EMP Educação Matemática Pesquisa
ISSN 1983-3156
Qualis A1

http://dx.doi.org/10.23925/1983-3156.2025v27i2p392-417

Mathematical knowledge for teaching in the context of inclusive education: an analysis of descending bifurcations in the teaching of the arithmetic mean

Conocimiento matemático para la enseñanza en el contexto de la educación inclusiva: un análisis de las bifurcaciones descendentes en la enseñanza de la media aritmética

Connaissance mathématique pour l'enseignement dans le contexte de l'éducation inclusive : une analyse des bifurcations descendantes dans l'enseignement de la moyenne arithmétique

Conhecimento matemático para o ensino no contexto da educação inclusiva: uma análise de bifurcações descendentes no ensino de média aritmética

Wuallison Firmino dos Santos[1]
Universidade Estadual da Paraíba (UEPB)/Instituto Federal de Educação da Paraíba (IFPB)
Mestre em Educação Matemática
https://orcid.org/0000-0002-2354-6896

Marcus Bessa de Menezes[2]
Universidade Federal do Pernambuco
Doutor em Educação
https://orcid.org/0000-0003-0850-1793

Abstract

This study analyzes a specific episode in the teaching arithmetic mean in a 9th grade class, composed of sighted students and a blind student, in a state public school in Paraíba. Using an analytical approach inspired by the middle structure of the Theory of Didactic Situations, especially the concept of didactic bifurcations, it examines how the professor's didactic knowledge adapted to meet the needs of the visually impaired student. It is observed that the professor introduces different strategies for students access to knowledge, showing sensitivity to the classroom context in which there is a visually impaired student. However, these adaptations can result in a temporary disconnection between the student and the objective set for the class. The conclusions highlight the importance of an analytical approach focused on didactics in the teaching of mathematics in inclusive contexts, which explores the impacts of didactic practices on professors' knowledge.

[1] E-mail: wuallison13@hotmail.com
[2] E-mail: marcus.bmenezes@ufpe.com

Keywords: Teacher knowledge, Inclusive mathematics education, Arithmetic mean, Mathematics didactics.

Resumen

El presente estudio analiza un episodio específico de enseñanza de la media aritmética en una clase de 9º grado, compuesta por alumnos videntes y un alumno ciego, en una escuela pública estatal de Paraíba. Utilizando un enfoque analítico inspirado en la estructuración del ambiente de la Teoría de la Situación Didáctica, especialmente el concepto de bifurcaciones didácticas, se examina cómo los conocimientos didácticos del profesor se adaptan para atender a las necesidades del alumno con discapacidad visual. Se observa que el profesor introduce diferentes estrategias para que los alumnos accedan al conocimiento, mostrando sensibilidad al contexto del aula en que hay un alumno con discapacidad visual. Sin embargo, se identifica que estas adaptaciones pueden resultar en una desconexión temporal del alumno del objetivo trazado para la clase. Las conclusiones destacan la importancia de un enfoque analítico centrado en la didáctica en la enseñanza de las matemáticas en contextos inclusivos, explorando los impactos de las prácticas didácticas en los conocimientos de los profesores.

Palabras clave: Conocimientos docentes, Educación matemática inclusiva, Media aritmética, Didáctica de las matemáticas.

Résumé

Cette étude analyse un épisode spécifique d’enseignement de la moyenne arithmétique dans une classe de 9ème année, composée d'élèves voyants et d'un élève aveugle, dans une école publique de l'Etat de Paraíba. En utilisant une approche analytique inspirée par la structuration du milieu de la Théorie de la Situation Didactique, en particulier le concept de bifurcations didactiques, il est examiné comment les connaissances didactiques de l'enseignant s’adaptent pour répondre aux besoins de l'élève malvoyant. Il est observé que l'enseignant introduit différentes stratégies d'accès au savoir pour les élèves, faisant preuve de sensibilité au contexte de la classe, qui comprend un élève aveugle. Cependant, il est identifié que ces adaptations peuvent entraîner une déconnexion temporaire de l'élève par rapport à l'objectif fixé pour la classe. Les conclusions soulignent l'importance d'une approche analytique centrée sur la didactique dans l'enseignement des mathématiques dans des contextes inclusifs, en explorent les l'impacts des pratiques didactiques sur les connaissances des enseignants.

Mots-clés : Connaissances de l’enseignant, Éducation mathématique inclusive, Moyenne arithmétique, Didactique des mathématiques.

Resumo

O presente estudo analisa um episódio específico de ensino de média aritmética em uma turma do 9º ano, composta por alunos videntes e um aluno cego, em uma escola pública estadual da Paraíba. Utilizando uma abordagem analítica inspirada na estruturação do meio da Teoria das Situações Didáticas, especialmente o conceito de bifurcações didáticas, examina-se como os conhecimentos didáticos do professor se adaptam para atender às necessidades do aluno com deficiência visual. Observa-se que o professor introduz estratégias diferenciadas de acesso ao saber para os alunos, demonstrando sensibilidade ao contexto da sala de aula em que há um aluno com deficiência visual. No entanto, identifica-se que essas adaptações podem resultar em uma desconexão temporária do aluno no objetivo traçado para a classe. As conclusões destacam a importância de uma abordagem analítica centrada na didática no ensino da matemática em contextos inclusivos, que explore os impactos das práticas didáticas nos conhecimentos dos professores.

Palavras-chave: Conhecimentos docentes, Educação matemática inclusiva, Média aritmética, Didática da matemática.

Mathematical knowledge for teaching in the context of inclusive education: an Analysis of descending bifurcations in the teaching of the arithmetic mean

Didactic action in inclusive mathematics education often leads to challenging situations that require in-depth reflection on mathematical knowledge for teaching and its interactions with school inclusion. In this context, this study analyzes a specific episode in which a professor (**P**) teaches arithmetic mean to a 9th grade class, composed of partially seeing students (**A**$_v$)nd a blind student (**A**$_c$)), in a public state school in Paraíba.

This episode was analyzed based on an excerpt from the transcript of the lesson described in Santos' master's thesis (2020). This analysis details the professor's knowledge in relation to the practice of teaching mathematics, allowing us to examine how he applies didactic knowledge in the classroom and how this interacts with the students' responses.

Our approach is inspired by the concept of didactic bifurcations proposed by Margolinas (2004), which denotes the moments when the didactic situation planned by the professor to achieve certain learning objectives diverges due to the way students invest in the proposed task.

Margolinas (2004) uses this term to explain the divergence that can occur between the professor's intentions and the students' realization of the activity. According to this author, students can get involved in problems that, although they seem similar to those envisaged by the professor, lead them along unforeseen paths, making it difficult to construct the planned knowledge.

Appropriating this idea, this study focuses on bifurcations as a concept that reveals different possibilities, now from the perspective of the professor, analyzing how this professional deals with these divergences in the context of the inclusion of a visually impaired student. We have observed that these bifurcations reflect not only the professor's response to the difficulties that arise from the interaction between the student and the knowledge, but also his attempts to adapt the planned didactic situation to inclusive demands, even if these adaptations often don't lead to the desired learning.

These bifurcations can be explained by the professor's limited understanding of school inclusion, especially with regard to the specificities of the students and the knowledge at stake. Although the professor seeks to explore the content of arithmetic mean that should be taught to all students, we observed a lack of reflection on how adaptations to help the disabled student require knowledge in an inclusive context that also considers the knowledge involved.

Our analysis explored the complexity of this interaction between mathematical knowledge for teaching, school inclusion and didactic actions, contributing to a deeper understanding of mathematics teaching in inclusive contexts.

Professor's knowledge

The practice of teaching mathematics requires not only a mastery of the mathematical content but also a deep understanding of the nuances of teaching this specific content. This includes the ability to analyze student errors, explain concepts in an accessible way, and make didactic decisions. An analysis of the knowledge essential to the profession of mathematics professor reveals the intrinsic complexity of this task.

The training of professors has long been the subject of research in Mathematics Education. Shulman (1986) highlighted the importance of teachers' knowledge of the content they teach, as well as the reasons why professors transform their representations of this knowledge throughout their careers.

Mizukami (2004), following this same principle, presents two models on the thinking and knowledge of professors: one related to the knowledge of teaching and the other related to the process of learning this knowledge during training in formal modalities and in professional practice. These studies indicate the existence of a mathematics for teaching.

Shulman (1986) outlined the fundamental elements of this professional knowledge, highlighting the need to understand not only the content, but also the way it is taught by the professor and internalized by the students. The distinction between *Content Knowledge* (CK) and *Pedagogical Content Knowledge* [3](PCK) is essential for understanding the complexity of mathematics teaching and its implications for research in the field of Mathematics Education and for teaching practice.

We have adopted the name "didactic knowledge" for the concept previously known as pedagogical content knowledge (PCK) by Shulman (1986). This change reflects an emphasis on the specificity and application of studies aligned with the French-influenced didactic approach to mathematics. While the term "pedagogical content knowledge" highlights the intersection between content and pedagogy, we opted for "didactic knowledge" to highlight its association with the teaching of specific content, such as mathematics, similar to the distinction between "pedagogical contract" and "didactic contract". In this way, didactic knowledge refers to that aspect of PCK that is related to the mathematical knowledge to be taught.

Ball, Thames, and Phelps (2008) expanded this distinction, proposing a more detailed classification of mathematical knowledge for teaching (MKT). This includes not only *Common Content* Knowledge (CCK), but also *Horizon Content* Knowledge (HCK) and *Specialized Content* Knowledge (SCK).

---

[3] Respectively, the corresponding translation is "content knowledge" and "pedagogical content knowledge".

This SCK is crucially different from common mathematical knowledge (CCK), requiring a depth and detail that goes beyond simply executing algorithms reliably. This mathematical knowledge is not available to other professionals who use mathematics, and is specific to the teaching context. It represents an intersection between mathematical knowledge and pedagogical knowledge, requiring a deep understanding not only of mathematical concepts but also of how to teach them.

Ball *et al.* (2008) emphasize the need for a kind of depth and detail in mathematical knowledge that goes beyond what is needed just to execute algorithms. A particular situation that exemplifies the need for specific mathematical knowledge for teaching is the calculation of arithmetic mean, which is usually understood as knowledge in the domain of the algorithm (Cazorla, Santana & Utsumi, 2019): add up the values and divide by the number of data involved in the sum.

As Ball, Hill, and Bass (2005) point out, mathematical knowledge for teaching encompasses both mathematical and didactic reasoning, establishing a critical intersection between these two domains. This type of knowledge requires a depth and level of detail that goes beyond the simple, reliable execution of algorithms, is specific to the educational context, and is fundamental to teaching practice.

The professor's professional training is essential to deal with the new demands of the educational context, such as the inclusion of students with disabilities in ordinary schools. This adds challenges to their career, requiring them to acquire new answers that were not covered in their initial training, or even to question whether the knowledge they have acquired throughout their career is suited to this new context.

Structuring the environment

We used the model of structuring the environment to examine the professor's didactic knowledge and its impact on both planning and classroom interactions. Margolinas (1995, 2002, and 2004) adapted the model proposed by Brousseau (1986, 1998) to capture the complexity of teaching practices.

Based on the Theory of Didactic Situations (TSD), the model combines the theoretical with the empirical. For reasons of space and the purpose of this paper, we will not explore in detail this modification, which can be found in Margolinas (2004).

Although it may seem linear, this model reflects the complexity of professors' activities, which are more similar to the planning of classroom experiments than to the actual work of teaching.

In order to analyze the daily didactic actions of professors and students in an interconnected way, Brousseau (1986) developed a model of structuring the environment, in which situations nest into each other, creating an onion-like structure. Margolinas (1995, 2004) to include the role of the professor improved this model. A simplified version of this model is presented below.

Table 1.

Levels of professor activity (Margolinas, 2004, own adaptation)

Level +3 - Values and conceptions about teaching (and) learning - Educational project: educational values, conceptions of learning and teaching (in general and/or especially in mathematics).
Level +2 - Global didactic project t - Planning and idealizing the sequence of lessons which is made up of the notions which is made up of the notions to be studied and the knowledge to be built
Level +1 - Lesson Project - Specific didactic project for the lesson: Specific didactic Project for the lesson: objectives and work planning.
Level 0 - Didactic situation - Carrying out the lesson, interacting with the students, making decisions in actions
Level -1 - Observation of student activity - Perception of student activity, regulation of work delegated to students

This representation allows us to characterize the professor's knowledge in different situations of his activity, at different levels. These levels correspond to the different positions that the professor acquires during his or her didactic action in the teaching of mathematics and can be used in the analysis of teaching decisions in order to describe them in a reflective way.

We can describe and interpret the aforementioned levels as follows: at **level +3**, we find a professor at the noospheric level, reflecting on the teaching and learning of mathematics comprehensively. **Level +2** refers to the organization of the teaching of mathematical knowledge, which is divided into concepts to be studied, properties, and applications. At this level, the teacher mobilizes knowledge about the organization of teaching situations and learning objectives.

At **level + 1**: planning is crucial because here the teacher mobilizes more specific knowledge, including knowledge of the students and their difficulties in relation to the mathematical notion being addressed. At **level 0**, we find the professor's implementation of the lesson, where his or her knowledge is manifested through interpretations and/or representations of the students, leading to immediate decisions.

Finally, at **level 1**, the professor observes the students' interaction with the environment he has prepared, identifying possible errors and difficulties in relation to the knowledge at stake.

The different levels interact with each other, i.e., they are not isolated but dynamically intertwined, reflecting the complexity of the professor's work.

A technique that integrates both the professor's point of view, represented by the teaching intention, and the student's point of view, represented by the stage in which the student accepts the situation as his responsibility (Brousseau, 1998), allows a comprehensive analysis of the interaction between the two.

By adopting the professor's point of view, it is possible to better understand the structure of the teaching situation, the professor's intentions in organizing the learning environment, and how knowledge is presented and organized. On the other hand, by considering the students' point of view, it is possible to understand how they receive and interpret the information, identify their difficulties, prior knowledge and level of engagement in learning.

Margolinas (2004) proposes an approach that analyzes both the professor's point of view (top-down analysis) and the student's point of view (bottom-up analysis), providing a holistic view of classroom dynamics. This makes it possible to identify not only what is taught, but also how it is received, understood, and assimilated by students.

Bottom-up analysis starts from the students' point of view, exploring the possibilities of their investment in mathematical problems. It shows that some students can get involved in situations unforeseen by the professor, creating bifurcations in the didactic situation and making it difficult to achieve the planned learning objectives.

This reflection highlights the need to take into account the diversity of students' reactions in the classroom, since we can no longer adopt a generic approach. The schematic representation of *didactic bifurcations*, with a main branch and peripheral branches, illustrates how the tension between the professor's perspective and that of some students can arise when they follow unplanned paths, making it difficult to construct the expected knowledge.

Challenges and strategies: the professor in mathematics education for visually impaired students

In the context of inclusive education, the role of the professor is crucial in organizing learning situations that are accessible to all students, including those with visual impairments. By recognizing the specific challenges faced by these students, the professor plays a key role in adapting teaching to their needs, thus ensuring that everyone has access to knowledge.

To do this, it is essential to understand the different types of access to knowledge (Castillan, 2020) that can be provided to visually impaired students. One of these types is direct access, where the student listens to the professor explain the content to the class. Another form is access through assistive tools, such as apps or devices that allow the student to hear or read the content in an alternative way, such as spoken text or Braille. In addition, adapted access involves providing the student with an adapted version of the material, such as a Braille text, while adapted access through technology involves digitally adapting the material so that it can be accessed on a computer. Finally, third party access occurs when the student receives the information orally from an assistant or colleague.

It is important to note that in addition to providing different ways of accessing knowledge, the professor must also create an inclusive classroom environment in which all students feel valued and capable of learning. This may involve adjusting the presentation of statements and choosing variables that legitimize the difference between the visually impaired students (Morás, 2023), as well as providing additional support when necessary and encouraging collaboration among students.

However, the organization of inclusive teaching situations is not only a question of individual adaptation but also of adapting them to make them accessible to all students, thus promoting the progress of the class as a whole (Nogueira, 2020). This requires a careful understanding of the needs of visually impaired students and the obstacles that all students may face in learning a particular piece of content.

Therefore, organizing learning situations in the context of inclusive education requires a balance between recognizing the individual needs of students and developing strategies that promote the progress of the whole class. By taking a careful and sensitive approach to the needs of visually impaired students, the professor can play a key role in promoting truly inclusive education.

Methodological aspects

Inspired by the models of Ball et al. (2008) and the structuring of the environment proposed by Margolinas (2004), this study uses data from the dissertation of Santos (2020), which focuses on the teaching of statistical concepts to a 9th grade class in which there is a blind student we call Jorge. We selected a specific episode in which the professor addresses the concept of arithmetic mean, using transcript protocols of observed lessons and a semi-structured post-interview.

To enrich our analysis, we will take a top-down approach. We will examine how this perspective manifested itself during the lesson, taking into account the organization of situations for the students.

Top-down analysis is an approach that focuses on understanding the didactic situation from the professor's point of view, starting from the most general levels and moving to ones that are more specific. Margolinas (2004) describes this approach as a process of investigation that starts at level +3 and gradually explores the lower levels down to level 0.

In the section on the structuring of the environment, we present a diagram adapted to the structuring of the environment proposed by Margolinas (2004). However, in the analysis we evoke in more detail elements corresponding to each level, such as the environment (M), the professor (P) and/or the student (E) and the situation (S).

We start at level +3, the noospheric environment (M+3), in which the professor mobilizes naturalized knowledge about teaching and learning, configuring the noospheric situation (S+3). The noospheric professor (P+3) works intuitively, without the need to question this knowledge, since it is deeply rooted in his practices.

At level +2, we have the means of construction (M+2), where the professor makes decisions about teaching the subject in general. They take on the role of constructor (P+2), organizing and structuring pedagogical activities that prepare the content. The construction situation (S+2) reflects these choices and focuses on the preparation of the topics to be worked on.

At level +1, the didactic environment (M+1) refers to the specific conditions for planning lessons. Here, the professor acts as a planner (P+1) who organizes the detailed plan of activities. The reflective student (E+1) does not yet interact directly with the knowledge, but is involved in the activities. The project situation (S+1) involves making decisions about how to conduct the lesson based on this planning.

Finally, at level 0, the learning environment (M0) is the space in which direct interactions between professor and student take place, aimed at the construction of knowledge. The professor (P0) adjusts his actions in real time, while the student (E0) tries to solve the tasks proposed in the didactic situation (S0).

Due to space limitations, the bottom-up analysis from the student's point of view is not covered in this paper. However, based on the a priori analysis from the professor's point of view, we define the didactic situation, highlighting how the professor's knowledge manifests itself at each level and influences his or her didactic actions.

General description of the class

The lesson took place on July 31, 2018, during which the professor focused his instruction on measures of central tendency. We highlight the episode in which the focus was on understanding the arithmetic mean, and provide an overview of the sequential development of the lesson.

The professor used the textbook as his main resource during the lesson. For Jorge, he used chess pieces to represent mathematical concepts, a strategy organized based on the quantities and types of pieces available.

To approach measures of central tendency with the whole class, the professor organized the lesson sequentially, starting with the mean, followed by the median, and finally the mode. In Jorge's case, the professor took a different approach, familiarizing him with the chess pieces and exploring the idea of variables and absolute frequency. Only after this introduction did the professor cover the mode, followed by the median, and finally the mean.

Although the data collected in Santos' research (2020) do not provide an explanation for this difference in sequence, we speculate that the professor's choice to start with the mode is related to the implication of this concept with the frequency of the data, due to the differences in the resources used, which led him to develop this strategy adapted for Jorge.

Below we present the professor's dialogues with the class and with Jorge, specifically in situations involving the arithmetic mean. For the class, he explored the algorithm for calculating the arithmetic mean with a numerical list. Later, he presented and explored this algorithm with Jorge using chess pieces.

The beginning of the interaction between the professor and the class was marked by a quick and confident explanation of measures of central tendency. However, the blind student did not actively participate in the dialogues because the professor separated the moments of knowledge presentation: first for the sighted students and then for Jorge. It was only at this last moment that Jorge had the opportunity to express himself. Below is an excerpt from the episode dealing with the mean.

P: *3, 4, 5, 6, 7, 8... How can I calculate their arithmetic mean?*
A_v: *3+ 4 + 5 + 6 + 7 + 8 = 33.*
P: *Divided by how much? By the amount.*
A_v: *6*
P: *Psst, please be quiet! I need to go over to Gabriel, I mean Jorge. Can I continue?*
A_v: *Yes.*
P: *Thank you! In order for him to understand, Jorge needs to listen. ?* $\frac{33}{6}$
A_v: *5,5.*
P: *Was it understandable?*
A_v: *Yes* (Santos, 2020, p. 122).

Below is the dialog with Jorge about the study of the arithmetic mean:

P: *Now I'm going to teach you the arithmetic mean, which will give you the notion of adding up, analyzing the quantity of all, pawns... and dividing by each piece presented.*

A_c: *Ok.*
P: *You're going to add up the quantity... in this case, the frequency of these variables.*
A_c: *But can you tell? Can you tell?*
P: *How do we know, you're going to separate the variables again, properly.*
P: *Any questions, guys?* [Addresses the class]
A_v: *No.*
P: *I have six classes, six variables, the arithmetic mean is divided by each species. Pass the hand! How many pieces?*
A_c: *Sixteen*
P: *How many variables?*
A_c: *Six.*
P: *These sixteen divided by the variables... If I have sixteen divided by eight, can you do the math?*
A_c: *Two.*
P: *Very good* (Santos, 2020, pp. 126-127).

We observed a misunderstanding in the instruction on division, showing a lack of preparation for dealing with decimal results with Jorge. Below we will explore this aspect and others from the professor's point of view.

The professor's point of view: top-down analysis

Based on the excerpt from the lesson on arithmetic mean, combined with data from the interview, we carried out a descending analysis, highlighting for each level, the environment M, the subjects P and E (representing the role that the subject can recognize or imagine) and the situation S (interaction of the subject with the environment).

We start at level +3, from which we have M3, the noospheric environment, a P-noospheric subject that can be invested in by the professor, which gives us S3, the noospheric situation. Based on our observations of the lesson and the interview, we can refine our analysis of the noospheric situation (S3) and its components. The noospheric situation is made up of the following elements: M3 and P3.

The medium M3' includes the statement "Not having been trained in this, in this part, I have to look for it every day" (Santos, 2020, p. 113), when asked about how to teach a blind student. This suggests that the professor was trained to teach a class as a whole (in the generic sense), which we can consider as M3[4].

The constitution of M3 leads us to believe that P3 may have the following knowledge:

C+3: Orality allows sighted and blind students to learn together. The students must remain silent so as not to interfere with Jorge's learning. He has been trained to teach all

[4] We will use the apostrophe to indicate a variation of the element, considering the complexity of the activity for the professor when dealing with the collective x individual logic (Toullec-Théry, 2015).

students, so oral verbalization is a resource available to the professor, just as listening is available to all students. In principle, everyone can learn from the same situation, Jorge, through listening.

(C+3)': The orality of the proposed situation may be insufficient and the use of other resources is useful for the student to learn mathematics, especially concrete resources, given the student's situation. Since the professor feels that he is not trained to teach students with disabilities, he has to look for additional resources on a daily basis. This student may need individualized activities at certain times that are different from the tasks of his classmates.

This composition helps us to better understand the complexity of the noosphere situation and the knowledge and challenges faced by the professor and the students in an accessible learning environment.

At level, +2 we have M2, the means of construction, P-constructor, the professor who organizes the sequence of content, which gives us S2: the construction situation.

According to the nesting, we have S2 = M3, so the construction situation in this case is "getting students to calculate measures of central tendency". When we analyze the lesson, especially the situation of calculating the mean, we can see that the professor's construction plan began with exploring the mean, then the median, and finally the mode. P2 then has to develop an appropriate mode of didactic intervention.

The corresponding M2 medium will include the professor's knowledge of presentation methods in this order with the data he will explore in class. We can say that this medium includes at least the situations that P2 is familiar with in the context of calculating central tendency. P2's interaction with M2 leads to the following didactic situation: presentation of the calculation of the mean, revisiting the notion of the absolute frequency of a variable from the previous lesson.

For Jorge, M2 is a priori the same, but the professor's construction plan began with a review of the "variables", the absolute frequency of each type (chess pieces), then the exploration of the mode, including the exploration of the aspect of more than one mode, then the median and finally the mean. P2 must then develop an appropriate didactic intervention mode.

The corresponding M2 medium will include the professor's knowledge of the presentation methods in this order with the data they will explore in class. We can say that this medium includes at least the situations that P2 is familiar with in the context of the calculation of central tendency, which, according to the interview, was based on the organization of the situation for the students themselves.

P2's interaction with M2 leads to the following didactic situation: presentation of the calculation of the average, review of the concept of "variable" and the absolute frequency of a variable from the previous lesson. P2 invests in knowledge:

C+2: refers to the sequence of content organized by the professor to build instruction around the calculation of measures of central tendency. This includes the study of the mean, mode, and median, in that order.

(C+2)': this knowledge, branched from the main knowledge, represents a variation of the sequence of contents organized by the professor, specifically for Jorge's case. In this case, the sequence begins with a review of variables, followed by an exploration of the absolute frequency of each class (pawns), then the mode (including multiple modes), and finally the median and mean.

This composition shows how the professor organizes the content according to the needs and characteristics of the students, demonstrating an adaptation of the lesson plan to the specific learning needs of each group of students, which has implications for the knowledge at stake.

At level +1, we have that S1 = M2. The project situation S1 is to get students to calculate the arithmetic averages of a list of numbers. P1, the teacher planner, analyzes in advance (or at the time of the lesson) the impact of the constructed situation on the students' learning. The M1 (didactic medium) with which he interacts are the students' reactions that he observes in the S0 situation or that he has observed in situations that he considers similar.

In Jorge's case, the S1 situation is modified, although he listens to the oral verbalization of the collective moment, the calculation of the average is explored using the quantity of each type of chess piece. P1 invests in knowledge:

C+1: relates to knowing how to calculate arithmetic averages of a list of numbers. This includes understanding the concept of average and the procedures for calculating the average of a set of data.

(C+1)': is a variation of the Core Knowledge related to adapting the S1 design situation to meet the specific needs of a student, such as Jorge, who uses different sensory strategies to perform average calculations. This could involve modifying the task to make it more accessible to him, such as calculating the average using the quantity of each type of chess piece that he hears or feels.

At level 0, S0 = M1, whose design situation is one in which interaction takes place between the professor and the students to teach the concept of arithmetic mean. M0 (learning environment) includes the strategies and methods used by the professor to teach the concept of arithmetic mean. This includes explaining the concept, interacting with the students, and adapting the approach to Jorge's specific needs.

P0 is responsible for conducting the lesson and explaining the concept of arithmetic mean to the class. He controls the flow of the lesson, asking questions of the students and providing clear explanations of the topic. He also interacts directly with George to ensure that he understands the concept through the situation of the chess pieces.

E0 are the students in the class who are learning about arithmetic mean. They interact with the professor during the lesson, answering questions and participating in the activities suggested. Among the students, there are those who use the visual method of learning and Jorge, who uses other strategies adapted to his specific needs. For P0, the following knowledge is mobilized:

C0: the concept of the arithmetic mean and its mathematical properties. This includes understanding how to calculate the average of a set of numbers by dividing the sum of all the values by the total number of values.

(C0): is a variation of the Core Knowledge and involves adapting the explanation of arithmetic mean to meet the specific needs of a student, such as Jorge, who may need a different approach to understanding the concept. This could include the use of concrete examples or manipulatives to aid understanding.

***A posteriori* analysis: the professor's point of view**

By observing how the lesson unfolded, we will deepen the analysis by comparing the professor's point of view with what happened in the lesson, highlighting agreements and/or misunderstandings, and possible new insights from these observations.

At level +3, the professor recognizes the need to look for additional resources on a daily basis to teach the visually impaired student, indicating a lack of specific training in this area. During the lesson, the professor uses tactile resources, such as chess pieces, to make the content accessible to Jorge.

The professor's knowledge of seeking additional resources to meet Jorge's specific needs is reflected in the lesson, in which he uses chess pieces to explain mathematical concepts. This indicates an agreement between the professor's didactic intention and his practice in the observed lesson.

However, we believe that tensions can arise due to the professor's lack of specific training to deal with visually impaired students. This can result in an improvised approach during the lesson. A new insight that can emerge for the professor is the importance of adaptation and the continuous search for resources to ensure inclusion, but more than this, the study of this material and the knowledge targeted for the lesson is essential (Morás, 2023).

At level +2, the professor organizes the sequence of content and resources according to the needs of the class. In the lesson, the professor adapts the sequence of the presentation for Jorge, starting with a review of “variable” and frequency using chess pieces before introducing the concept of mode, median and arithmetic mean.

We highlight a tension between the professor's point of view and the student's understanding in the case below:

> **P:** *Look, here we have the chess pieces; I first want you to sort them into variables.*
> $\boldsymbol{A_c}$: *What do you mean?*
> **P:** *The pawns, the knight, each with their allies, separate them into piles*. (Santos, 2020, p. 123)

The student's misunderstanding represented by “what do you mean?” can be attributed to a lack of clarity in the professor's communication. When the professor mentioned “variables” when referring to chess pieces, the student may have misinterpreted that the professor was referring to different categories or types of objects, rather than understanding that each individual piece represented a specific value of the variable “type of chess piece”.

This confusion may have occurred due to the ambiguity of the term “variable”, which can be interpreted in different ways, especially by students who are not familiar with statistical concepts.

A clearer approach might have been for the professor to explicitly explain that the chess pieces represent the values of the variable "type of chess piece", which is a nominal qualitative variable in which each piece is categorized into a specific class.

We think that there is a partial correspondence between the professor's plan and what happened in class, in which he adapted the sequence of content to Jorge's specific needs. However, when this adaptation is not well planned, tensions can arise that result in a fragmented or disconnected approach to what was developed for the class as a whole. For example, while the numerical list situation was used to outline a sequence for exploring mean, median, and mode, for Jorge the sequence was organized in reverse.

Our discussion does not focus on the appropriateness of this sequence, but highlights, first, that the situation presented to the students may have influenced the professor's decisions regarding the sequence adopted in each case. Second, in Jorge's case, the professor evokes other elements that were not present in the others, such as an explicit relationship between frequency and trends or a discussion about the existence of another mode, for example.

A new insight that can emerge is the importance of a careful and planned adaptation of the content sequence to ensure a learning experience for the student with a disability that, as far as possible and when necessary, dialogues with the situation for the class as a whole (Nogueira, 2020).

At level +1, the professor plans the interaction between him and the students to teach the concept of arithmetic mean, taking into account the specific needs of the class and of the student Jorge. During the lesson, the professor interacts with the students, explaining the concept of arithmetic mean and adapting the approach to ensure that Jorge understands the content.

There is a match between the professor's plan and what happened in class in terms of interaction to teach the concept of arithmetic mean. However, tension may arise if the accommodation is not effective and Jorge's understanding is compromised. Although P2's goal of understanding the arithmetic mean algorithm was achieved, the professor made some adjustments that will be analyzed in Situation S0.

At level 0, the professor implements the strategies outlined, including explaining the concept and algorithm of calculating the mean, interacts with all students through oral verbalization, and makes adaptive adjustments to these interactions to meet Jorge's specific needs. During the lesson, the professor used instructional strategies such as explaining the algorithm and interacting directly with Jorge using chess pieces. The approach of the lesson was adapted to ensure that Jorge understood the concept and the algorithm.

At this level, there is agreement between the professor's plan at level +1 and what happened in class in terms of the teaching strategies used. The professor showed adaptation and sensitivity by using tactile resources and explaining to Jorge, seeking didactic accessibility (Assude, Perez, Suau, Tambone & Vérillon, 2014).

The students followed the professor's instructions - add up the numbers and divide by the number of numbers - in a straightforward manner, performing the calculations as instructed. They added the given numbers and divided the result by the total. This approach reflects the application of a previously taught procedure without necessarily including a deep understanding of the operations (Cazorla et al., 2019).

Understanding the concept of arithmetic mean involves performing the operation of adding the values in a set and then dividing that sum by the total number of values in the set. This operation produces a mean value that represents the central tendency of the data, balancing the contributions of each value.

Interpreting the result involves understanding its meaning in the specific context in which it is applied. This understanding is not limited to performing mathematical calculations, but also to interpreting the results in different situations. In the exemplified case, there is no contextualized interpretation of the situation that would allow the student to recognize the importance of the average, only an understanding of the procedure for calculating the average (Cazorla et al., 2019).

Nevertheless, in the context of the situation presented by the professor, students recognize the relationship between the numbers presented and the expected results. They correctly associate the sum of the numbers with the first part of the operation and the total number of numbers with the second part. This correspondence allows them to arrive at the correct answer for the arithmetic mean.

In Jorge's case, the situation is different; it's not a list of numbers. He can calculate the arithmetic mean of the chess pieces simply by adding up all the pieces (8 pawns + 2 rooks + 1 queen + 1 king + 2 knights + 2 bishops) and dividing by the total number of pieces (6). In this way, he could arrive at the arithmetic mean of the chess pieces (2.6 pieces).

This result, being "continuous", already points to a confusion in our understanding of the nature of the numbers that represent the number of chess pieces. The result of 2.6 as the average number of chess pieces makes no sense in the context of the number of pieces, since we are dealing with a discrete count of pieces.

However, we have even more concerns. When dealing with numbers in a list, calculating the arithmetic mean is a common and useful practice (Cazorla et al., 2019). However, when dealing with different sets of items, such as chess pieces, the context changes.

In Jorge's case, he is working with different types of chess pieces, each of which has an associated quantity. Instead of calculating the average of the chess pieces as a whole, it might make more sense to calculate the average quantity of each type of piece individually. For example, Jorge might do the following:

a) Add up the number of each type of piece (e.g., 8 pawns, 2 rooks, 1 queen, 1 king, 2 knights and 2 bishops).

b) Divide the total sum by the number of a piece type.

This approach would give Jorge an average that would represent the average quantity of each type of piece in his set. For example, if he did the calculations and found that the average number of pawns is 2, this would mean that, on average, he has 2 pawns in his set of chess pieces[5].

This more detailed analysis allows George to better understand the distribution of his chess pieces, and is more appropriate than calculating a general average of piece types representing numerical values. Here, a challenge for Jorge is to identify the frequency of each type of piece. He can recognize that chess pieces are different and that each one has a specific quantity.

We believe that a new insight that can emerge for the professor is the importance of their creativity in adapting their teaching approach to meet the specific needs of the students, but that they cannot underestimate the transformations that the meanings of the mathematical knowledge at stake can undergo by changing the organization and didactic material used.

If for the students, the rational nature of the average did not seem to cause tensions with the teacher's objective, in the situation involving the chess pieces, this became evident when the professor changed the values without explaining to Jorge the reason for the change from dividing 16 by 6 to dividing by 8.

The professor's knowledge and downward bifurcation

Based on the previous analyses, we want to bring to light an interpretation of the professor's knowledge and highlight some observable phenomena that are relevant to our discussions in this paper.

During the lesson, the professor uses direct instructions to explain the concept of arithmetic mean. For example, he says: "You are going to add the quantity, in this case, the frequency of these variables" (Santos, 2020, p. 126). However, when addressing the class, he does not specify the "variables".

In the context of calculating the average of a list of numbers, each die has a frequency equal to 1, while in the situation of chess pieces, this uniformity is not observed; there may be frequencies other than 1, depending on the type of piece, such as pawns, rooks, bishops, or knights. We note that the term "variables" used by the professor actually refers to the different types of pieces, which reveals a misunderstanding in the use of terminology, especially since it has a specific meaning in the statistical context.

This approach demonstrates not only the professor's mastery of the mathematical content, but also his knowledge of how to teach it in a way that is accessible to students. The

[5] Bearing in mind that the pieces presented to him only corresponded to one side of the board.

presence of a visually impaired student requires the professor to publicly adapt his or her knowledge. The particular situation of this student requires the professor to adapt the teaching of mathematics, and perhaps without this particular need, he or she would not feel the urgency to make these adaptations explicit in the teaching of the subject.

Therefore, this raises the question that inclusion opens an area of interest for more detailed studies that explore the adaptations that the professor makes to his or her didactic knowledge in order to make the learning environment more inclusive through different ways of accessing knowledge (Assude et al., 2014; Castillan, 2020).

The professor shows sensitivity to the instructional context by adapting the procedure for calculating the arithmetic mean from dividing 16 by 6 to dividing 16 by 8. We don't have enough information to understand whether this "abrupt" adaptation of the divisor value in this division was for reasons that take into account the discrete nature of the parts, or whether the professor changed it because he thought the student wouldn't be able to do this division.

In either case, this change reflects her understanding of specific mathematical knowledge for teaching (SCK|), which takes into account the nature of the knowledge in question in the first case, or the student's knowledge of Jorge's knowledge in the second case. The professor makes an adaptation, which in any case is the result of the context of the chess pieces used in the activity.

In this scenario, and based on the top-down analysis, we have identified a phenomenon that has essential similarities with the concept of didactic bifurcations outlined by Margolinas (2004). The author points out that students can find himself or herself in mathematical situations that seem similar at first glance, but are unforeseen by the professor, which can make it difficult to achieve the learning objectives.

This observation leads us to reflect on how the professor, aware of the need to make knowledge accessible, breaks down the domain of "teaching the arithmetic mean algorithm from a list of numbers" into other knowledge that takes into account the specificities of the visually impaired student.

In this sense, we understand that we can no longer adopt an exclusively generic approach from the professor's point of view; it is imperative to consider the adaptations he makes to the situation for the whole class, in an approach that particularizes the learning experience for the disabled student. In the case analyzed, we can represent this phenomenon as follows:

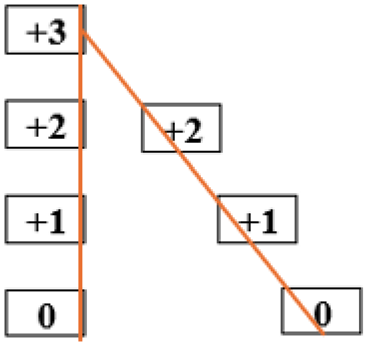

Figure 1.
Downward bifurcations from the professor's point of view (own elaboration)

To detail the path shown in Figure 1, we used the logic of descending branches from the professor's point of view, starting from the highest level (+3) to the lowest (0). The main path (on the left) represents the generic approach intended for the whole class, while the marginal branch (on the right) illustrates a specific adaptation made for Jorge.

At level +3, the professor draws up the general plan for the whole class, using oral verbalization as the main means of common access to knowledge for all students. The idea is that all students, including Jorge, can understand the subject being studied by listening to the professor's explanations.

However, recognizing that access for Jorge should occur in a different way, at level +2 the professor introduces an adapted access to teach measures of central tendency. At first glance, we might think that this is the same knowledge: teaching measures of central tendency. However, observing how the lesson unfolded, we noticed, for example, that the order in which these trends were presented was different for the class as a whole from for George, as well as the repetition of the absolute frequency, which was explained only to George.

We think that there is a connection at level +3, because we think that the access through the oral verbalization of the professor was accessible to Jorge. We even think that the professor doesn't realize that the situation adapted for Jorge has already changed in principle, since the verbalized lesson, which was aimed at the whole class, has already made the student develop knowledge about the knowledge in question.

We are not questioning the motives of the professor to separate the time for individual support for Jorge. That is possible for any student. Our purpose here is to point out that it is often imperceptible to the professor that this individual attention can become disconnected from the "thread" of the lesson he has planned for the class, and that there are risks in his control of situations that are built "in parallel".

Limitations and potential, especially the former, often lead the teacher to separate a group, even a single student. However, in these cases, it is difficult to achieve the established objective in the didactic time of the class, as they often have to resort to concepts they have seen before (Toullec-Théry, 2015).

In the case analyzed, we noticed that the professor only reviewed the frequency for Jorge. In the data found in Santos' dissertation (2020), we did not find information about whether the professor worked simultaneously with the other students, considering a list of data with a frequency other than 1.

In this sense, Jorge was temporarily disconnected from the didactic time of the class. In this case, the professor has no didactic knowledge of the implications of dealing with similar, perhaps different, situations. This is problematic for professors who constantly claim to be incapable of teaching students with disabilities because they have a perspective that focuses on the class as a whole (Costa, 2007; Mesquita & Sousa, 2023).

What seems to prevail for the professor in these cases is: how to connect or stay connected to the main "thread" that the professor is aiming at in these cases? Where possible, it is necessary to understand that the professor is also a subject of the inclusive policy and, supported by the knowledge that everyone learns by participating together, must recontextualize the situation that seemed separate for one student for the whole class. Why couldn't the situation with the pieces have been applied to everyone? The principle of inclusion of sets applies: the situation of the chess pieces is included in the situations that can be set up by the professor for everyone.

Therefore, we treat this case as a descending bifurcation. Margolinas (2004) notes that students may invest in situations not planned by the professor. In the case analyzed in this study, we identified a branch that bifurcates when the professor seeks solutions to the tension between collective work and individual work, especially in the case when there are students with disabilities in the class, which is a visible difference for the institution.

By this, we mean that other bifurcations could have occurred in the classes, but they may be visible to the professor who pays careful attention to his students' production. In the case analyzed, Jorge is an institutional subject with a visible difference: a visually impaired student.

In general, the interaction between the teacher's point of view, who expects the students to understand and correctly apply the procedure taught, and the student's point of view, highlights the importance of the professor's continuous monitoring of the students' progress and of adapting his teaching approach to the individual needs of the students, as in the case of Jorge, who may need a more adapted explanation due to his visual impairment.

Final considerations

The analysis carried out in this paper, anchored in the elements observed during the lesson, reinforces the importance of the different types of teacher knowledge and highlights the need for an adaptive and context-sensitive approach to teaching mathematics.

Our analysis sought to explore the complexity of this interaction between mathematical knowledge, school inclusion and didactic actions, with the aim of contributing to a deeper understanding of mathematics teaching in inclusive contexts.

In analyzing the adaptation of the lesson to the needs of a visually impaired student, we observed that the teacher used different strategies to access knowledge, demonstrating sensitivity to the classroom context, which points to the need for research that discusses mathematical knowledge for teaching in inclusive contexts.

For example, at several points in this study, the need for didactic knowledge for inclusion or even about inclusion was identified. The inclusion of assistive devices can lead teachers to believe that inclusion is achieved simply by the presence of these devices, without necessarily understanding how this affects their teaching actions and without recognizing the need to develop specific knowledge to deal with this situation.

For example, although oral verbalization was a common form of access for all students, the teacher introduced specific adaptations for Jorge, recognizing the need for differentiated approaches. However, we found that the teacher chooses a "different" situation when looking for an accommodation. These accommodations can result in the student being temporarily disconnected from the rhythm of the class, highlighting the importance of a more inclusive approach.

This observation raises questions about how teachers can improve their practice to ensure collective participation, recognizing that inclusion is not only an individual responsibility, but also a systemic issue that requires a holistic approach.

We still lack comprehensive studies that explore the organization of didactic situations by teachers in inclusive environments, with a didactic analysis that seeks to identify the impact of these practices on teachers' knowledge. Such research could stimulate important debates and serve as a knowledge base for teachers facing challenges such as the inclusion of students with visible, institutionally recognized differences.

The study of descending bifurcations in this context seems promising because, although we are still in a case study analysis, further studies can be conducted considering the diversity that inclusion brings, as well as the different approaches and responses of teachers to inclusive education policies.

More generally, we would like to point out that the use of the theoretical-methodological tool of structuring the environment offers us the possibility of analyzing teachers' knowledge in all its complexity.

Acknowledgment

This work was carried out with the support of the Cordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brazil (CAPES) - Funding Code 001, through the Doctorate-Sandwich Program Abroad (DSPA), which provided the first author's research internship at the *ACTé* laboratory of the Université Clermont Auvergne, in France, under the supervision of Professor Claire Margolinas and Professor Olivier Rivière. We would also like to thank the Instituto Federal de Educação, Ciência e Tecnologia da Paraíba (IFPB) for its institutional support, made possible by the leave for qualification granted by Ordinance 1791/2023 of the Rectory/IFPB of October 3, 2023. We thank Silvia Iacovacci for translating this version of the article into English. Her revision and translation work significantly contributed to the quality and clarity of this text.

References

Assude, T., Perez, J.-M., Suau, G., Tambone, J., & Vérillon, A. (2014). Accessibilité didactique et dynamique topogénétique : Une étude de cas. *Recherches En Didactique Des Mathématiques*, *34*(1), 33–57. https://revue-rdm.com/2014/accessibilite-didactique-et/

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing Mathematics for Teaching: Who Knows Mathematics Well Enough To Teach Third Grade, and How Can We Decide ? *American Educator*, Fall *14*–46. http://deepblue.lib.umich.edu/handle/2027.42/65072

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content Knowledge for Teaching: What Makes It Special? *Journal of Teacher Education*, *59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Brousseau, G. (1986). La relation didactique : le milieu. *Actes de la IVème Ecole d'Eté de didactique des mathématiques*, Paris : Irem Paris 7.

Brousseau, G. (1998). *Théorie des situations didactiques*. Grenoble : La pensée sauvage Editions.

Castillan, L. (2020). *Améliorer l'accessibilité des manuels scolaires pour les élèves déficients visuels* [Tese de Doutorado]. Université Toulouse le Mirail - Toulouse II]. https://theses.hal.science/tel-03431384

Cazorla, I. M., Santana, E. R. dos S., & Utsumi, M. C. (2019). O campo conceitual da média aritmética: Uma primeira aproximação conceitual. *Revista Eletrônica de Educação Matemática*, *14(1)*, 1–21. https://doi.org/10.5007/1981-1322.2019.e62827

Costa, M. C. S. (2007). *Sentimentos de professores frente às dificuldades na prática da educação inclusiva de alunos com deficiência no ensino fundamental* [Dissertação de Mestrado em Psicologia da Educação, Pontifícia Universidade Católica]. https://tede.pucsp.br/bitstream/handle/16330/1/Maria%20Cristina%20%20Sanchez%20da%20Costa.pdf

Margolinas, C. (1995). La structuration du milieu et ses apports dans l'analyse a posteriori des situations. *Recherches en Didactique des Mathématiques*, La pensée Sauvage, 89-102.

Margolinas, C. (2002). Situations, milieux, connaissances : Analyse de l'activité du professeur. Actes de la 11éme Ecole d'Été de Didactique des Mathématiques: Grenoble: La Pensée Sauvage.

Margolinas, C. (2004). *Points de vue de l'élève et du professeur. Essai de développement de la théorie des situations didactiques* [Habilitation à diriger les recherches en Sciences de l'Éducation, Université de Provence - Aix-Marseille I]. https://theses.hal.science/tel-00429580

Mesquita, D., & Sousa, C. (2023). Sentimentos dos professores perante situações de inclusão. *CADERNOS DE PESQUISA: PENSAMENTO EDUCACIONAL*, *18(1)*, 115–128. https://doi.org/10.35168/2175-2613.UTP.pens_ed.2023.Vol18.N50.pp115-128

Mizukami, M. da G. N. (2004). Aprendizagem da docência: Algumas contribuições de L. S. Shulman . *Educação*, 33–50. https://periodicos.ufsm.br/reveducacao/article/view/3838

Morás, N. A. B. (2023). *Um Dispositivo Didático Com Potencialidades Inclusivas: Um estudo a respeito de problemas de estruturas aditivas com números naturais*. [Tese de Doutorado em Ciências e Educação Matemática, Universidade Estadual do Oeste do Paraná]. https://tede.unioeste.br/handle/tede/6618

Nogueira, C. M. I. (2020). Educação Matemática Inclusiva: Do que, de quem e para quem se fala? In A. M. M. R. Kaleff & P. C. Pereira, *Educação Matemática: Diferentes Olhares e Práticas*. Editora Appris.

Santos, V. L. (2020). *Análise sobre o fenômeno da transposição didática interna no ensino de estatística: Um estudo com a inclusão de um aluno cego em uma sala de aula regular*. [Dissertação de Mestrado em Educação Matemática, Universidade Estadual da Paraíba]. http://tede.bc.uepb.edu.br/jspui/handle/tede/3922

Shulman, L. S. (1986). Those Who Understand: Knowledge Growth in Teaching. *Educational Researcher*, *15*(2), 4. https://doi.org/10.2307/1175860

Toullec-Théry, M. (2015). *Des politiques françaises en matière d’éducation centrées sur l’individualisation, la personnalisation plus que sur le collectif: Quels effets sur les apprentissages des élèves ?* França: CNESCO. https://www.cnesco.fr/wp-content/uploads/2016/09/toullec1.pdf