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Towards a shared reference epistemological model?

¿Hacia un modelo epistemológico de referencia compartido?

Vers un modèle épistémologique de référence partagé ?

Rumo a um modelo epistemológico de referência compartilhado?

Ana Maria Veloso Nobre[1]
Pontifícia Universidade Católica de São Paulo
Mestre em Educação Matemática
https://orcid.org/0000_0002_8151_2643

Vera Helena Giusti de Souza[2]
Universidade de São Paulo
Doutora em Educação Matemática
https://orcid.org/0000-0003-0705-0839

Abstract

This text was written to disseminate a proposal of the Study Group on Calculus in Secondary and Higher Education, headed by researchers Dr. Pierre Job (ICHEBrussels Management School-Belgium) and Dr. Luiz Márcio Santos Farias (UFBA-Brazil), whose objective is to create a joint work Brazil-Belgium in search of a reference epistemological model for calculus and analysis teaching. In 2023, at the Integrated Studies Seminar-Calculus in Secondary and Higher Education at PUC/SP, Professor Job presented aspects of the research carried out in Belgium based on a reference epistemological model, the theory of didactical situations and the anthropological theory of the didactic regarding the limit concept. To motivate calculus teachers to engage in this research, it summarizes the events of two of the three-day seminars. The first part reports the questions proposed on the first day, followed by a reflection on some. Those questions served as a model for the initial survey of difficulties that could provide answers to three open questions: Are the premises used in Belgium valid in the Brazilian context? Is it possible to build a shared theoretical framework that allows us to question the status of epistemological obstacles and the main difficulty encountered in teaching the concepts of limits, derivatives, and integrals? What kind of experiment can we plan to test this model? The second part summarizes Prof. Job's presentation on the second day about the research carried out in Belgium.

Keywords: Epistemology, Teaching, Limit, Model.

[1] anobre@pucsp.br
[2] vhgiusti@usp.br

Resumen

Este texto fue escrito para divulgar la propuesta del Grupo de Estudios sobre Cálculo en la Enseñanza Media y Superior, liderado por los investigadores Dr. Pierre Job (ICHEBrussels Management School-Bélgica) y Dr. Luiz Márcio Santos Farias (UFBA-Brasil), cuyo objetivo es crear un trabajo conjunto Brasil-Bélgica en busca de un modelo epistemológico de referencia para la enseñanza del cálculo y del análisis. En 2023, en el Seminario de Estudios Integrados - Cálculo en la Enseñanza Media y Superior, en la PUC/SP, el profesor Job presentó aspectos del trabajo realizado en Bélgica, basado en un modelo epistemológico de referencia, en la teoría de las situaciones didácticas y en la teoría antropológica de lo didáctico, en relación al concepto de límite. Con el fin de motivar a los profesores de cálculo a comprometerse en esta investigación, se resumen los acontecimientos de dos de los tres días del seminario. En la primera parte del texto, se relatan las preguntas propuestas en el primer día, seguidas de una reflexión sobre algunas de ellas, que sirven de modelo para el relevamiento inicial de las dificultades que podrían dar respuesta a tres preguntas abiertas: ¿Las premisas utilizadas en Bélgica son válidas en el contexto brasileño?, ¿Es posible construir un marco teórico común que nos permita cuestionar el estatuto de los obstáculos epistemológicos y la principal dificultad encontrada en la enseñanza de los conceptos de límite, derivada e integral?, ¿Qué tipo de experimento podemos planear juntos para probar este modelo? La segunda parte del texto resume la presentación del Prof. Job en el segundo día sobre la investigación realizada en Bélgica.

Palabras clave: Epistemología, Enseñanza, Límite, Modelo.

Résumé

Ce texte a été préparé pour faire connaître la proposition du Groupe d'étude sur le Calcul dans l'enseignement secondaire et supérieur, dirigé par les chercheurs Dr. Pierre Job (ICHEBrussels Management School-Belgique) et Dr. Luiz Márcio Santos Farias (UFBA-Brésil), dont l'objectif est créer un travail commun entre Brésil et Belgique, à la recherche d'un modèle épistémologique de référence pour l'enseignement du calcul et de l'analyse. En 2023, lors du Séminaire d'études intégrées – Calcul dans l'enseignement secondaire et supérieur, à la PUC/SP, le professeur Job a présenté des aspects des travaux menés en Belgique, basés sur un Modèle Épistémologique de Référence, la Théorie des Situations Didactiques et la Théorie Anthropologique de Didactique, concernant la notion de limite. Pour motiver les enseignants de Calcul à s'engager dans cette recherche, les événements de deux des trois jours du séminaire sont résumés. Dans la première partie du texte sont rapportées les questions proposées le premier jour, suivies d'une réflexion sur certaines d'entre elles, questions qui servent de modèle à un premier état des lieux des difficultés pouvant apporter des réponses à trois questions ouvertes : ?Le les prémisses utilisées en Belgique sont-elles valables dans le contexte brésilien?, « Est-il possible de construire une structure théorique commune qui permette de s'interroger sur le statut d'un obstacle épistémologique et la principale difficulté rencontrée dans l'enseignement des concepts de limite, de dérivée et d'intégrale?, ?Quel type d'expérience peut-on prévoir pour tester ce modèle? Dans la deuxième partie du texte, est rapporté un résumé de la présentation faite par Prof. Job, dans le deuxième jour, sur la recherche menée en Belgique.

Mots-clés : Épistémologie, Enseignement, Limite, Modèle.

Resumo

Este texto foi elaborado para divulgar a proposta do Grupo de Estudo do Cálculo no Ensino Médio e no Ensino Superior, liderado pelos pesquisadores Dr. Pierre Job (ICHEBrussels Management School-Bélgica) e Dr. Luiz Márcio Santos Farias (UFBA-Brasil), cujo objetivo é criar um trabalho conjunto Brasil-Bélgica, em busca de um modelo epistemológico de referência para o ensino de cálculo e análise. Em 2023, no Seminário Estudos Integrados – Cálculo no Ensino Médio e no Superior, na PUC/SP, o Professor Job apresentou aspectos da pesquisa realizada na Bélgica, baseada em um modelo epistemológico de referência, na teoria das situações didáticas e na teoria antropológica do didático, com relação ao conceito de limite. Para motivar professores de cálculo a se engajarem nessa pesquisa, resumem-se os acontecimentos de dois dos três dias do seminário. Na primeira parte do texto, relatam-se as questões propostas no primeiro dia, seguidas por uma reflexão sobre algumas delas, questões essas que servem de modelo para o levantamento inicial de dificuldades que poderão dar resposta a três questões abertas: As premissas usadas na Bélgica são válidas no contexto brasileiro? É possível construir uma estrutura teórica comum que permita questionar o status de obstáculo epistemológico e a principal dificuldade encontrada no ensino dos conceitos de limite, derivada e integral? Que tipo de experimento é possível planejar em conjunto para testar esse modelo? Na segunda parte do texto, apresenta-se um resumo da exposição feita pelo Prof. Job, no segundo dia, sobre a pesquisa desenvolvida na Bélgica.

Palavras-chave: Epistemologia, Ensino, Limite, Modelo.

Towards a shared reference epistemological model?

The Study Group for Secondary and Higher Education Calculus (Grupo de Estudos de Cálculo no Ensino Médio e Superior - GECEMS) was concretized in 2019 by professors Pierre Job[3] and Luiz Marcio Santos Farias[4] to establish a joint Brazil-Belgium project, in search of a Reference Epistemological Model for calculus and analysis teaching. In Brazil, it is formed by six poles:

- Federal University of Mato Grosso do Sul (UFMS - Central-West Region), coordinated by Professor Dr. Sonia Maria Monteiro da Silva Burigato
- Pontifical Catholic University of São Paulo (PUC/SP - Southeast Region), coordinated by Professor Dr. Sonia Barbosa Camargo Igliori
- Federal Technological University of Paraná (UTFPR - Southern Region), coordinated by Professor Dr André Luis Trevisan
- Federal University of Pará (UFPA - Northern Region), coordinated by Professor Dr. Jose Messildo Viana Nunes
- State University of Feira de Santana (UEFS - Northeast Region 1), coordinated by Professor Dr. Jany Santos Souza Goulart
- Federal University of Bahia (UFBA - Northeast Region 2), coordinated by Professor Dr. Joseph Nee Anyah Yartey

The Integrated Studies Seminar—Calculus in Secondary and Higher Education, held at PUC-SP in September 2023 and led by Professor Dr. Pierre Job, aimed to present the research proposed by GECEMS in search of a possible Reference (R) Epistemological (E) Model (M) for the teaching of Calculus and Analysis shared by Brazil and Belgium (Schneider, 2019).

The seminar lasted three days, with a workshop on the first day (morning and afternoon), an online seminar on the second day (morning), and two workshops on the third day (morning and afternoon). In the first two days, Prof. Job only introduced the ideas of the GECEMS for the concept of limit and, on the third day, the concept of didactic engineering (Chevallard, 2011, p. 23), a methodology used by the group in the search for a typical model (M), which serves as a reference (R) for teaching calculus, and which is based on the epistemology (E) of concepts. This text will not cover the third-day workshops on didactic engineering and its assumptions.

The Atelier

To encourage calculus teachers' participation in this search for REM, this text disseminates the main ideas discussed in the first two days of the seminar, without intending to exhaust them. Regarding the first day (the Atelier), we summarize below the activities

[3] Prof. Pierre Job, from the Institut Catholique des Hautes Études Commerciales (ICHEBrussels Management School)
[4] Prof. Luiz Marcio Santos Farias, from the Federal University of Bahia (UFBA)

developed, accompanied by our reflections; regarding the second day, we summarize Prof. Job's presentation.

The activities proposed in the Atelier aimed to address the theme of the second-day seminar, “Towards a shared reference epistemological model (REM)?” The participants were divided into four groups: G1 through G4, with G1 and G2 formed by undergraduate students from PUC/SP, G3 by calculus teachers, and G4 by researchers in mathematics education. It is important to note that, in the case of the research proposed by GECEMS, groups G1 and G2 had to be formed in Brazil: G1 comprised basic education mathematics teachers and G2 included teaching degree mathematics students.

In the case of the research, each group had the task of answering questions related to the concept of limit proposed by Job, which served as a model for part of the preliminary analyses of didactic engineering that aimed to answer the title question of the seminar.

More specifically, G1 was responsible for four questions (G11 to G14) and G2 for five (G21 to G25). The answers to these questions should be handed to groups G3 and G4, who should subsequently analyze them. For this reason, all these questions were also answered with an a priori analysis by G3 or G4.

Group G3 was responsible for six questions (G31 to G36). The first two were common to all groups; the third was also proposed for G2; the fourth was also for G1; the fifth was only for G3 (a demonstration for sequences); and the sixth was also for G1. The answers given by G3 would be the feedback for groups G1 and/or G2.

Group G4 answered seven questions (G41 to G47). The first two were common to all groups; the third and fourth were also proposed for G2; the fifth, sixth, and seventh were only for G4 (demonstrating the limit for a composite function). The answers given by G4 was the feedback for group G2.

In what follows, we present the statements of the questions proposed to groups G1 through G4, followed by a reflection based on our understanding and, without wanting to exhaust the subject, on the two questions common to all groups and the others from group G4.

Questions for Group G1 (four)

G11. Define the limit of a sequence.

G12. Suppose a colleague has introduced the concept of the real limit of a real function along the following main lines.

- He has students calculate the values of a function f for values increasingly closer to 5.
- He shows that images $f(x)$ are getting closer to 12 as x approaches 5.
- He then announces that in such circumstances, we say that 12 is the limit of $f(x)$ when x tends to 5.
- After that, he generalizes by introducing the following definition:

"b is the limit of $f(x)$ when x tends to a when $f(x)$ can be as close as b as you want when x is sufficiently close to a."

A student who followed this teaching challenges you and asks you to explain the following quote, since he did not understand the notion of limit.

"Hello, professor, sorry to bother you, but I don't understand how you get from the expression "the closer x is to a, the closer $f(x)$ is to b" to "$f(x)$ is as close to b as we want when x is sufficiently close to a." I feel like there is a connection, but it seems too vague to me. Could you give me some explanation, please?"

G13. Here is an excerpt from an interview with a professor. He explains his reasons for adopting the following definition of the limit of a function.

- I think the definition of the limit of a real function in $\varepsilon - \delta$ is complicated for students to understand.
- To better convey this, I divide it into two parts.
- I define "x tends to a", which I write $x \to a$, by $\forall \varepsilon > 0$: $|x - a| < \varepsilon$
- Likewise, I define "$f(x)$ tends to b", which I write $f(x) \to b$ and by $\forall \varepsilon > 0$: $|f(x) - b| < \varepsilon$.
- This allows me to say that the limit of $f(x)$ in a is b if and only if $f(x) \to b$ when $x \to a$.
- This way, students understand better than through the classical definition in $\varepsilon - \delta$.

Given the question, we ask: Write a text detailing your position regarding your colleague's practice. Do you agree with his way of proceeding?

G14. Suppose a colleague introduced the notion of a finite limit of a sequence of real numbers to the students, giving the following definition:

$$\forall \varepsilon > 0 \; \exists n \geq 1 \; \forall \boldsymbol{m} > \boldsymbol{n} : |a_m - a| < \varepsilon \; (*^{1})$$

A student who has followed this explanation questions you and does not understand why we need $\forall m > n$ in the definition. It seems to him that taking

$$\forall \varepsilon > 0 \; \exists n \geq 1 : |a_n - a| < \varepsilon \; (*^{2})$$

would be enough because this expression $(*^2)$ clearly indicates that "a can get as close as you want" which is, according to him, the basic idea of the limit of a sequence.

Given the question, we ask: What does $(*^1)$ mean in Portuguese? And $(*^2)$? What distinguishes them from each other? Detail your position to the student. Do you agree with his argument? Can't we be contented with $(*^2)$ instead of $(*^1)$?

Questions for Group G2 (five)

G21. Define the limit of a function.

G22. Question **G12**.

G23. A colleague uses the following definition of a limit:

"b is the limit of f in a when we can make $f(x)$ get as close as we want to b, taking x sufficiently close to a" $(*^2)$

Students are divided and understand this definition in different ways. You will find below the different interpretations debated. For the sake of clarity, they are formulated in predicate logic.

- « $\forall \varepsilon > 0\ \exists \eta > 0\ \forall x \in Dom(f)\ : |x - a| < \eta \Rightarrow |f(x) - b| < \varepsilon$ » (A)
- « $\forall \varepsilon > 0\ \exists \eta > 0\ \forall x \in Dom(f)\ : 0 < |x - a| < \eta \Rightarrow |f(x) - b| < \varepsilon$ » (B)
- « $\forall \varepsilon > 0\ \exists x \in Dom(f)\ : |f(x) - b| < \varepsilon$ » (C)

Given the question, we ask:

1. Complete the following table by crossing the equivalent interpretations.

It is equivalent to ↷	A	B	C	$(*^2)$
A				
B				
C				
$(*^2)$				

2. Explain in writing to students your table and justify which interpretations are or are not equivalent and for what reasons.

3. What interpretations can be adopted to clarify the meaning of $(*^2)$?

G24. Take a function $f: \mathbb{R} \to \mathbb{R}$ and $a, b \in \mathbb{R}$. In what order will you present the different limit cases to students and why?

a. $\lim_{x \to a} f(x) = b$

b. $\lim_{x \to a} f(x) = \pm\infty$

c. $\lim_{x \to \pm\infty} f(x) = b$

d. $\lim_{x \to \pm\infty} f(x) = \pm\infty$

G25. Here is an excerpt from an interview with a professor where he explains his reasons for adopting the following definition of the limit of a function.

Consider a function $f: \mathbb{R} \to \mathbb{R}$ and $a, b \in \mathbb{R}$.

I say that b is the limit of f in a when

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in dom(f) : 0 < |x - a| < \delta \Rightarrow 0 < |f(x) - b| < \varepsilon$$

My reasoning is as follows.

I start with the definition that is often adopted as a limit

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in dom(f) : 0 < |x - a| < \delta \Rightarrow |f(x) - b| < \varepsilon$$

I write down that condition $0 < |x - a|$ means that x tends towards a without ever reaching it.

But if x tends to a without ever reaching it, $f(x)$ will tend to b without ever reaching it.

This is the very principle of a limit.

It would therefore be more coherent to add the condition $0 < |f(x) - b|$ for the definition of limit.

It is this reasoning that leads me to adopt

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in dom(f) : 0 < |x - a| < \delta \Rightarrow 0 < |f(x) - b| < \varepsilon$$

Given the question, we ask: Write a text detailing your position regarding your colleague's practice. Do you agree with his definition?

Questions for Group G3 (six)

G31. The question given is **G12** or **G22**

G32. The question given is **G11**

G33. The question given is **G25**

G34. The question given is **G14**

G35. This question continues the previous question (**G14**) and invites reflection.

Here is an accurate statement and an attempt to prove the property**.**

¨Let us take two sequences of real numbers (a_n) and (b_n) and two real numbers a and b such that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$.

Then,

$$\lim_{n\to\infty}(a_n + b_n) = a + b.$$

Tentative proof.

Consider $\varepsilon > 0$.

There are i and j such that for all $n \geq i$, $|a_n - a| \leq \frac{\varepsilon}{2}$ and for all $n \geq j$, $|b_n - b| \leq \frac{\varepsilon}{2}$. (E_1)

Taking $n \geq \max\{i, j\}$, we obtain $|(a_n + b_n) - (a + b)| = |a_n - a + b_n - b| \leq |a_n - a| + |b_n - b| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. (E_2)

We can therefore conclude that $\lim_{n\to\infty}(a_n + b_n) = a + b$. (E_3)

1. Is this tentative proof correct if we adopt

$$\forall \varepsilon > 0 \ \exists n \geq 1 \ \forall m > n : |a_m - a| < \varepsilon \ (*^1)$$

as a limit definition? If yes, justify steps (E_1), (E_2), (E_3). Otherwise, explain very precisely which step(s) of this proof is/are false and give counterexample(s).

2. Is this tentative proof correct if we adopt

$$\forall \varepsilon > 0 \ \exists n \geq 1 : |a_n - a| < \varepsilon (*^2)$$

as a limit definition? If yes, justify steps (E_1), (E_2), (E_3). Otherwise, explain very precisely which step(s) of this proof is/are false and give counterexample(s).

3. Based on the answers to the previous points, will you review the answer given to the previous question and your explanation to the student who wanted to adopt $(*^2)$ as defining the limit of a sequence instead of $(*^1)$? Detail your answer.

G36. The question given is **G13**.

Questions for Group G4 (seven)

G41. The question given is **G12** or **G22** or **G31.**

G42. The question given is **G21.**

G43. The question given is **G24.**

G44. The question given is **G23.**

G45. Consider the following property of so-called transitivity limits and a tentative proof.

Property.

Be $f, g : \mathbb{R} \to \mathbb{R}$ two functions and a, b, c three real numbers such that

- $Im(f) \subseteq Dom(g)$
- $\lim_{x \to a} f(x) = b$
- $\lim_{x \to b} g(x) = c$

So

$$\lim_{x \to a} g\big(f(x)\big) = c$$

Tentative proof.

Consider $\varepsilon > 0$.

There is $\delta > 0 \ \forall y \in Dom(g) : 0 < |y - b| < \delta \Rightarrow |g(y) - c| < \varepsilon.$ (E_1)

There is $\gamma > 0 \ \forall x \in Dom(f) : 0 < |x - a| < \gamma \Rightarrow |f(x) - b| < \delta.$ (E_2)

Then $\forall x \in Dom(f) : 0 < |x - a| < \gamma \Rightarrow |g(f(x)) - c| < \varepsilon$ (E_3)

Then $\lim_{x \to a} g\big(f(x)\big) = c.$ (E_4)

Given the question, we ask: This tentative proof is correct if we adopt the definition that b is the limit of the f function in a?

- $a \in adh(dom(f) \setminus \{a\})$
- $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in dom(f) : 0 < |x - a| < \delta \Rightarrow |f(x) - b| < \varepsilon\ (*^1)$

If yes, justify the steps (E_1), (E_2), (E_3), (E_4). Otherwise, explain very precisely which steps in this proof are false and give a counterexample.

G46. The given question is equal to the question **G45**.

Given the question, we ask: Based on the analysis of the tentative proof, explain which hypothesis should be added to f to save the proof tentative and ensure its legitimacy by using the following definition of limit.

- $a \in adh(dom(f) \setminus \{a\})$
- $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in dom(f) : 0 < |x - a| < \delta \Rightarrow |f(x) - b| < \varepsilon\ (*^1)$

The tasks are to make an a priori analysis of the given question; to write an answer that serves as a model/correction for the students.

G47. The given question is equal to the questions **G45** and **G46.**

Given the question, we ask:

1. Show that, by adding the hypothesis "g is continuous in $f(a)$" to the property of transitivity, this becomes true, and it is possible to adapt the proof so that it becomes a correct proof.
2. Explain very precisely where the differences are between the proof that uses the continuity hypothesis and the one in the previous question that used the hypothesis "There is an open interval I such that $a \in I$ and $\forall x \in I \setminus \{a\}, f(x) \neq b$"
3. Explain the purpose of the G4 group question from your point of view. Did you learn anything new? Yes, no? If so, what?

Below are our reflections on the two common issues and the others from the G4 group. We chose to focus on this group because the participants are, in theory, the most qualified to conduct an *a priori* analysis of the questions and are supposed to analyze the responses of the participants in groups G1 and G2 and give feedback to group G2, all essential tasks for those interested in GECEMS research.

Reflections on shared issues

One shared question for the groups is to define a sequence limit (G1 and G3) or a function limit (G2 and G4). The answers to this question allow us to observe whether group participants opt for a formal or pragmatic approach and how they justify it. For example, if a participant chooses the definition in words, "*b* is the limit of f(x) as x tends to *a* when f(x) can be as close as one wants to b when x is sufficiently close to a," one must be careful about the interpretation of "sufficiently close" or "can be as close as one wants," because these phrases depend on a previous agreement (only known to the teacher), which will echo in the student's lack of autonomy. We could say that a+5, a+4, a+3... or a-5, a-3, a-1... satisfy any cited sentences, but they are not the values the teacher wants. Another precaution with the text in words is in interpreting "when x tends to a," which has a dynamic connotation that does not always happen and that can justify difficulties in understanding that the limit of a constant function is the constant itself and that the value of the limit does not need to be assumed by the sequence or the function.

> In the case of the concept of limit, we observe that the words 'tends to' and 'limit' have meaning for students before any class on the subject and that students continue to rely on these meanings after they have had the formal definition. Investigations have revealed many different meanings for the expression 'tends to':
> - To approach (eventually moving away from)
> - Approach... without reaching
> - Approach... reaching
> - Opinion (without any variation, such as 'this blue tends towards violet') (Schwarzenberger & Tall, 1978, apud Cornu, 1991, p. 154, our translation)[5]

These ideas justify our reflection on the other issue shared by the four groups, which aims to analyze the teacher's approach and the student's resulting doubt.

> Suppose a colleague (a professor) has introduced the concept of the real limit of a real function along the following main lines.
> - He has students calculate the values of a function f for values increasingly closer to 5.
> - He shows that images $f(x)$ are getting closer to 12 as x approaches 5.
> - He then announces that in such circumstances, we say that 12 is the limit of $f(x)$ when x tends to 5.
> - He then generalizes by introducing the following definition:
>
> "b is the limit of $f(x)$ when x tends to a when $f(x)$ can be as close to b as you want when x is sufficiently close to a ."
>
> A student who followed this teaching challenges you and asks you to explain the following quote since he did not understand the notion of limit.
>
> "Hello, professor, sorry to bother you, but I don't understand how you get from the expression "the closer x is to a, the closer $f(x)$ is to b" to "$f(x)$ is as close to b as we want when x is sufficiently close to a." I feel like there is a connection, but it is too vague for me. Could you give me some explanation, please?" (our emphasis)

[5] In the case of the limit concept, we observe that the words 'tends to' and 'limit' have significance for the students before any lessons begin and that students continue to rely on these meanings after they have been given a formal definition. Investigations have revealed many different meanings for the expression 'tends toward': • to approach (eventually staying away from it) • to approach ... without reaching it • to approach ... just reaching it • to resemble (without any variation, such as "this blue tends toward violet") (Schwarzenberger; Tall, 1978, apud Cornu, 1991, p. 154)

The doubt may arise from how a single table is filled out for a single “a,” with values conveniently “chosen” by the teacher, immediately followed by the definition of the function limit adopted. The responses of the four groups to this question allow us to observe whether the participants understand and justify why the doubt arises, how difficult generalization can be for both values of “a” and different approximations of “a,” and how they propose to answer it. Such an approach can be considered harmful because, by focusing first on the variation of x and then on the variation of f(x), if a student can generalize, he/she can solve exercises such as “calculate the limit of,” but not demonstrate sentences such as “show that the limit of this and that is this and that.” The latter sentences depend on the formal definition, which begins with the variation of f(x) to find the variation of x, ideas that may be important in some situations, both in and out of school life.

Reflections on G4 questions

Section 1.5 already presented reflections on questions G41 and G42, which are common to all groups. G41 corresponds to table filling, and G42 to the definition of limit of a function.

G43: (or G24**).** Take a function $f\colon \mathbb{R} \to \mathbb{R}$ and $a, b \in \mathbb{R}$. In what order will you present the different limit cases to students and why?

a. $\lim_{x \to a} f(x) = b$

b. $\lim_{x \to a} f(x) = \pm\infty$

c. $\lim_{x \to \pm\infty} f(x) = b$

d. $\lim_{x \to \pm\infty} f(x) = \pm\infty$

The answers given by group G2 will allow group G4 to compare them with their own answers and reflect on the students' possible difficulties in infinity questions involving infinity. As part of our reflections, we hypothesized that the pragmatic approach, for example, when filling in tables, can cause difficulties that are not easily overcome. For the calculus teacher, if students already know the behavior of sequences (for example, AP, GP, or Fibonacci), the order can be c, a, b, d, or a, c, b, d, to take advantage of the idea of n tending to infinity, without the use of tables. If the teacher wants to start by filling in tables, we see this as possible in cases a and b since the approximations of x are finite values for a finite a, and one can see what happens to the values of f(x). We consider cases c and d more problematic because the values of x go to plus infinity or minus infinity, with the power of the continuum.

G44. (or G23) A colleague uses the following definition of a limit:
“b is the limit of f in a when one can make $f(x)$ as close to b as one wants, taking x sufficiently close to a” $(*^2)$

Students are divided and understand this definition in different ways. You will find below the different interpretations that are debated. For the sake of clarity, they are formulated in predicate logic.

- « $\forall \varepsilon > 0\ \exists \eta > 0\ \forall x \in Dom(f)\ : |x - a| < \eta\ \Rightarrow |f(x) - b| < \varepsilon$ » (A)
- « $\forall \varepsilon > 0\ \exists \eta > 0\ \forall x \in Dom(f) : 0 < |x - a| < \eta\ \Rightarrow |f(x) - b| < \varepsilon$ » (B)
- « $\forall \varepsilon > 0\ \exists x \in Dom(f)\ : |f(x) - b| < \varepsilon$ » (C)

Complete the following table by crossing the equivalent interpretations.

It is equivalent to ↷	A	B	C	$(*^2)$
A				
B				
C				
$(*^2)$				

The answers given by group G2 will allow group G4 to compare them with their own answers and highlight possible difficulties students may have with sentence $(*^2)$. In our reflections, the student has no reason to interpret "as close as" differently from "sufficiently close," which means that he/she will accept A as equivalent to $(*^2)$, because in this language, x can be equal to a. It may also happen that they do not notice the difference between A and B and consider them equivalent to each other and $(*^2)$. We still consider it possible that a student interprets sentence C as equivalent to $(*^2)$, because they think that a single value of a is sufficient and that the vicinity of a is not necessary, which can be induced by filling in a single table. The question is relevant because its analysis will highlight the difficulties that may arise in a non-formal approach with the use of the mother tongue and, depending on the number of incorrect answers, that the concept of limit was not learned and should be revisited with the G2 participants. One more reason to advocate discussing this issue and its answers with the G3 group.

Questions 5, 6, and 7 share part of the statement; they were only proposed to group G4 and bring to the discussion the formal demonstration of the limit of the composite function, which is one of the most important because most functions are composite. This discussion is critical because it shows that the choice for pragmatism cannot be closed. There are fundamental results that will need a formal approach, and this is one of them.

Joint statement for questions 5, 6, and 7

Consider the following property of so-called transitivity limits and a tentative proof.

Property.

Be $f, g : \mathbb{R} \to \mathbb{R}$ two functions and a, b, c three real numbers such that

- $Im(f) \subseteq Dom(g)$
- $\lim_{x \to a} f(x) = b$
- $\lim_{x \to b} g(x) = c$

So $\lim_{x \to a} g\big(f(x)\big) = c$

Tentative proof.

Consider $\varepsilon > 0$.

There is $\delta > 0\ \forall y \in Dom(g) : 0 < |y - b| < \delta \Rightarrow |g(y) - c| < \varepsilon$. (E_1)

There is $\gamma > 0\ \forall x \in Dom(f) : 0 < |x - a| < \gamma \Rightarrow |f(x) - b| < \delta$. (E_2)

Then$\forall x \in Dom(f) : 0 < |x - a| < \gamma \Rightarrow |g(f(x)) - c| < \varepsilon$ (E_3)

Then $\lim_{x \to a} g\big(f(x)\big) = c$. (E_4)

The questions associated with this statement are

G45. This tentative proof is correct if we adopt the following definition that b is the limit of the function f in a?

- $a \in adh(dom(f) \setminus \{a\})$
- $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in dom(f) : 0 < |x - a| < \delta \Rightarrow |f(x) - b| < \varepsilon\ (*^1)$

If yes, justify the steps (E_1), (E_2), (E_3), (E_4). Otherwise, explain very precisely which steps in this proof are false and give a counterexample.

G46. Based on the analysis of the tentative proof, explain which hypothesis to add to f to save the tentative proof and ensure its legitimacy by using the following definition of limit.

- $a \in adh(dom(f) \setminus \{a\})$
- $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in dom(f) : 0 < |x - a| < \delta \Rightarrow |f(x) - b| < \varepsilon\ (*^1)$

G47. 1. Show that, by adding the hypothesis "g is continuous in $f(a)$" to the property of transitivity, this becomes true, and it is possible to adapt the proof so that it becomes correct.

2. Explain very precisely where the differences are between the proof that uses the continuity hypothesis and the one in the previous question that used the hypothesis "There is an open interval I such that $a \in I$ and $\forall x \in I \setminus \{a\}, f(x) \neq b$"

When organizing the feedback that needs to be given to G2 participants, the researcher can use questions G45, G46, and G47 to give an example of the importance of placing and using $0 < |x - a| < \delta$ in the definition of the limit and G23's resolution. And that the choice for pragmatism can and must be alternated with formalism.

Although we have not presented our reflections on question G35, it brings similar ideas to be discussed in the feedback from G3 to G1 or G3 to G2. For this reason, we argue that, although it is not part of the REM proposal, these results could also be discussed in Group G3.

We conclude, without exhausting, our reflections on the questions proposed to group G4 and conclude by highlighting that the dynamics proposed by GECEMS, with the a priori analysis of the questions and the feedback, will allow the researcher and/or the calculus teacher to develop research instruments to be tested and analyzed and move towards answering the questions posed by the study group: Are the premises used in Belgium valid in Brazilian context? Is it possible to build a shared theoretical framework that allows questioning the status of epistemological obstacle and the main difficulty encountered in teaching the concepts of limits, derivatives, and integrals? What kind of experiment can be planned together to test this model?

The answers suggest that it is affirmative the answer to question posed by Prof. Job on the second day of the seminar.

Toward a shared reference epistemological model?

Given the diversity of the audience, Job begins with a quick introduction to didactics, highlighting that it is part of the exact sciences and that, in Belgium, there is a big difference between didactics and pedagogy. Didactics considers that an essential factor in explaining learning difficulties lies in the knowledge itself and that the specificity of the content is a determining factor in the appropriation of knowledge. In pedagogy, as a science of education, the relationships between teachers and students and students with each other are considered.

As teachers, the Study Group on Calculus in Secondary and Higher Education (GECEMS) members are interested in the process of questioning knowledge, which leads to the notion of epistemological obstacles or, in other words, those obstacles that are intrinsically linked to the nature of knowledge. For mathematics, this idea was developed especially by Guy Brousseau (1998), the founder of the theory of didactical situations.

But what is epistemology? It is a systematic analysis of how and why knowledge is created and how it evolves in the institutions.

The work of the didactician consists of constructing didactic models of knowledge, which will be the case of the targeted REM. To encourage its creation, Job proposes to present what he and his team in Belgium have so far as an epistemological model for teaching calculus and analysis. After presenting this model, Job proposes analyzing assumptions and reasons for choosing questions that are relevant in the Brazilian context, taking as an example those used in the workshop the previous day, since Belgian and Brazilian cultures are very different and the diversity of Brazilian culture is impressive, which makes it difficult, a priori, to make valid assumptions for the entire country without having investigated on site. It is necessary to talk to people and obtain experimental data to confirm or refute the choice of questions to create the Brazilian REM.

As in any scientific discipline, didactics can use a variety of research approaches, such as a *didactic approach* or an *evidence-based approach*. Without intending to force an absolute truth –whether about didactics or issues for teaching calculus–, Job proposes to introduce the *didactic approach* he conducts in Belgium to convince Brazilian researchers that it is an excellent option for developing the REM.

To this end, he aims to create a contrasting effect by exposing why *evidence-based approaches* are highly questionable. Although I accept that this type of approach represents a great advance in scientific research, there are a whole series of gray areas in using this type of methodology in the biomedical world or educational research.

Based on the methodology of *evidence-based approaches*, political decisions were made, such as abolishing repetition in Belgium and other countries because it meant the student would have to attend again a class in which he did very poorly. A big problem with these decisions is that they do not consider causality, and analyzing the various reasons that explain failure should be mandatory. If grade repetition is abolished, students no longer perceive the meaning of learning because they are no longer rewarded for success; they work more to obtain grades than to appropriate knowledge, and one can reach the opposite conclusion: removing repetition leads to failure. Policies promoting success by decree and using standardized tests such as PISA are open to discussion and questioning.

All of this leads back to knowledge, a central aspect of didactics. It is necessary to question knowledge. How do we question it? How do we proceed?

Job proposes two main theories to answer these questions: Guy Brousseau's theory of didactical situations (1998) and Yves Chevallard's anthropological theory of didactics (2011, p. 23).

Based on these two theories and with the support of the didactic transposition tool (Brousseau, 1998), a researcher can build a reference epistemological model for the knowledge involved, which, in this text, is the limit of functions of a real variable. Didactic transposition (Brousseau, 1998) is understood in a broad sense, including notions of didactic contract, which is a set of tacit rules that regulate the interaction between teacher and students regarding knowledge, which is what differentiates the didactic contract from the pedagogical contract (ibidem).

When didactic transposition is combined with the construction of a reference epistemological model, a fundamental methodology for didactics is induced: didactic engineering (Schneider, 2009). This model will then be built based on historical analyses, epistemological analyses, and existing research and used to analyze the transposition of knowledge in different institutions. This allows to build a Fundamental Situation, a set of questions for which knowledge will provide an ideal answer. This is a specific way of characterizing knowledge.

This type of methodology is qualitative, therefore considered phenomenological, and is part of the refutability structure (Popper, 2014), which means that the data produced, and all hypotheses and interpretations, are tested experimentally, refuting the models and research results obtained with *evidence-based approaches.* This is an essential feature because if there are not enough experiments, you are not being scientific. In other words, the phenomenological nature of the methodology used in teaching ensures that the result obtained by research demonstrates the link between an engineering approach and the observed data.

To defend this position, Job says that groups as homogeneous as possible are formed *in evidence-based approaches*. However, when we discuss homogeneity, what exactly are we talking about when working with human beings? Brousseau places this question at the center of the theory of situations with the following question: Are constructivist teaching methods superior to other teaching methods? He concluded that they are not based on an experimental approach of a scientific nature, which eventually introduced a different point of view that contrasts in several conditions with the results in *evidence-based approaches*.

Brousseau (1998) reports that his study object is the interaction of knowledge with the collective. What matters is not the individual psychology of people but rather the notion of an epistemological obstacle, which characterizes the difficulties that will inevitably be encountered in this collective and links them to the specific nature of knowledge.

So, in the constitution of homogeneous groups in *evidence-based approaches*, what is the problem? When students answer knowledge-related questions, one might ask: Why do they answer them correctly? Is it because they look at the teacher's behavior to see if they are on the right path? And reciprocally, the teacher agrees to engage in this approach when he announces: “Yes, you did understand what this is about,” but in fact, it is just a game of connivance between teacher and student.

The above leads us to the notion of a didactic contract (Brousseau, 1998), a series of codifications of the reciprocal behavior of teachers and students concerning the knowledge at stake. These codifications may be implicit but will govern how each party assumes knowledge.

The didactic contract (ibidem) in the didactics of mathematics is a theorization, modeling of human behavior in the face of knowledge. It models interactions between people and knowledge within the classroom, which makes it possible to understand why there are learning difficulties and typical classes of these difficulties. With this, it is possible to highlight epistemological obstacles related to knowledge in mathematics.

Brousseau also highlights the need to consider in this search the functioning of institutions, whose study of development and systematization is one of the aspects proposed by Chevallard (2011, p. 23) in the anthropological theory of the didactic (ATD), so-called because it postulates that institutions take the knowledge produced by scientists, transform it into knowledge to teach and, after that, into that which will be taught. This distortion is called didactic transposition (Schneider, 2008; Chevallard, 1991) and makes us think about the institutional relativity of knowledge.

For the concept of limit, for example, there is nothing in common in the didactic transpositions practiced in some institutions, reinforcing the institutional relativity of knowledge. One approach proposed by the ATD is to study the practices related to a given knowledge in different institutions to identify distortions and points of overlapping or contradicting to explain teaching phenomena and learning difficulties.

In the ATD, the idea of institution is broad and can even be a school of thought, such as the so-called movement of competencies. Based on the triad of wise knowledge, knowledge to teach, and knowledge taught, it is possible to study how currents of thought, visible or invisible, structure the way in which knowledge must be incorporated in different institutions, which, in ATD, is called the scale of levels of codetermination.

When studying the history and epistemology of the notion of limit, one realizes that there were different periods and the notion of epsilon/delta was not present from the beginning. From the distinction between how the limit was used at one time and how it is used today, the main epistemological characteristics of this notion can be extracted, and for this, a new notion is used, that of *praxeology*, which is a set consisting of a task (what?), a technology (how?) and a justification (for what?). The task is what we want to accomplish, the technology is how that task will be accomplished, and the justification is how an institution will legitimize the technique.

In human practices, if you do something, you do it in a particular way, for which there will be a particular level of justification that depends on the context. These different levels of justification are called levels of rationality. For example, in mathematics, levels of rationality are used in a deductive way, strictly deductive way, and others.

Regarding the notion of limit, the reference model will be composed of two praxeologies, a *pragmatic* and a *deductive pragmatic* one. Historically, the limit was given as a technique, among others, to determine geometric or physical measures (especially in kinematics). The way of justifying this technique was of the pragmatic justification type, as the notion was applied to cases that had been resolved by other methods, and there was an agreement between the two.

When this is done in many examples, confidence in the technique increases and reaches a threshold where it is accepted as legitimate because it has been experimentally verified. This level of rationality was standard to much of mathematics until the 20th century, during which formal methods took precedence over any other level of rationality. A feature of pragmatic praxeology is that objects are not constructed deductively or axiomatically, as in current mathematics. They have the status of pre-constructed, according to Chevallard (1991), who says: “A pre-constructed is something that is not constructed but presented by a 'deixis',” which is something that is shown in an appeal to complicity, in ontological recognition. The existence of the object seems to be obvious, not susceptible to doubt; it escapes questioning that presupposes it, and it is the unassailable support point for reflection.

The deductive praxeologies are in the second praxeology of the reference model. The level of rationality is strictly deductive, that is, only deductive justifications are acceptable. The main task is to give calculus a deductive architecture as impeccable as that of Euclid's elements. In particular, all pre-constructed objects will be explicitly constructed, deductively and axiomatically. One technique for this deductive architecture, according to epistemologist Lakatos (1984), is a dialectic of evidence and refutation. Definitions (of mathematical notions and objects), theorem statements, and demonstrations are worked on together.

A theorem is stated, and an attempt is made to prove it. To do this, a series of definitions are chosen, and it is verified whether they allow the demonstration. If not, the definitions or the statement are changed in search of an agreement between these elements. In the case of the constitution of a definition of limit, in deductive praxeology (dialectic of proof and refutation), the definitions are taken, and one that serves as a non-intuitive tool in a deductive architecture is established.

In a historical analysis of the notion of limit, one can take Cauchy’s example, in his courses from 1820 to 1823 at the École Polytechnique, to attest that he used the dialectic of proof and refutation based on the works of his predecessors Ampère and Lagrange. Until recently, this notion of limit was considered the optimal answer to give a deductive structure to calculus in the sense of Brousseau's theory (1998). A fundamental situation regarding the notion of limit has just been described.

It is possible to structure the calculation deductively in a different way, with a non-standard analysis, which proves the idea of the institutional relativity of knowledge and explains why it is complicated to make the notion of limit an object of teaching.

The two pragmatic and deductive praxeologies form a complete cycle and constitute the Belgian reference model. Although disjoint, they work in a dialectical way.

For example, a technique whose justifications are pragmatic and empirical is initially developed to determine measurements in geometric or physical matters. As the discipline evolves, one wants to place these empirical elements on an increasingly solid basis and transform them into something deductive. This constitutes a first cycle of pragmatic modeling of things external or internal to mathematics, at the end of which one tries to make an inversion to constitute a purely deductive theory, with a validation that is necessarily internal to mathematics.

Once this purely deductive theory has been established, physics, geometry, or other subjects are subordinated to it. There is a reversal that only makes complete sense in relation to the preliminary work of determining measurements because it becomes extremely difficult to understand in depth how and why this deductive structuring is reached without it

This is an extremely important aspect because teachers are involved. They get stuck between two extremes, one purely deductive and the other intuitive. Even though the intuitive definition does not mean much, it is chosen by those who consider deductive mathematics too difficult for students and, perhaps, because they do not believe in the possibility of doing pragmatic and, therefore, experimental mathematics.

Job says that he has just given the general outlines of the reference model used in Belgium regarding the notion of limit and that certain aspects of this notion have been left aside. This model was chosen because it brings an interesting perspective and can be tested experimentally, but it is not definitive; it is subject to evolution. Exchanges with Brazilian colleagues can help this model evolve.

Job considers the presentation of the Belgian REM finished and proposes to put one of the questions from the Atelier up for discussion to explain the choice of such a question.

In this question, you can experience a realistic situation in which you are forced to explain to a student who has difficulty with the notion of limits. This situation is realistic because student misunderstandings are encountered in the field. It is not just a figment of our imagination. So, thank you for taking this seriously because it is part of your job as a teacher to learn how to deal with these situations.
Suppose a colleague has introduced the concept of the real limit of a real function along the following main lines.

- He has students calculate the values of a function f for values increasingly closer to 5.
- He shows that images $f(x)$ are getting closer and closer to 12 as x approaches 5.
- He then announces that in such circumstances, we say that 12 is the limit of $f(x)$ when x tends to 5.
- He then generalizes by introducing the following definition:

" $b$is the limit of $f(x)$ when x tends to a when $f(x)$ can be as close to b as you want when x is sufficiently close to a."
A student who followed this teaching challenges you and asks you to explain the following quote since he did not understand the notion of limit.
"Hello, professor; sorry to bother you, but I don't understand how you get from the expression "the closer x is to a, the closer $f(x)$ is to b" to "$f(x)$ is as close to b as we want when x is sufficiently close to a." I feel like there is a connection, but it is to vague to me. Could you give me some explanation, please?" (emphasis added)
When asking the question highlighted in the quote to teachers and students in Belgium, what happened? Why?

To answer these questions, we must know the preliminary context to explain why they were asked in the first place. Provided for in didactics, this procedure is called a priori analysis. The first element of this analysis to be highlighted is that, in Belgium, there are many variations in the use of value tables and the choice of function types.

It is relevant to show the structure of the education system in Belgium, where secondary school lasts six years (around 12 to 18-year-olds). In the fifth year (around 17-year-olds), teachers impart sequence limits, function limits, and derivatives; in the sixth year (around 18-year-olds), integrals, always with functions of a real variable. The textbook adopted for several decades in Belgium is *Espace Math 56*, from which it is possible to take the following example:

« On donne la fonction f : R→R : x→(x^4-x^3)/(4x-4).
a. Détermine le domaine de f.
b. Le réel 1 n'est pas dans le domaine de f. Cependant des réels de plus en plus proches de 1 ont une image par f puisqu'ils sont dans le domaine de f. À cet égard, complète le tableau suivant

x	0.9	0.99	0.999	0.9999		1.0001	1.001	1.01	1.1
f(x)									

c. Les valeurs de f(x) deviendraient-elles proches d'un réel particulier lorsque x prend des valeurs de plus en plus proches de 1 ? Quel serait cet éventuel réel ?
d. [...] » (Espace Math 56, Livre de l´élève, 2000, apud Job, 2011, p. 204)

Initially, students are asked to identify the domain of f and complete the table of f values. Based on the completed table, they are asked: Do f(x) values become closer to a particular real number as x takes on values increasingly closer to 1? What is that real number? The student is completely teleguided, there is no research activity. The phrase "You are deprived of seeing anything; I will tell you what you need to see" is valid. We are in the middle of a contractual effect, in the sense of Brousseau's theory of situations (1998) and the effects of the didactic contract. There is no real knowledge; there is control on the part of the teacher over what the student should see behind those tables. The textbook provides a table with what the student must observe, and, indeed, he should observe that the values of f are close to a specific real number when x takes values increasingly closer to 1. Some other examples allow a freer observation. Anyway, it is a pretend game, because the teacher still tells the student what he/she was supposed to see.

So, the question is: On what grounds can the student rely on whether his/her observation is acceptable? That is not possible. There is a considerable level of indeterminacy with this type of table. For example, one can observe that the values of f are all positive, or that they are increasing at one moment and decreasing at another, or that they are integers or numbers with at most three decimal places. There are infinite possibilities.

Job conducted experiments in Belgium in which students were completely free regarding this type of table and concluded that it does not in any way hinder what the teacher wants. Let them observe and go in all directions, and that is normal.

If the only level of rationality that the student can trust when put into action is the teacher answering yes or no to the question "Is it okay like this, teacher?", this action is jejune, without rationality in the scientific sense of the word. If one takes seriously the epistemological model sought in the activity described for the notion of limit, nothing resembles proof and refutation. The problems of incomprehension observed may be related to the activity type and not the students.

To validate what is appropriate for teaching, one must have a reference epistemological model (Schneider, 2019) and clear ideas, without which the door is open to all kinds of derivations.

Returning to the example in *Espace Math 56*, after noting that the closer x is to 1, the closer f(x) is to 3, the teacher or the textbook introduces a definition of the notion of limit, which he considers intuitive: “If the values of f(x) become as close as you want to *b* when x assumes values sufficiently close to *a*, we say that the limit of f(x) is *b* when x tends to *a*."

There is a leap between the activity and the proposed definition. A limit is defined as the conversion between visual observations (graphical and/or numerical) and a written text. It is not a testing tool; it is, at most, qualified as intuitive, which makes it free from questioning because, if something is considered intuitive, someone who does not understand it feels incompetent and does not speak out.

Job discusses visual observations. Do two people see the same thing? Yes, how could they not? The problem is that, in the case of graphs or tables of values, to understand the notion of limit, you must see and accept what the teacher wants you to see.

In the case of the example of *Espace Math 56*, there are graphs with arrows everywhere: how should a student interpret them? This requires visual conditioning. All you need to do is place your finger on the graph and follow the arrows, especially the ones on the function graph itself. How could one not conclude that the notion of limit and its definition are intuitive? In this view, the notion of a limit is “cinematized,” reinforced by expressions such as “bringing the values of ... ever closer to...”. Students will have a positivist empiricist relationship with the notion of limit (in the sense of the epistemology of science) because this is the *exact copy* of the student's visual behavior scanning over a graph (The term in italics arises from the fact that in positivist empiricism the laws, notions, and concepts of science are an *exact copy* of the objects of the sensitive world).

For the epistemologist Fourez (1988), scientific laws can be discovered independently of any context or project. Scientific models, notions, and laws exist by themselves and are an exact copy of the world. Physical laws exist in and of themselves and are in no way models humans design to understand the world around them. This shows the gap between the modern scientific approach and positivist empiricism fostered by textbooks such as *Espace Math 56*. Next, the author describes what he considers a rigorous definition by epsilons and deltas, followed by a small graph to explain what this means. Something intuitive is moving towards something rigorous, but there is no explanation of how and why this transition is made. There are only graphs with arrows in all directions and occasional gaps to explain them.

From an epistemological point of view, it makes no sense to proceed in this way, but this way of proceeding was brought about by the curriculum counter-reform of Modern Mathematics and textbooks, and teachers follow the curriculum recommendations. Everything that is deductive has been progressively replaced by an epistemology called graphic epistemology, which postulates that, whenever possible, mathematical notions should be accompanied by graphic work.

The above shows the incredible distortion between a Lakatosian point of view and the epistemology defended by the official curriculum, which allows us to understand why graphs and numerical tables are everywhere in the textbook example. Graphic epistemology is terrible because it develops an empiricist relationship in students from a very early age. There is a deep contradiction in education in Belgium between a graphical epistemology, which makes no sense, and teaching based on the deductive structuring of mathematics.

From the history of mathematics, the basis of the reference epistemological model sought, the pragmatic notions of derivative and integral existed long before the notion of limit. But many teachers are still guided by the deductive architecture, inherited from Cauchy and learned during initial education, which does not consider teaching derivative before limit, which can create great contradictions as they try as much as possible to hide the deductive aspects considered too complex.

To put these contradictions into discussion, Job returns to the issue of the Atelier, whose objective was to make prospective calculus teachers question the type of practice found in textbooks –legitimized by teachers and defended by official guidelines– and test whether the participants have an empirical relationship with the sensitive world. This hypothesis arose from a didactic engineering study in Belgium to show how this empirical obstacle manifests in students.

And what results did Job obtain with twelve prospective teachers in Belgium who will work with the notion of limit, a mandatory subject –unlike in Brazil– in the fifth grade (students aged 16/17)? Eight of the twelve interviewees think that the expression "the closer x is to a, closer $f(x)$ is to b" (1) implies "$f(x)$ is as close to b as we want when x is sufficiently close to a" (2). Two of them do not take a position and only two say that the first expression does not imply the second (correct).

What argument is adopted by those who try to prove that 1 implies 2? "I know that the closer x gets to a, the closer f(x) gets to b. So, if we take x sufficiently close to a, I will always get an image f(x) as close as I want to b. That is the reasoning. Then, I can deduce that f(x) is as close as we want to b when x is sufficiently close to a."

Job claims the argument is false and continues the interviews to convince them. To illustrate, in the seminar, Job presents two examples of how the participants expressed themselves.

First example

When we say f(x) approaches b, we understand that f(x) has a value that is increasingly close to b. Thus, f(x) can be as close to b as we want if we take x to be adequate, that is, close to a, it is because it is necessary for x to approach a. So, at some point, x is sufficiently close to a to make f(x) close to b, as we wanted.

Second example

In the sentence: "f(x) is as close to b as you want", there is a notion of precision. You can choose a precision of 0.1 or 0.01 or 10 ^ (-12), or whatever you want. Ultimately, the error is admitted between b and f(x). Thus, "if the closer x gets to a, the closer f(x) gets to b, we can deduce that, if the desired precision has not yet been achieved, it is sufficient to approximate the precision, to bring x closer to a. There will then be a moment when we will be sufficiently close to a."

The hypothesis of why prospective teachers expresses themselves this way and propose this false demonstration is the "hypothesis of the optimistic relationship with the sensitive world."

The context of the engineering experiment with high school students was not the same, but they used similar arguments, which originated from an improper assimilation between the properties of real numbers and the perceptions of the sensitive world. For students, the first expression implies the second because, in the sensitive world, getting increasingly closer to an object means that you always get to that object and always as close as you want.

Is this hypothesis valid for students and prospective teachers? Job proposes three arguments to justify the hypothesis of empiricism.

First argument

Prospective teachers have now been students and, as such, have found this optimistic relationship, which remains even in the face of another type of approach. This duplication characterizes a double discontinuity, which means that secondary mathematics will not be replaced by higher education mathematics or vice versa. Both can live in separate spaces; depending on the context, one type of mathematics or the other will be activated. What is the moral of this? There is no reason for a student to change the relationship they developed with the notion of limits in secondary school during higher education.

Indeed, the academic knowledge acquired in higher education does not allow students to position themselves on high school mathematics. For example, if they had understood the notion of density in different ways, in the case of real numbers, they would know immediately that the first expression does not imply the second one.

Second argument to justify the hypothesis of empiricism

Consider the production of one of the prospective teachers, who explicitly relies on the sensitive world. He says that if x approaches a, the distance separating f(x) from b becomes increasingly smaller and makes an analogy: If you are five meters away from a tree and approach it to one meter, you are no more than four meters away. From there, the distance between the tree and you become smaller as you get closer. And concludes, "You are there." Job says that he makes a table of values there, accompanied by arrows, which effectively suggests that we will be able to bring f(x) as close to b as we want. It is sufficient to do this by taking the x cursor and getting sufficiently close to two.

The third argument to justify the hypothesis of empiricism

Job states that, in the research he conducts in Belgium, this empiricist relationship has also been observed in prospective teachers in initial education, in a whole series of other mathematical knowledge, not only regarding the notion of limit but also regarding derivatives, integrals, complex numbers, and vectors. This type of observation leads him to the hypothesis that it is an epistemological obstacle, as it is persistent over time and is not found in just one knowledge but in many of them. A whole range of difficulties must be researched to see if they are epistemological obstacles.

It is interesting to highlight the interaction of these three arguments and the globality they form. There are others, some of which are based on experimental data acquired over several years. When you put it all together, it is hard to deny the obvious. Job states that didactics is part of the human sciences and sciences.

It is important to continue accumulating experimental data, even on known subjects, because the work involves human beings. It is necessary to update and test research results in new institutions to see how they behave. If one does not do this, one risks becoming blind to one's prejudices.

Due to the phenomeno-technical nature of the tools, they are not neutral and influence what is observed, especially when working with humans. Hence, there is a need to constantly review experimental data in light of the analysis of the tools that made it possible to obtain this data. For this reason, Job's motivation for coming to Brazil is that the reapplication of the research he developed in Belgium could present completely different results.

In conclusion, Job adds that he did not intend to say that one cannot use kinematic perceptions and that using value tables and graphs is forbidden. The sole intention was to draw attention to a use that may deviate from the intended results. These tools can be invaluable, but it is necessary to subtly analyze how they can be used in a pertinent and coherent way. Specifically, regarding the notion of limit, Job claims that using these tools is completely at odds with a Lakatosian understanding of the notion of limit.

Job thinks that secondary and higher education teachers must state in black and white which epistemological project they are part of. There are several that may be coherent. If this is not done, there remains an implicit, graphic epistemology, implicit in some aspects and explicit in others, which infiltrates all levels of the educational system and causes damage as it tends to develop this empiricist position in students.

Another consequence of not studying epistemology enough is that there can be extreme polarization between two points of view: people who want to be rigorous and those who want to be intuitive. Problems on both sides must be discussed because, according to Job, neither side is right.

Why? Firstly, it has to do with rigor, that is, with levels of rationality. There is not only the strictly deductive level of rationality but at least one more, the pragmatic level, which has been used for several centuries. Why discard it? Mainly because both correspond to different and legitimate epistemological positions. In engineering sciences, the pragmatic mindset will prevail over the strictly deductive aspects, which is normal. But it is necessary to assume and live this consciously. The query arises between these two projects and possibly others. Which one will we choose? For what type of students? Detailing, explaining, and developing courses coherently with the chosen project and logic is necessary. For prospective teachers, Job defends acculturation through the integrality of both reference models, that is, sometimes at the pragmatic level and sometimes at the deductive level, because, in secondary education and some parts of higher education, this integrality would offer an alternative stance for teachers to teach calculus.

Instead of implicitly having a conditioning that tends to make them move toward the strictly deductive, they would have the possibility of doing something with their students that is not purely deductive, but that would still be valid from an epistemological point of view. In other words, it is possible to move away from the polarization between being purely deductive and being intuitive, which we do not want to talk about often if, in teacher education, history, and epistemology courses are not superficial. Job spoke about rigor in practice but remembered that intuition can take its place and that, in fact, this promotes the empiricist obstacle.

Is it saying that one does intuitive teaching necessarily better? What are we talking about? This question puts into perspective a naive, widespread idea that any intuition will necessarily be good to follow. Indeed, there is no such risk. The objective is not to condemn everything that is intuitive but, again, to study each type of intuition, each intuition, its relevance, and its validity. Again, this presupposes experimental didactic work, a project that seems to me very underdeveloped, probably due to the difficulty of implementation. It involves studying, on an experimental basis, the institutions and the people in them, their contexts, and which ones are interesting to consider, and which are not. And, once installed, which ones are likely to evolve and under what conditions. If we do not do this work, in ten, 20, or 30 years, we will be in the same sterile debate between "I am an intuitive person" and "I am a rigorous person," which may lead to absolutely nothing.

Conclusion

We understand the importance of concern with calculus and analysis teaching since several studies (Anjos et al., 2023, pp. 1-26; Garcia & Gomes, 2022, pp. 937-957; Godoy &

Almeida, 2020, pp. 50-74; Thomas et al., 2012, pp. 90-136; Igliori & Silva, 2001, pp. 39-67; Lachini, 2001, pp. 146-190; de Guzman et al., 1998, pp. 747-762; Fischbein et al., 1979, pp. 3-40) have shown for years the difficulties in Brazil with specific concepts of the subject and/or with the lack of knowledge of basic education content, such as polynomials, factorizing, real numbers, all fundamental for understanding the ideas present in those subjects. Such research also highlights the high repetition and dropout rates in subjects and courses that include calculus in the first year.

This text highlights the research and questions proposed by the Study Group on Calculus in Secondary and Higher Education (GECEMS): *Are the premises used in Belgium valid in the Brazilian context? Is it possible to build a shared theoretical framework that allows questioning the status of epistemological obstacles and the main difficulties encountered in teaching the concepts of limits, derivatives, and integrals? What kind of experiment can be planned together to test this model?* are valid and relevant and that those interested in participating in this research will find, in the activities proposed in the workshop on the first day, a path towards the elaboration of a shared reference epistemological model for teaching calculus and analysis.

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