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Integration of pragmatic and deductive approaches in teaching limits: concept and theorem in action

Integración de enfoques pragmáticos y deductivos en la enseñanza de límites: concepto y teorema en acción

Intégration des approches pragmatiques et déductives dans l'enseignement des limites : concept et théorème en action

Integração de abordagens pragmáticas e dedutivas no ensino de limites: conceito e teorema em ação

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Abstract

This study investigates the construction of the concept of the limit of a function, a central challenge in calculus and analysis teaching. Using two empirical studies, one completed, based on the theory of conceptual fields, and another in progress, based on the anthropological theory of didactics, we developed tasks that integrate pragmatic and deductive epistemological models. Preliminary results suggest that this integration can help students understand the subject and reduce the fragmentation of scientific knowledge in the area. We conclude that applying these theoretical approaches provides a more cohesive and effective vision in teaching the limits of a function.

Keywords: Limits, Theorems and concept in action, Pragmatic and deductive praxeologies.

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Resumen

Este estudio investiga la construcción del concepto de límite de una función, un desafío central en la enseñanza del cálculo y el análisis. A partir dos estudios empíricos, uno concluido, basado en la teoría de campos conceptuales, y otro en curso, basado en la teoría antropológica de lo didáctico, desarrollamos tareas que integran modelos epistemológicos pragmáticos y deductivos. Los resultados preliminares sugieren que esta integración puede ayudar a los estudiantes a comprender la materia y reducir la fragmentación del conocimiento científico en el área. Concluimos que la aplicación de estos enfoques teóricos proporciona una visión más cohesionada y efectiva en la enseñanza de los límites de una función.

Palabras clave: Límites, Teoremas y concepto en acción, Praxeologías pragmáticas y deductivas.

Résumé

La présente étude examine la construction du concept de limite de fonction, un défi central dans l'enseignement du calcul et de l'analyse. À l’aide de deux études empiriques, l’une achevée, basée sur la Théorie des Champs Conceptuels, et l’autre en cours, basée sur la Théorie Anthropologique de la Didactique, nous avons développé des tâches visant à intégrer des modèles épistémologiques pragmatiques et déductifs. Les résultats préliminaires suggèrent que cette intégration peut minimiser les difficultés de compréhension des étudiants et réduire la fragmentation des connaissances scientifiques dans le domaine. Nous concluons que l’application de ces approches théoriques fournit une vision plus cohérente et plus efficace de l’enseignement des limites fonctionnelles.

Mots-clés : Limites, Théorèmes et concept en action, Praxéologies pragmatiques et déductives.

Resumo

Este estudo investiga a construção do conceito de limite de função, um desafio central no ensino de cálculo e análise. Utilizando duas pesquisas empíricas, uma finalizada, com base na teoria dos campos conceituais, e outra em andamento, fundamentada na teoria antropológica do didático, desenvolvemos tarefas que visam integrar modelos epistemológicos pragmáticos e dedutivos. Os resultados preliminares sugerem que essa integração pode minimizar as dificuldades de compreensão dos alunos e reduzir a fragmentação do conhecimento científico na área. Concluímos que a aplicação dessas abordagens teóricas proporciona uma visão mais coesa e eficaz no ensino de limites de função.

Palavras-chave: Limites, Teoremas e conceito em ação, Praxeologias pragmáticas e dedutivas.

Integrating pragmatic and deductive approaches in the teaching of limits: concept and theorem in action

This study is part of the project *Conception of an Epistemological Reference Model (ERM) for the Concept of Limits as a Tool for Federating Research in Didactics on Calculation and Analysis,* a multi-institutional research project funded by the National Council for Scientific and Technological Development (NCST), which aims to propose an epistemological reference model (ERM) for limits that will allow an in-depth filtering of the epistemological coherence of research in didactics and provide a grid for reading the epistemological coherence of institutional practices for teaching this knowledge.

The actions of this project, developed through tasks proposed in 5 higher education institutions, in the 5 regions of Brazil, were designed to try to observe how students, teachers, and researchers understand and develop their actions regarding the teaching of limits.

The researchers' choice of this object is salutary, since it has been a central concern of mathematics education researchers for several decades (Tall & Vinner 1981; Cornu, 1983, 1991; Schneider, 1988; Tall, 1991; Job, 2011). This centrality is not accidental. If, on the one hand, limits constitute the cornerstone of the integration between calculus and analysis, on the other hand, the recurrent and persistent difficulties in understanding and appropriating this concept are attested by several researchers around the world (Tall & Schwarzenberger, 1978; Tall, 1991; Williams, 1991; Bezuidenhout, 2001; Parameswaran, 2007; Schneider, 1988; Job, 2011; Job & Schneider, 2014).

Despite decades of research on the topic, researchers have yet to find a teaching situation that addresses all of these difficulties (Job & Schneider, 2010). At the same time, this body of research has led to something of a paradox. Researchers have not been able to mitigate the gaps in this problem, but they have multiplied the visions and approaches related to the study of teaching limits, leading to a kind of fragmentation of scientific knowledge in the field (Job & Schneider, 2010; 2014). We believe that this fragmentation occurs because the theoretical frameworks of this research can be difficult to articulate, which raises the question of the meaning of the research results obtained and how to link them.

As a result, we have brought to this discussion elements produced in specifically two studies, one already completed and the other in progress, which are part of the above-mentioned project. These studies were carried out as a way of constructing the above-mentioned ERM, but it considers work already completed and that is in progress so that they can contribute to the process of constructing the concept of function limit with students in Brazil, in the subject of Calculus I, in the mathematics course, both at the undergraduate and graduate levels.

The concept of the function limit has been the subject of several studies because it is considered difficult to understand, a fact with which most experienced teachers of differential and integral calculus agree (Baldino, 1995; Giraldo, 2004; and Tall, 1991). There are even arguments against teaching it to students in courses such as engineering, which are considered more appropriate for mathematics courses (Fernandes, 2015). However, studies indicate that even in these courses, students have great difficulty with various concepts that are mobilized to deal with situations involving the construction of the concept of limit, such as the set of real numbers, functions, and the concept of infinity, concepts that are considered difficult to understand (Artigue, 1995).

In one of our studies, we studied the actions of students in dealing with situations to introduce the concept of the limit of a function, based mainly on the Theory of Conceptual Fields (TCF). First, we delimited the concepts involved in the construction of the concept of limit in the institutions, in this case Brazil and France, which would be studied. As well as the level of teaching at which this concept is introduced and the difficulties listed in the research. The methodology of this completed study was based on the application of activities, questionnaires, and interviews. The actions of the students in dealing with the tasks implemented by these instruments were modeled according to the analysis methodology constructed. In this specific case, they are the rules in action, which are responsible for the temporal management of the subject's actions, involving means of acquiring information and, at the same time, controlling the action, and the theorems in action, which are the mathematical knowledge imbricated in the rules in action, responsible for their organization. These theoretical elements allowed us to study important filiations and ruptures in the students' schemes, and thus to investigate the process of constructing the concept of limit.

The other, ongoing research, is based on the Anthropological Theory of Didactics (ATD), since this theory makes it possible to study mathematics through human anthropology, based on activities, considering institutions, people, objects, and their relationships with each other and between them. These activities are considered as praxeologies (Chevallard, 1999), in other words, it is a theory that seeks to explain the logic behind human actions, based on the idea that people act intentionally and with purpose, especially human actions in the process of understanding the concept of limits of functions. The study by Job and Schneider (2014) characterizes the difficulties in both teaching and learning limits as related to epistemological obstacles, given the primacy of the dominant epistemological models in institutions. In this sense, these authors point out that the epistemological models that structure the teaching of limits are basically composed of two praxeological models: pragmatic and deductive. The pragmatic praxeological model lives in the intuitive notion of limits through the manipulation of numbers, tables, and graphs. The deductive praxeological model is structured by a formal definition using ε and δ.

The methodology of this second study, which is part of the larger study that is the construction of a ERM, was based on the development of proposed tasks, through Ateliers and Engineering (Artigue, 1988), and the confrontation of the praxeologies expected by the institution (teachers, researchers, universities) with the personal praxeologies developed by the students in the proposed tasks. These results have made it possible to construct didactic sequences that allow the integration of the praxeological models mentioned above, minimizing the rupture of the teaching of limits by the moments of teaching by intuitive notion and formal definition.

Theoretical elements involved in this study

Constructing the concept of the limit of a function involves several concepts that are considered difficult for students to understand, such as the function and the set of real numbers. Artigue (1995) argues that it is important to consider this when we are interested in learning the concept of limit. Thus, one of the studies carried out in this article is based on Vergnaud (2009). For this author, a concept cannot be analyzed in isolation; if we are interested in learning it, we need to consider a set of situations involving this concept, as well as the set of concepts involved in these situations. Vergnaud therefore proposes the study of the conceptual field.

It is through these situations that the concept acquires meaning for the student, and it is by studying it in action—in other words, by dealing with these situations—that we can study the process of knowledge construction (Vergnaud, 2009). To solve a situation, the learner mobilizes what the author calls a scheme, which is a way of organizing action for a set of situations that they recognize as similar to others they have already experienced. A schema includes: “An objective, sub objectives, and anticipations; rules in action for acquiring information and control; operative invariants: concepts in action and theorems in action; and possibilities for inference in a situation” (*ibid*., 2009, p. 21).

Thus, when faced with a situation, the student identifies what needs to be done, the goal, and through the rules in action for taking information and control, the operative invariants are selected or activated, which are the mathematical knowledge needed to deal with the activity. In other words:

Conduct is not only made up of actions, but also of the information needed to continue the activity, and the controls that allow the subject to be sure that they have done what they thought they were going to do and that they are continuing along the chosen path.

[...]

Even more decisive from a cognitive perspective are the operant invariants, since concepts in action make it possible to extract the relevant information from the environment and select the theorems in action necessary for calculating, at the same time, the objectives and sub-objectives that can be formed, and the rules in action, for taking information and for control, which make it possible to achieve them (Vergnaud, 2009, p. 22).

It was these operative invariants mobilized in the student's actions that allowed us to study the knowledge mobilized by the students when dealing with situations to construct the concept of function limit. Thus, we modeled the student's productions in terms of rules in action and theorems in action. A summary of this modeling analysis can be seen in Figure 1.

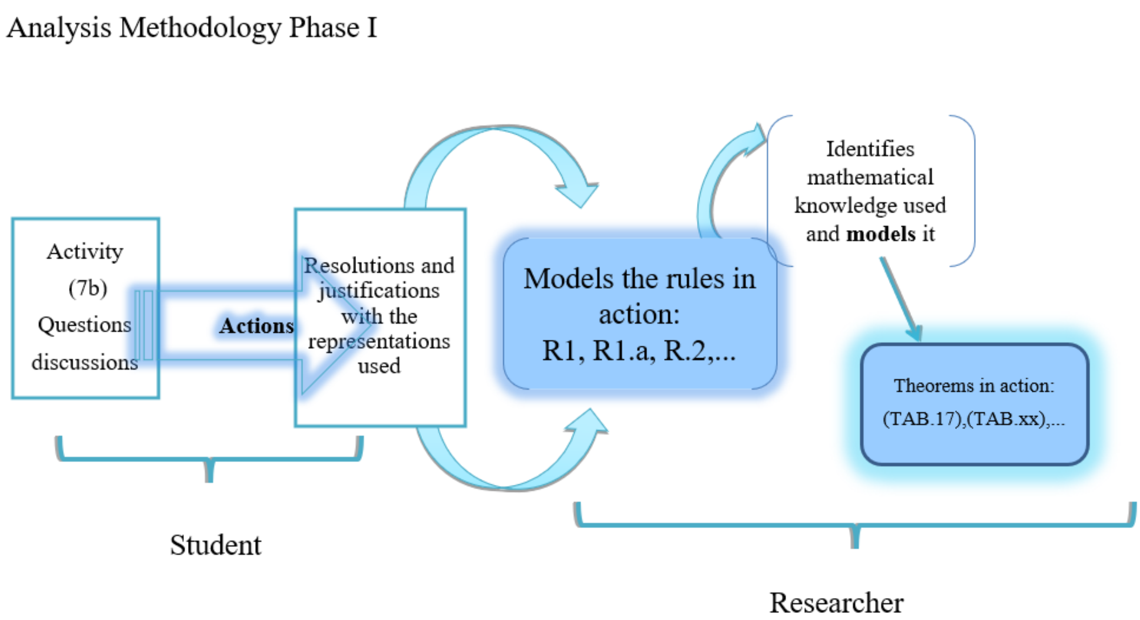

Figure 1:

Research analysis methodology (Source: Phase I of Burigato's Analysis methodology, 2019, p.88).

We can refine our analysis of student's actions using ATD (Chevallard, 1999). Within this theory, we find elements for deconstructing and reconstructing mathematical praxeologies. Mathematical praxeologies can be represented by [T, τ, θ, Θ] and structured by the blocks of knowledge (praxis) [T, τ], which focuses on practical knowledge, and knowledge (logos) [θ, Θ], which is the epistemic discourse that justifies the choices of praxis. We understand that the integration [T, τ]$\leftrightarrow$[θ, Θ] makes it possible to construct a model of (mathematical) knowledge that postulates that any activity that can be conceptualized as a task (T), for example, calculating a limit (f(x)=L), is a technique (τ) that is generally one of many ways of calculating a limit, depending on the task (T). If the task involves calculating the limit using a graphical representation of a function, the student can use intuition (τ_1), in this case visual intuition. In addition, to determine the limit of an algebraic function, a student can perform this calculation using algebraic expressions (τ_2).

The technique (θ) is a justification of the technique (τ) used to solve the task (T), or rather, a technique (θ_1) for (τ_1) is intuition, i.e., visually, the closer *f* is to *L*, the closer *x* is to *a*. The technology (θ_2) must be the logic of predicates ($\exists$ - exists; $\forall$ - for all) and modular inequalities ($0 < |x- a| < \delta \rightarrow |f(x) - L|< \varepsilon$). Theory ($\Theta$), on the other hand, which is a more abstract level of justification than technology for these examples, is the density of real numbers, i.e., there will be a real number between any two real numbers.

Based on the ideas proposed by Chevallard (1992) and the praxeological elements mentioned above, it is possible to represent the actions of the subjects in the institutions in relation to the knowledge developed in the classes, understanding the mathematical activity as any other human activity incorporated in the institutions. These theoretical elements help us to understand how boundary knowledge has been developed, thought, and its teaching organized over time. It is through these observations that a praxeological reconstruction becomes possible to minimize the gaps in the understanding of the concept of limits (Doumbia, 2020).

These observations are essential for the constitution of a ERM based on historical and epistemological aspects of limits, which allows us to raise some questions: What knowledge was part of the knowledge of limits? What structures do not yet exist? Why was knowledge about limits organized in this way? How did this knowledge change over time? These questions are fundamental to understanding human action in the process of the evolution of the knowledge in question and, consequently, the institutional practices of the subjects who teach and study limits.

For the process of praxeological reconstruction, we approached the work of Job and Schneider (2010, 2014). These authors point out that there are two praxeological models that can be used in the teaching of limits: pragmatic and deductive. The first has a more dynamic proposal through the manipulation of the intuitive notion, using the expressions “tends to” or “approaches” loaded with large reinforcements in tables, graphs, algebraic expressions, etc., mobilizing the general argument of “lack of precision” and “lack of symbolization”. The second, on the other hand, has a static proposal, using the formal definition through ε and δ, and starting precisely with the following definition: Let f be a function and a be a point contained in the domain of f. We say that f has a limit L, at point a, if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that, for any x belonging to the domain of f, the following condition is satisfied ($0 < |x - a| < \delta \rightarrow |f(x) - L| < \varepsilon$). The limit L, if it exists, is unique, and we represent it by: $\lim_{x \to a} f(x) = L$.

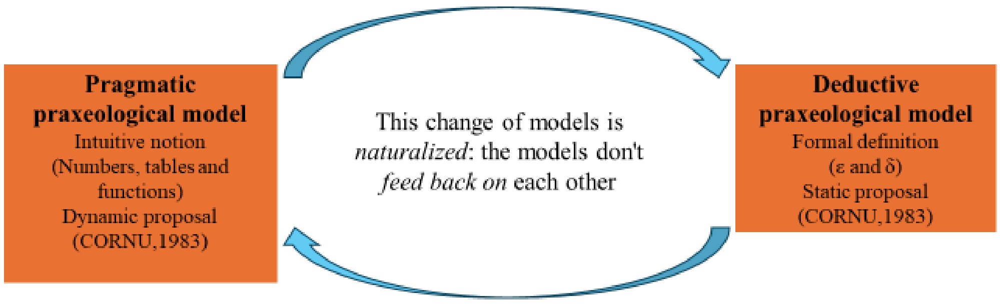

Figure 2.

Naturalization of change in Praxeological Models: pragmatic and deductive (The authors)

In this sense, while in pragmatic praxeology the task (T) consists in evaluating the characteristics that exist as objects that have not yet been formally defined, the techniques (τ) are justified with pragmatic arguments and the validation of the technique, the technology (θ), with deductive arguments, that is, the justification of pragmatic praxeology is based on deductive praxeology. This phenomenon is shown by Farias, Carvalho and Teixeira (2018) as *the loss of the raison d'être* for teaching the concept of limits, which is characterized by the "disappearance of the meaning of knowledge in the transpositive process (in institutional mathematical praxeologies)" (Farias, Carvalho & Teixeira, 2018, p. 104) in institutional teaching practices (Bosch & Gascón, 2010) when concepts, both pragmatic praxeologies and deductive praxeologies, are not properly explored in situations where teachers study and teach institutional knowledge.

Therefore, we understand that this shift between praxeological models that do not feedback on each other is treated with a certain naturalness (Figure 2) and is dominant in institutions, which Job (2011) refers to as the phenomenon of naturalization (Job & Schneider, 2014), which in a way "makes it impossible" for teachers to problematize the concept of limits to make significant progress.

Methodological guidelines

The study presented here is of a qualitative nature (Garnica, 2001), since this type of research, which is close to the research method in Mathematics Didactics, is of an experimental type, given the descriptions and interpretations of both the perspectives expected by the teachers and/or researchers and the variations in the analyses reported by the researcher. In addition, the author adds that the essential elements of qualitative research lie in

> [...] the preponderance of inductive processes, the predominance of descriptive data, the emphasis on process over product, the need for generating questions and well-defined rules of action for analyzing the data collected, public evaluation criteria, discussed and agreed upon by the community, and the responsibility of the researcher in relation to his research [...] defined in some theoretical-methodological context (*ibid*., 2001, p.8-9).

In this context, this investigation has considered the proposed analysis of tasks from two perspectives: the light of TFC, especially in the construction of theorems in action, and the second, from the perspective of how these theorems in action make it possible to understand the pragmatic and deductive models that, developed in an integrated manner, can minimize the obstacles to learning limits.

Consider the limit $\lim_{x\to 1}\frac{x^2-1}{x-1}$ and suppose that we find the value of this limit by investigating the interval of the y-axis close to 2 but now we choose the "size" of this proximity, in this case with amplitude 0.5 which we will call epsilon (ε), that is, ε = 0.5. Thus, we are looking at the values of the function on the y-axis where $2-\varepsilon < f(x) < 2+\varepsilon$, or otherwise $1{,}5 < f(x) < 2{,}5$.

a) Find values on the x-axis that exactly correspond to the values of the function with amplitude $\varepsilon = 0{,}5$ and explain how you found them.

b) Write the answer found in item (a) in inequality form and see if you can determine an amplitude for the interval found and call it δ (delta).

c) Now that we are interested in choosing values for x so that every corresponding $f(x)$ is $2-\varepsilon < f(x) < 2+\varepsilon$, could we have other values for δ? Justify your answer.

d) Adding 2 to all members of the inequalities $2-\varepsilon < f(x) < 2+\varepsilon$ we obtain $-\varepsilon < f(x)-2 < \varepsilon$ and writing in modulus form, we have $|f(x)-2| < \varepsilon$. Write in modulus form the inequality found in item (b).

Figure 3.

The first activity chosen, task 1 (Taken from Burigato, 2019, p. 100)

We have chosen to present in this article situations involving tables, the observation of graphical representations and algebraic manipulations. So that we can discuss the productions with the representations most commonly used in the classroom (Burigato, 2019). We'll call them Task 1, Figure 3, and Task 2, Figure 4.

Knowing that $\lim_{x \to 3} 2x - 5 = 1$ by intuitive notion, check if the value found for the limit can be verified by the formal definition and answer the following questions.

a) Relate the inequalities found, or the moduli, of the definition by epsilon and delta to the intuitive definition of limit and justify how you made these relationships.

b) Graph the given function with the intervals you found.

c) Explain the limit of the function in your own words, taking into account items (a) and (b), using both the formal and intuitive definitions.

Figure 4.

Second chosen activity, task 2 (Taken from Burigato, 2019, p. 102).

The student's productions were modeled on theorems in action, considering the results of research on the concept of the limit of a function at a point (Burigato, 2019), as well as the lists of activities and the didactic book used by the teacher of the subject, in Figure 5 we have an image illustrating this. The students solved the activity on a sheet of paper and then discussed their choices for dealing with the activity with their classmates, who were organized in pairs. In this way, the data was produced using the resolutions written on the activity sheets and the audio obtained from the discussions held.

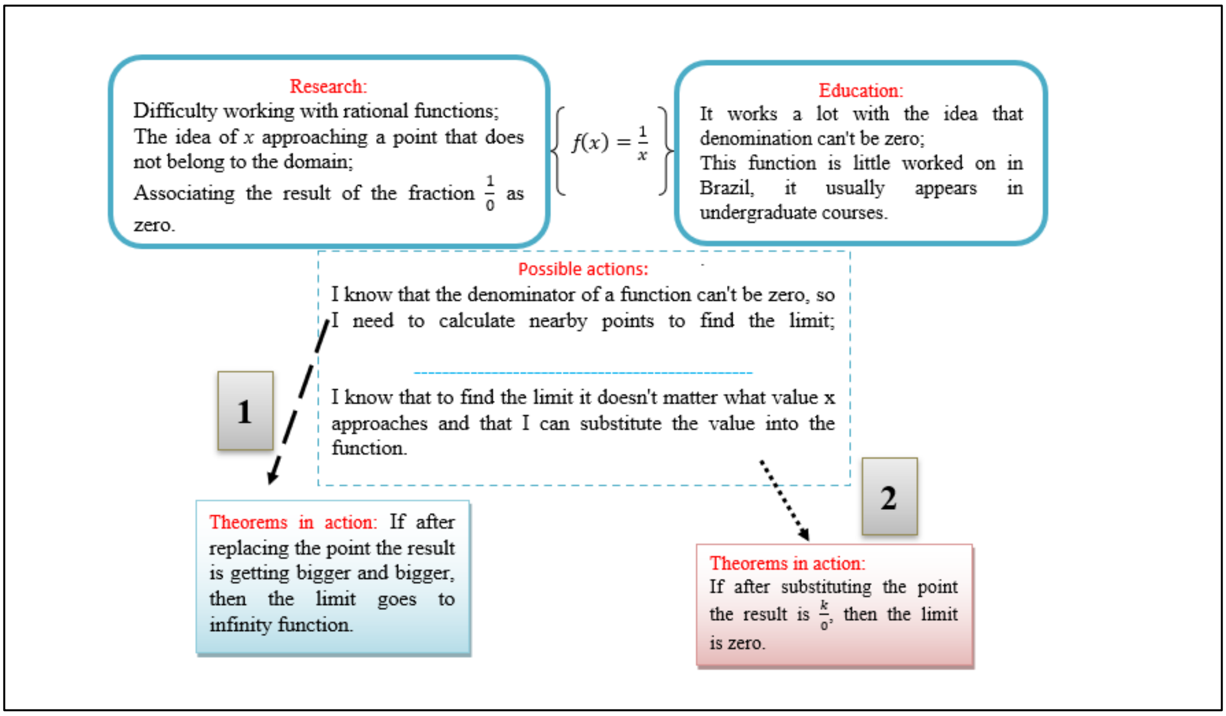

Figure 5

Synthesis exemplifying how the modeling was carried out (Example of the modeling carried out by Burigato, 2019, p.115).

The elements that will be presented in this text followed the indexes as explained in Figure 1, in this case theorems in action and rules in action

Discussions and analysis

The first activity, task 1, involved elements of the intuitive definition with the quantifiers of the formal definition and, in this situation; the student had not yet been introduced to the formal definition of the limit of a function at a point. We were interested in investigating the adaptations of the schemes mobilized by the subject when dealing with notions of proximity involving a "size" for the elements of a range, which in this case was a particular epsilon.

The representation of the task was $\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$, and the student initially made the simplification of the function to solve item (a), which involved a given interval for the values of the function, and by manipulating the inequalities he was able to answer the other items. We present one of the modeled theorems in action, with its respective rule in action.

> Theorem in action (TAB.vii): *If a function f can be simplified, then the limit of the simplified function f will be equal to the limit of the function f*. Linked to the rule in action (RB.4): When I need to find the limit of a rational function, I know that I can simplify the algebraic expression of the function to work with a simpler expression, to make the calculations easier. (Burigato, 2019, p.119).

He simplified the algebraic expression of the function given for the limit and substituted in the inequalities $2 - \varepsilon < f(x) < 2 + \varepsilon$, obtaining $2 - 0{,}5 < x + 1 < 2 + 0{,}5$, and did the correct manipulation obtaining $1.5 < x < 2.5$. In this item, one of the problems the student discussed with his colleague was the fact that the answer, in this case the range, was in terms of *x*. We modeled the student's actions on an incorrect theorem in action:

> Theorem in Action (TAB.3): *When studying the limit of functions, it is always necessary to work with elements of the domain to obtain the answer in terms of f(x).* Linked to the rule in action: If I am solving any activity with functions, I know that I need to work with the elements of the domain. (Burigato, 2019, p.120).

This was one of the student's difficulties that we consider important when working with the formal definition. In general, the intuitive definition is presented as working on the approximation of points in the domain to a given point, observing what happens to the values of the function, while the formal definition takes the opposite approach, starting from the analysis of an interval, given by an epsilon size, in which the values of the function approach the value of the function, which is the limit. For this student, it doesn't make sense, he argues with his colleague, who has done it differently, "*but I think you have to leave it as* $f(x)$".

The transition from intuitive to formal aspects is naturalized (Job, 2011; Job & Schineider, 2014), so in this case, operating with the elements of a function made the student unable to understand the result found in the algebraic manipulation. He knew that he had performed the algebraic operations correctly, but he simply ignored them and replaced them with *f(x)*. The schemes that he mobilized for previous situations and that were effective are the ones he mobilizes for those in which he finds similarities. These are the necessary breaks that would allow him to build schemes for the new concept he is constructing, but he has been unable to break with these operative invariants and make the new connections that are fundamental to dealing with the formal definition of limit Vergnaud (1990).

The aforementioned gaps in the understanding of limits become evident when the student adopts a praxis [T, τ], from a pragmatic model, and tries to justify this practice, a logo [θ, Θ], in the deductive model.

In this regard, Artigue (1995) points out that students do not have a well-constructed concept of function when they have to deal with activities involving the concept of the limit of a function. She argues that it is these situations involving the limit that bring in aspects not yet experienced that will make students expand their understanding of functions. Deconstructing the idea that they arrive at this point in the lesson, where the concept of the limit of functions is presented, with the concept of function well constructed.

For the new elements that were presented about the definition of limit, we modeled the following theorem:

> Theorem in action (TAB.i.a) *If the values of the function approach a value L when x becomes close to a given point p, then the limit of the function is L when x tends to p*. Linked to the rule in action: When I'm solving an activity about limit I need to deal with elements of the domain, I know I need to work with values close to the point we're dealing with the limit. (Burigato, 2019, p.121).

Another pertinent point to analyze was the representations used: the student did the graphical representation correctly, even though this is a case of function in which studies indicate that they have difficulties in both representing and understanding what is happening with the function (Segadas-Vianna; Tall & Vinner, 1981). In fact, with his discussion with his colleague obtained through the audios, we infer that it was this representation that helped him to choose the operative invariants mobilized in the activity and, in this case, to change the result obtained in the algebraic manipulation of the inequalities.

Task 2 was proposed after the correction of Task 1, on another day, together with the presentation and discussion of the formal definition of the limit of a function at a point. The student substituted the given function into the expression modulo $|(2x - 5) - 1| < \varepsilon$ and arrived at the inequality $|x - 3| < \frac{\varepsilon}{2}$. He realized that there is part of this inequality in the other inequality, $0 < |x - 3| < \delta$, but he didn't get the delta in terms of the epsilon, as he did in the previous task. He argued that he thought he should find a specific number and that it was strange to leave the number with a letter. We modeled the theorem in action as incorrect (TAB.2): *The result of an activity with an algebraic expression will always be a number*. And when he needs to relate the epsilon to the delta, he returns to the idea that it is always necessary to analyze the values of *x*, of the domain to know what happens with the values of the image, “[...] to find <the *interval of* $|f(x) - 1| < \varepsilon$ *, it is necessary to find the values of f(x) by finding the x's close to 3*>.” (Burigato, 2019, p.124), again mobilizing the theorem in action (TAB.3) that we mentioned earlier.

In this situation, the result obtained through algebraic manipulation was once again problematic, especially when the student tried to make connections with their knowledge of functions. The operative invariants mobilized, in this case the two theorems in action that we mentioned (TAB.2) and (TAB.3), come into conflict with the new aspects that he needs to deal with in order to construct the new concept, yet he had difficulty breaking with these elements of his scheme.

These ruptures are necessary and, at the same time, it is necessary to construct appropriate invariants for the concept of limit. An incorrect theorem in action mobilized in this activity was (TAB.1) *If the limit exists, then δ must equal the given ε.* At this point, the student was trying to reflect on the “size” needed for the limit to exist and, again, the graphical and algebraic representations caused confusion. At one point in the discussion, the student argued that he thought he should “create” a delta to relate to the epsilon. We believe that the incorrect theorems in action (TAB.2) and (TAB.3) contributed to this problem, and it is important to invest in situations to deconstruct this mistaken knowledge.

These elements mobilized in the situations, involving the conceptual field of function limit, can be analyzed using TAD by discussing the aspects of pragmatic praxeology, the tasks, and the schemes mobilized to solve these types of praxeologies, with the schemes that can help the immersion and development of deductive praxeologies. In Chart 1 below, we show some elements identified in the students' actions, according to the representations they mobilized.

Elements identified in the mobilized schemes **Obtained from the answers in algebraic form:** ➢ If the function on which I am calculating the limit has no restriction on the point of investigation of the limit, then I can substitute the point in the variable x of the function;

- If I'm dealing with the limit of functions, then I always need to work with elements of the domain to get the answer in terms of $f(x)$;
- I always work with elements of the domain of functions.
- If the limit exists, then δ must be equal to the given ε;
- I calculate several points close to the point to see the limit when x tends to p.
- If a function f can be simplified, then the limit of the simplified function f
- function will be equal to the limit of the function f;
- I simplify the algebraic expression of the function to make calculations easier;
- I work with values close to the point where we are dealing with the limit;
- The result of an activity with an algebraic expression will always be a number.

Obtained through responses in the form of natural language, written or oral:

- To find the limit, I find the y for values of x close to the point of investigation of the limit.
- The limit of the function is how the function behaves when it approaches a point x or $f(x)$.
- To get the limit tending to a point p we need to find values of $f(x)$ near the point p.
- The limit can be different from the value that the function takes on at the point where the limit is investigated.
- The function when x tends to p tends to L, but $f(p) \neq L$.

Figure 6.

Elements of the conceptual field identified in the situations according to (Part of the research chart (Burigato, 2019, p.145)).

The elements of the conceptual field presented in Figure 6 indicate the predominance of the pragmatic model through the intuitive notion. In addition, the algebraic domain prevails over the numerical and graphical domains, as can be seen in Figure 7.

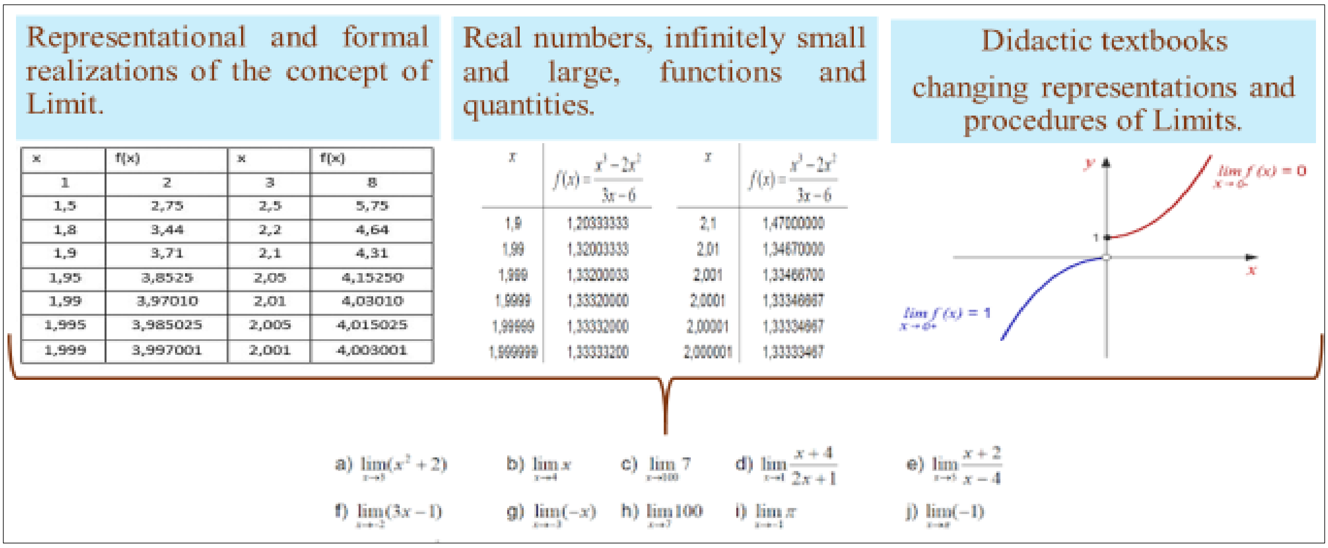

Figure 7.

The algebraic domain stands out over the numerical and graphical domains.

We observed a strong influence of the pragmatic model in students' interactions and responses in the numerical, algebraic, and graphical domains, in that order. The first domain is strongly associated with table work. The second, algebraic manipulations and the third, graphical constructions of the functions developed in the previous domains.

The lack of understanding of the elements involved in functions is quite evident, especially in the manipulations of the function's domain. By not understanding this concept and its properties, which feed the intuitive idea of the concept of limit, the student is unlikely to understand the relationship that must exist between the delta (δ) and the epsilon ε. Consequently, if there is no understanding of the pragmatic model, students will not understand the deductive model and will use it only to show the possible values for δ and ε to show the existence of a limit, ignoring the elements of the formal definition of limits themselves. Consequently, the transition between these two models is naturalized.

By the formal definition, if $x \in D(f)$, if $\forall \varepsilon > 0, \exists \delta > 0$, we have that if $0 < |x - a| < \delta \rightarrow |f(x) - L| < \varepsilon$. We say that f, if it exists, has limited L, at point a, is unique, and we represent it by: $\lim_{x \to a} f(x) = L$, the quantifiers ∀ (for all) and ∃ (exists) are fundamental for the understanding that one can assume arbitrary values of ε to determine a value of δ to have smaller and smaller approximations, that is, to determine the limit of a function near certain points in the domain.

These elements aid deductive praxeologies. However, the representations hinder and limit the mobilization of knowledge for the development of deductive praxeologies.

Final considerations

The discussion of the concept of functional limits presented in this article, based on two research projects, one already completed and the other under development, presents initial ideas for an epistemological reference model that is dynamic and seeks to consider epistemological aspects that allow the transition from pragmatic and deductive praxeological models, minimizing the phenomenon of naturalization. This transition process is still being developed for this project.

We have observed that students lack a series of concepts that live in the pragmatic praxeological model, such as the articulation of the numerical, algebraic and graphical domains of functions, with only the algebraic domain prevailing, due to the lack of integration between the representations, which consequently hinders and limits the mobilization of knowledge.

With regard to the deductive model, we observed that in the formal definition, students believe that they must look for values that are closer and closer to x in order to find image values that are closer and closer to $f(x)$. In fact, Doumbia's research (2020) characterized this way of working as an epistemological obstacle, since the definition indicates the opposite idea: as ε gets closer and closer to zero ($f(x)$ is closer to L), δ will be much closer to zero (x will be closer to a). Therefore, it is essential that those involved in teaching limits understand this praxeological model, which makes it possible to assimilate the properties and theorems that follow this definition.

Therefore, we understand that it is essential to modify the teaching proposals to consider tasks that allow the integration between the domains and their representations and, consequently, the two praxeological models, pragmatic and deductive.

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