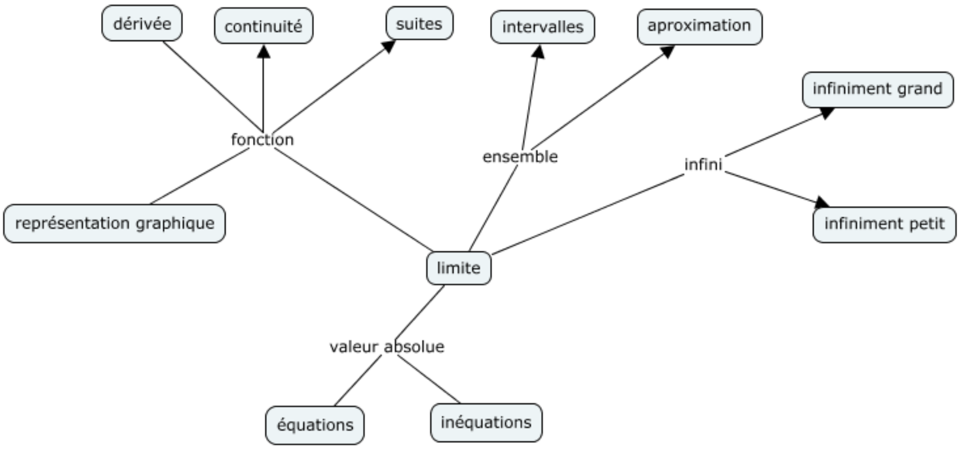

Figure 8[11].

The reference epistemological model taken from school textbooks (Doumbia, 2020, p. 184)

As can be seen in Figure 8, the notion of limit is not related to its complete conceptual field, and logic and other important aspects are missing. With this epistemological model, students learn to factor and simplify expressions of rational functions, to decompose a function into simple elements, to make the denominator and/or numerator of an irrational function rational, and to give meaning to epsilon. They do not have access to the meanings of delta, implication, and quantifiers that make it possible to give meaning to the formal definition of limit.

Conclusion

We can build a REM on a concept by relating it to its conceptual field and, with the help of Chevallard's quadruple, we provide the types of tasks, which allows us to know the reason for being for the existence of the concept and access its meaning. This is a powerful tool for textbook analysis, and the didactic reference model is built based on the REM.

[11] **Image translation**: Derivative; Continuity; Sequences; Intervals; Approximation; Function; Set; Infinity; Infinitely large; Infinitely small; Graphical representation; Limit; Absolute value; Equations; Inequalities

In our case, the construction of a REM allowed us to relate the notion of limit with the notions that give it meaning and direct our economic-institutional and ecological analyses. These analyses allowed us to highlight the dominant reference epistemological model (DEM) in the various institutions (school curricula and textbooks, etc.). This allowed us to show what meaning is given to the notion of limit in these different institutions (curricula, textbooks, etc.).

For the experimental phase (which we do not describe here), an alternative reference epistemological model (AREM) was constructed based on the comparison between the DEM and the REM, to reduce the gap between academic knowledge and taught knowledge. This work helped us update the knowledge related to the formal definition of the limit, such as graphical activities, resolution of inequalities with absolute values, intervals (notion of neighborhood), universal and existential quantifiers and an introduction to predicate logic. During the experimental phase (*cf*. Doumbia, 2020) based on the AREM, students learned to apply the formal definition, the construction of a sufficient condition, but also to solve a new task: to show that ℓ is the limit of f in a.

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