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# A proposal for an reference epistemological model for the study of limits through dialogue via attention mechanisms.

# Uma proposta de modelo epistemológico de referência para o estudo de limites dialogado via mecanismos de atenção.

# Una propuesta de modelo de referencia epistemológico para el estudio de los límites a través del diálogo a través de mecanismos de atención.

# Une proposition de modèle épistémologique de référence pour l'étude des limites en dialogue avec les mécanismes d'attention.

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## Abstract

In this theoretical essay, we intend to present task proposals and praxeological analyzes in an Reference Epistemological Model (REM) for teaching Differential and Integral Calculus, giving new meaning to the diffusion of the notion of limit of a function by definition. The aforementioned REM has as its epistemological and methodological assumption the Anthropological Theory of Didactics by Yves Chevallard and the ideas of Top Down and Bottom up attentional processing. Data production for this study occurs via praxeological analysis based on tasks extracted from textbooks and *a priori* analysis of the constructed REM. As a main result, it was observed through the a priori analysis of the tasks that made up the REM, that the tacit knowledge necessary to solve the tasks can be evoked by their structure,

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while in bottom-up processing the use of imagery resources helps in focusing paying attention to important conceptual aspects present in the proposed tasks.

***Keywords:*** Limit of a function, Attentional mechanisms, Epistemological model of reference.

## Resumen

En este ensayo teórico se presentan propuestas de tareas y análisis praxeológicos en un Modelo de Referencia Epistemológico (MER) para la enseñanza del Cálculo Diferencial e Integral, dando un nuevo significado a la difusión de la noción de límite de una función por definición. El mencionado MER tiene como presupuesto epistemológico y metodológico la Teoría Antropológica de la Didáctica de Yves Chevallard y las ideas de procesamiento atencional de arriba hacia abajo y de abajo hacia arriba. La producción de datos para este estudio se produce mediante un análisis praxeológico basado en tareas extraídas de libros de texto y un análisis a priori del MER construido. Como resultado principal, se observó, a través del análisis inicial de las tareas que conformaron el MER, que el conocimiento tácito necesario para resolver las tareas puede ser evocado por su estructura, mientras que, en el procesamiento ascendente, el uso de imágenes Los recursos ayudan a centrar la atención en aspectos conceptuales importantes presentes en las tareas propuestas.

***Palabras clave***: Límite de una función, Mecanismos atencionales, Modelo epistemológico de referencia.

## Résumé

Dans cet essai théorique, des propositions de tâches et d'analyses praxéologiques sont présentées dans un Modèle Épistémologique de Référence (MER) pour l'enseignement du Calcul Différentiel et Intégral, donnant un nouveau sens à la diffusion de la notion de limite d'une fonction par définition. Le REM susmentionné a pour postulat épistémologique et méthodologique la Théorie Anthropologique de la Didactique d'Yves Chevallard et les idées de traitement attentionnel *top-down* et *bottom-up*. La production de données pour cette étude se fait via une analyse praxéologique basée sur des tâches extraites des manuels et une analyse a priori du MER construit. Comme résultat principal, il a été observé, à travers l'analyse initiale des tâches qui composent le MER, que les connaissances tacites nécessaires à la résolution des tâches peuvent être évoquées par sa structure, tandis que, dans le traitement de *bottom-up* l'utilisation de ressources d'imagerie permet de concentrer l'attention sur des aspects conceptuels importants présents dans les tâches proposées.

***Mots-clés*** : Limite d'une fonction, Mécanismes attentionnels, Modèle épistémologique de référence.

## Resumo

Neste ensaio teórico, apresenta-se propostas de tarefas e análises praxeológicas em um Modelo Epistemológico de Referência (MER) para o ensino do Cálculo Diferencial e Integral ressignificando a difusão da noção de limite de uma função pela definição. O referido MER tem como pressuposto epistemológico e metodológico a Teoria Antropológica do Didático de Yves Chevallard e as ideias de processamentos atencionais *top-down* e *bottom-up*. A produção de dados desse estudo ocorre via análise praxeológica a partir de tarefas extraídas de livros didáticos e análise *a priori* do MER construído. Como principal resultado, observou-se, por meio da análise inicial das tarefas que compuseram o MER, que os conhecimentos tácitos necessários para a resolução das tarefas podem ser evocados por sua estrutura, enquanto, no processamento *bottom-up*, o uso de recursos imagéticos auxilia na focalização da atenção para aspectos conceituais importantes presentes nas tarefas propostas.

***Palavras-chave***: Limite de uma função, Mecanismos atencionais, Modelo epistemológico de referência.

**A proposal for an reference epistemological model for the study of limits through dialogue via attention mechanisms**

This work presents task proposals and praxeological analyzes in an Reference Epistemological Model (REM) for teaching Differential and Integral Calculus. Briefly, an REM can be understood as a network of praxeologies whose dynamics must be experimentally contrasted, given that it is a scientific hypothesis and, at the same time, specific to the researcher (Fonseca et al., 2014; Ruiz-Munzón et al., 2011).

Taking Mathematics Didactics as a starting point, this set of principles, concepts and theories guide the way teachers teach Mathematics and how students learn topics about this area of knowledge. An REM is certainly capable of defining beliefs about mathematical knowledge, the cognitive processes involved in learning Mathematics, the most effective teaching methods, among other important aspects for educational practice; therefore, the adopted REM serves as a basis for building teaching and learning strategies in Mathematics.

According to Sierpińska (1994), an epistemological model in Mathematics Didactics must consider both mathematical and pedagogical knowledge. This means that it is necessary to take into account not only the logical and formal structure of mathematical concepts, but also the difficulties and challenges that students face when learning these concepts.

> No educator can afford to omit knowledge content. It is the content of knowledge that is of interest to Didactics, as opposed to general pedagogical theories. It is clear that these "common mechanisms" can provide a theoretical basis for considerations and serve in the terminology used to describe conclusions regarding problems of understanding and decision-making in teaching. However, they are never the ultimate goal of research (Sierpińska, 1994, p. 98, our translation).

Following this line of thought, da Ponte et al. (2016) highlight the importance of considering the interactions between the different actors involved in the Mathematics teaching and learning process, such as teachers, students and teaching materials. An effective epistemological model must be able to integrate these different dimensions and promote a more dynamic and contextualized approach to teaching Mathematics. Therefore, we emphasize the need to reflect on students' previous conceptions regarding Mathematics, as these conceptions can significantly influence learning. Thus, an epistemological model, in Mathematics Didactics, must take into account not only the object of knowledge to be taught, but also the students' previous conceptions and aspects of cognition that will integrate the conditions and/or restrictions for learning knowledge in game.

Based on these works, we understand that an epistemological model, in Mathematics Didactics, must be flexible enough to account for the multiple forms of representation and communication present in the teaching context. In this way, it is possible to guarantee a more inclusive and diverse approach to teaching mathematics, promoting more meaningful and autonomous learning on the part of students.

One of our proposals is to discuss the tooling and theoretical support necessary to develop an REM. When we talk about "cognitive processes involved", "students' perceptions and mental representations" and "multiple forms of representation and communication", we realize that only the mathematical objects themselves, which live from "mathematical knowledge", are not sufficient in preparation of a REM. Therefore, we used some Neuroscience techniques to base the model, with an emphasis on attention mechanisms.

Attention is a fundamental cognitive process for learning, as it allows the selection and focusing of relevant stimuli for information processing. According to Posner and Petersen (1990), attention can be divided into three distinct neural networks: the alert network, responsible for monitoring the environment; the spatially oriented network, which directs attention to certain areas of space; and the executive network, involved in cognitive control and problem solving.

In the context of teaching Differential and Integral Calculus, attention mechanisms play a crucial role in understanding complex mathematical concepts. According to Zeki (2002), visual perception is one of the main forms of processing mathematical information, being essential for identifying patterns, relationships and properties of mathematical objects. Therefore, by directing students' attention to specific aspects of differential and integral equations, it is possible to activate prior knowledge (top-down processing) for solving tasks that can be combined with figural elements, which will, in turn, activate, the bottom-up mechanism (evoking visual sensory aspects), to facilitate students' understanding of the object studied.

Furthermore, emotions also play an important role in regulating attention during learning. According to Relvas (2023), they directly influence neural activity related to selective attention, modulating the prioritization of emotionally relevant stimuli. Thus, strategies that stimulate positive emotions in students can increase their motivation and engagement with mathematical content. However, the emotional aspect will not be the object of analysis in this work.

Another relevant point in developing an epistemological model for teaching Calculus is metacognition, that is, knowledge about our own cognitive processes. According to Schraw et al. (2006), students who have developed metacognitive skills are able to monitor and regulate their attention more efficiently during complex tasks, such as solving mathematical problems.

Therefore, encouraging students to reflect on their own thoughts and strategies can contribute to a better understanding of mathematical concept.

We are convinced that attention mechanisms play a fundamental role in learning Differential and Integral Calculus. By understanding how our ability to focus and concentrate works, educators can use more effective pedagogical strategies to facilitate the assimilation of content by students. Furthermore, considering emotional and metacognitive aspects when developing the epistemological model can further enhance students' mathematical learning.

## Attention mechanisms

Exact science subjects are often considered challenging and intimidating by many students. They require a high level of abstraction and logical reasoning, which can generate anxiety and stress in students. According to a study carried out by Pekrun et al. (2007), anxiety in relation to these subjects is associated with low self-efficacy and self-esteem among students.

Furthermore, such subjects can negatively impact the psychosocial aspect of students by reinforcing gender and racial stereotypes. According to research carried out by Steele and Aronson (1995), women and ethnic minorities tend to feel less capable in science subjects due to social stigmatization and cultural pressure. This could lead to a decrease in the interest and motivation of these groups in pursuing careers in Science and Technology.

On the other hand, some studies demonstrate that learning Exact Sciences subjects can have a positive impact on students' cognitive and emotional development. According to Boaler (2013), problem-based Mathematics teaching stimulates creativity, critical thinking and conflict resolution, contributing to the global development of students.

It is important to highlight that teachers play a fundamental role in promoting the psychosocial well-being of students in Exact Sciences subjects. A study carried out by Hembree (1990) highlights the importance of emotional support from teachers in reducing students' anxiety in relation to Mathematics. Pedagogical strategies that value diversity, promote gender equity and encourage collaboration among students are also essential to creating a more inclusive and welcoming environment.

Emotional security, in turn, is not a guarantee of success in the cognitive aspect, although it facilitates this path through training and improvement of attention mechanisms. Attention mechanisms play a fundamental role in the teaching-learning process in general and more precisely in Differential and Integral Calculus. Attention is responsible for directing the perception and processing of information, directly influencing the understanding and retention of mathematical concepts. In the context of learning Calculus, two theoretical models are often used to explain attention mechanisms: bottom-up and top-down. To this end, we will consider the approach taken in Sarter et al. (2001).

Bottom-up processing refers to spontaneous and automatic attention, directed by external sensory stimuli. In this case, the person may be attracted by the visual or auditory presentation of mathematical content, focusing their attention on specific elements, such as graphs, equations or verbal explanations. This type of attention is essential for identifying patterns and regularities in calculation problems, facilitating the resolution of the proposed questions.

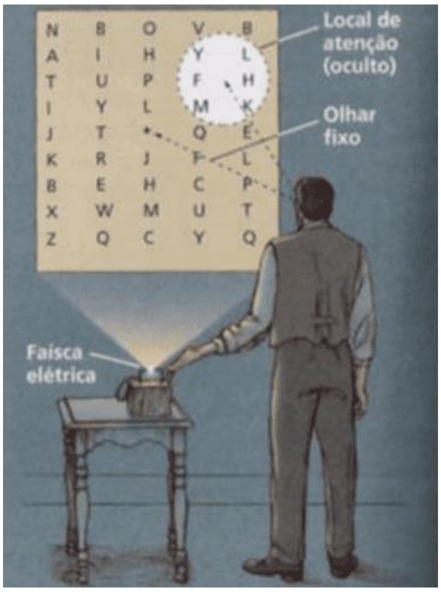


Figure 1a.

*Experimento atencional de Helmholtz realizado em 1894 (Gazzaniga et al., 2006)*

On the other hand, the top-down model involves controlled and voluntary attention, guided by the student's expectations, prior knowledge and individual goals. In this work, the student uses their previous experiences with Calculus to direct their attention to the most relevant aspects of the subject, such as fundamental concepts, problem-solving techniques or practical applications. Top-down processing allows the student to organize information in a meaningful and contextualized way, facilitating the understanding and memorization of mathematical concepts.

Together, the bottom-up and top-down models contribute to effective learning of Differential and Integral Calculus. While bottom-up attention helps to quickly identify important information in mathematical problems, top-down attention allows for a more structured and reflective approach to the content studied. By combining these two attentional mechanisms, teachers can promote deeper and more meaningful learning of mathematical concepts among students.

Table 1.

*Proposals: how the bottom-up mechanism can help in learning Differential and Integral Calculus*

| *Bottom-up* |
|---|
| 1. Use of concrete and practical examples to introduce the basic concepts of Differential and Integral Calculus.<br>2. Encouraging the resolution of simple problems before addressing more complex issues.<br>3. Encouragement of active participation by students in the construction of knowledge, through practical and experimental activities that demonstrate the applications of calculation in real life.<br>4. Gradual introduction of mathematical concepts, starting with the fundamentals of algebra and geometry before moving on to Calculus itself.<br>5. Carrying out periodic reviews and fixation exercises to consolidate student learning at each stage of the teaching-learning process.<br>6. Fostering the development of students' critical and analytical skills, encouraging them to question and explore different approaches to solving mathematical problems.<br>7. Application of gamification techniques in teaching differential and integral calculus, as a way to engage students and make learning more dynamic and interactive.<br>8. Use of technological resources, such as specific software for mathematical calculations, to help students visualize and understand the concepts presented in the classroom.<br>9. Promotion of interdisciplinarity in teaching calculus, relating mathematical content with other areas of knowledge, such as Physics, Chemistry and Engineering.<br>10. Establishment of a close relationship between teachers and students, creating a welcoming and motivating environment for learning Differential and Integral Calculus. |

Table 2.

*Proposals: how the top-down mechanism can help in teaching and learning Differential and Integral Calculus*

| Top-down |
|---|
| 1. Clear definition of learning objectives and goals: allow the teacher to precisely establish which are the fundamental concepts that students must master in Differential and Integral Calculus. |
| 2. Organization of the didactic sequence: it is possible to structure the content in a hierarchical manner, prioritizing the understanding of broader concepts before moving on to more specific details. |
| 3. Contextualization of mathematical concepts: relate the concepts of Differential and Integral Calculus with practical situations in students' daily lives, facilitating their understanding and application. |
| 4. Encouragement to solve complex problems: by presenting students with challenges that require the integration of different Calculus concepts, the top-down method favors the development of students' analytical and critical skills. |
| 5. Encouragement of autonomy and self-education: students are encouraged to seek new sources of information and develop independent research skills in the area of Differential and Integral Calculus. |
| 6. Use of technological resources: use of digital tools such as graphic software and simulators, expanding the possibilities of visualization and experimentation with mathematical concepts. |
| 7. Interdisciplinary integration: by adopting a holistic perspective in the Differential and Integral Calculus approach, the top-down method favors the connection between different areas of knowledge, enriching the educational process. |
| 8. Continuous and personalized feedback: provide individualized feedback to students, identifying their specific difficulties in learning Differential and Integral Calculus and proposing strategies to overcome them. |
| 9. Encouragement of metacognitive reflection: by promoting a reflective approach to the learning process itself, students develop greater awareness of their cognitive strategies when studying Calculus. |
| 10. Promotion of creativity and innovation: teachers can encourage students to explore new ways of thinking and solving problems in the context of Differential and Integral Calculus, encouraging creativity and originality in the construction of mathematical knowledge. |

Conscious use of the top-down mechanism allows students to be able to direct their attention to specific aspects of the content, filtering relevant information and ignoring distractors. By using mental organization and cognitive elaboration strategies, students are able to process information more effectively.

## Limits

The concept of limit is fundamental in Real Analysis; It results in several other important ideas, such as derivatives, integrals, continuity and optimization. Despite being considered basic, many people with training in the field of Exact Sciences have difficulty fully understanding this concept. The term “limit” has several meanings, but here it refers to the value to which a real function approaches as the elements of its domain accumulate around a specific value. For those who do not pursue careers in Mathematics, a lack of understanding of the concept of limits may not be as problematic; however, for higher-level Mathematics students and teachers, this understanding will directly affect their ability to advance into more complex courses or applications in a wide range of areas. Currently, the teaching of this notion – not the intuitive idea, but the formalization – in Calculus is addressed by the formalization based on quantifiers and developed by Weierstrass:

> The number L is the limit of the function f(x), where x tends to $x_0$, if given any arbitrarily small number ε, another number δ can be found such that for all values of x differing from $x_0$ by less than δ, the value of f(x) differs from L by less than ε. (Boyer, 1949, p. 287)

This is not exactly the way the concept of limit appears in Calculus books today, but it is very close to the current one. From here, the authors' experiences in teaching Differential and Integral Calculus, Real Analysis or Complex Analysis follow. In the initial stage of a Calculus or Analysis course, the context is that of real functions (its range is the set of reals) of one real variable (its domain is a subset of the reals); In this locus, the concept of limit is usually presented. Bringing more mathematical rigor to Weierstrass's definition, the notion of limit is formulated as follows: let f be a function (real and of one real variable) and p be an accumulation point for f (or for the domain of f), the limit of function f as x tends to p is equal to L if, and only if, given arbitrary $\varepsilon > 0$, there exists, from this fixed ε, $\delta > 0$ such that $0 < |x - p| < \delta \Rightarrow |f(x) - L| < \varepsilon$. In other words, if the distance from x to p is less than δ, where x is a point in the domain of f distinct from p (arbitrary, with this condition), implying that the distance from f(x) to L is less than ε.

This fact is represented graphically as follows:

$$\lim_{x \to p} f(x) = L$$

Figure 1b.

*Graphic-symbolic representation of the limit.*

The expression represented in Figure 1b summarizes the famous approach of limits by epsilons and deltas, which we will denote by ε-δ. Here it is worth explaining the starting point of the first way of analyzing this concept. Although it has been said that this concept is seen at an initial stage in Calculus subjects, the experiences reported follow this limit definition presented, at review level, in more advanced courses, such as Vector Calculus (for functions of several variables) and Analysis Real in institutions in three Brazilian states (Bahia, BA, Mato

Grosso, MT and São Paulo, SP) where the first author taught the aforementioned subject. When reviewing the concept of limit, the students formulated the following questions:

- What is an accumulation point?
- What does it mean for a point to accumulate in a value?
- Right after presenting the definition, the professor says that the idea is to make epsilon and delta tend to zero. Why?
- Is epsilon variable or constant? Is it fixed or is it arbitrary?
- Is Delta variable or constant? Is it fixed or is it arbitrary?
- Is x variable or constant? Is it fixed or is it arbitrary?
- Is f variable or constant? Is it fixed or is it arbitrary?
- What does "x - p" between the bars mean?
- Why $0 < |x - p|$? (After all, if $x = p$, we have $|x – p| = |p – p| = 0$.)

The aforementioned limit definition, simply recorded in Figure 1, mobilizes the following approaches:

1. Logical connectives ("imply").
2. Quantifiers ("exists", "for all", both can be considered from Naive Set Theory).
3. Variables.
4. Module function.
5. Distance (from the module function).
6. Accumulation point.
7. Actual function.
8. Function and its descriptors (domain, range, law).

In Calculus classes, the concept of limit is often considered one of the most complex by students (Karatas et al., 2011; Swinyard & Larsen, 2012). Identifying difficulties in understanding Limits is important for their relationship with other mathematical concepts. Equip students with the conceptual tools necessary to address a range of mathematical and applied problems, which involve the foundation of differential and integral calculus, the understanding of infinitesimal concepts, the development of abstract reasoning and the formalization of the concept of changes which, in turn, , will allow you to analyze how a function behaves as its variables approach certain values, involves the use of the limit of a function, theorems related to the concept of limit and determining whether the limit exists according to the formal definition. Students' ability to achieve these skills depends on adequate understanding of the formal definition of that notion..

However, the mathematical object lacks adequate experience within the institutional arrangement normally in force. Consider, for example, the graphical representations in Figures 2 and 3.

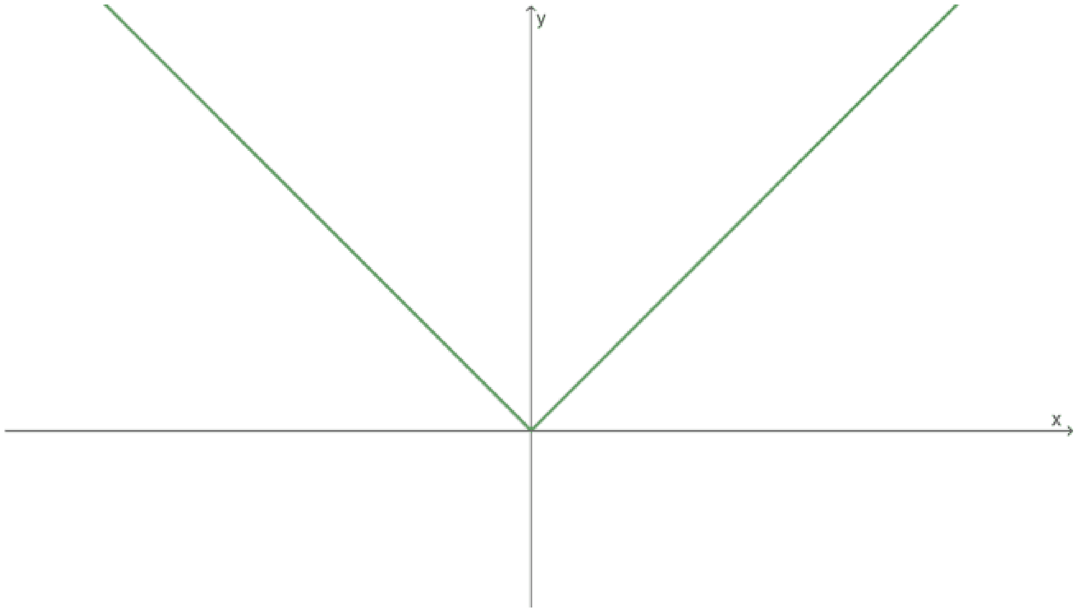


Figure 2.

*Gráfico no plano xy em forma de "V".*

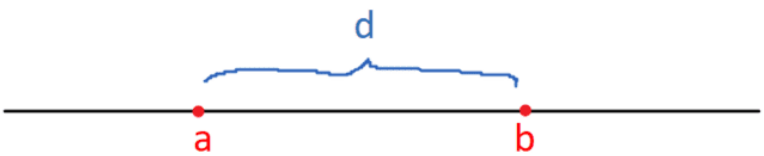


Figure 3.

*Representação gráfica da distância entre dois números na reta real.*

One of the authors, in didactic experiments in Differential Calculus classes at a federal university in Western of Bahia (a Brazil's State), raises the question: when the word "module" (in the context of real numbers) is pronounced, which of these figures appears in your "mental visualization": Figure 2 or Figure 3? The majority responds, without hesitation: Figure 2. Going into the investigative aspect, it was observed that there is no a priori association between "distance" and "module". This question, as well as the draft answers, guide the REM proposed here in a certain way, as they signal that there is a possibility for notions about attentional mechanisms to contribute to the learning of the aforementioned object of knowledge.

Furthermore, attention is a fundamental mechanism for learning in any area of knowledge. When it comes to the topic of "limits", the ability to concentrate and focus becomes even more relevant, as it requires reflection, self-control and the ability to establish personal and logical restrictions. Regarding the aforementioned question about the representation that students make in relation to the word module, an REM based on attentional mechanisms takes into account the role of the figures that represent the object without leaving aside the tacit knowledge of the students.

Studies in the field of neuroscience have demonstrated that attention is regulated by specific regions of the brain, such as the prefrontal cortex, responsible for executive functions such as planning, decision-making and emotional control (Cosenza & Guerra, 2009). Attention becomes even more important when it comes to learning about limits, as this topic demands a greater cognitive effort to understand the concept and procedures for calculating the limit of a function by definition, since the usual approach at the CDI institution occurs from more abstract levels to concrete applications or situations.

The concepts of “derivative” and “integral” allow us to analyze the global behavior of functions, going beyond the simple local study provided by the limit. The derivative measures the rate of change of a function in relation to its independent variable, providing information about the slope of the curve at a given point. The integral allows you to calculate the area under a curve, making it possible to find solutions to optimization problems and volume calculations..

While the limit focuses on the specific behavior of functions, the derivative and integral provide a more comprehensive and complete view of their global behavior. Also, through a change of framework, new functions can arise from a given real function. They allow you to analyze trends, identify critical points and carry out complex calculations that would be impossible with just the concept of limits. From the point of view of teaching organizations for the CDI, both the derivative and the integral arise from the concept of limit. In this way, it is possible to say that the concepts of derivative and integral are complementary to that of limit. They

> are conceived as pragmatic models of magnitudes considered as mental objects giving rise to a mathematical activity that has its own legitimate level of rationality, albeit different from the standard level underlying the formal aspects of limits, for example “static” definitions using quantifiers. This model allows us to defend the usefulness of taking into account an epistemological obstacle called empirical positivism as a grid for interpreting students' reactions to tasks that involve limits, either alone or in relation to other concepts, such as derivatives, integrals , etc. The scientific value of this epistemological obstacle lies in its ability to encompass and make sense of increasingly broad types of errors (Job & Scneider, 2014, p. 3).

As we see in the excerpt above, the outcome referred to in the previous paragraph is perhaps justified by the fact that the concept of limit of a function is conceived as a pragmatic model, which implies methodological strategies for dissemination with this bias of application to other objects of knowledge.

## Praxeological organization and didactic transposition

The Anthropological Theory of Didactics (ATD) is based on the idea that educational practice is influenced by specific praxeologies, that is, sets of knowledge and practices that guide the actions of the subjects involved in the teaching and learning process. According to Chevallard and Joshua (1985), praxeologies are composed of knowing how to do (technical knowledge) and knowing how to teach (pedagogical knowledge), which are related in a dialectical way.

In this sense, ATD highlights the importance of the relationship between knowing how to do and knowing how to teach, highlighting that both must be considered in an integrated way in the planning and execution of educational activities. This integration allows for a more holistic approach to knowledge, contributing to a greater understanding of the cognitive processes involved in learning.

Furthermore, ATD distinguishes between ostensive and non-ostensive dialectics. Ostensive dialectics refer to the explicit and conscious processes that occur during didactic interactions, while non-ostensive dialectics refer to the implicit and unconscious processes present in educational practices. According to Chevallard (1994), the recognition of these two dimensions is fundamental for a critical analysis of the pedagogical strategies adopted.

The institution is a central element in ATD, as it represents the social, cultural and political context in which praxeologies develop. According to Chevallard and Joshua (1982), institutions play a determining role in the configuration of school knowledge and in the organization of relationships between teachers, students and other educational agents. In this way, ATD proposes a complex and multidimensional approach to education, considering not only individual aspects, but also collective and social aspects.

## About the REM proposal

What praxeologies are evoked by students regarding the intuitive notion of "limit" of a function and the formal definition? Although pedagogical projects for Mathematics courses, for example, mention the presentation of the definition of limits as part of the syllabus of a Calculus course, there are no justifications or sufficient details to guide how this subject is dealt with in the classroom. With an experimental objective in Higher Education Didactics, the following task was presented: "How to build a graphic scheme coherent with the formal definition of limit? How can this scheme help in graphically representing the limit?" The aforementioned task is, in fact, the background for the question of this study: "How to build a graphical representation model for the limit of a function?". This question is generic, but will be limited to the context of real functions of one real variable, considering the usual topology of sets of real numbers.

### First hypotheses for model construction

Let's assume that students have already had contact with the formal concept of limits from previous courses or stages in a Calculus or Analysis course. The idea now is to graphically represent the limit, since the institutionality of Calculus Teaching brings, in addition to the analytical concept and algebraic manipulations, the graphic representation of various objects of the ecology of a regular Calculus course. It can even be seen that the presentation of the concept of various objects is motivated by graphic records justified in the face of a better understanding of the topic through imagery resources, for example: function (Figure 4); continuity and discontinuity (Figure 5); derivative (Figure 6) and integral (Figure 7). And for limits? What is the "canonical" graphical representation of a limit?

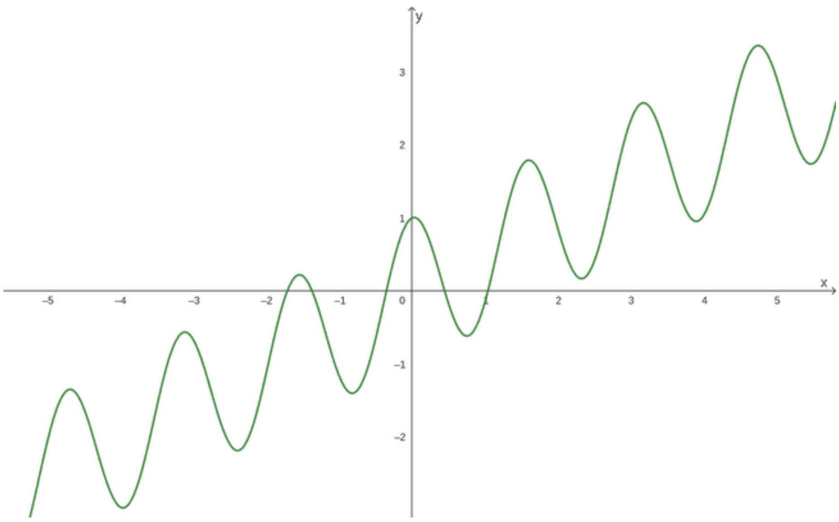


Figure 4.

*Graphing a real function of one real variable.*

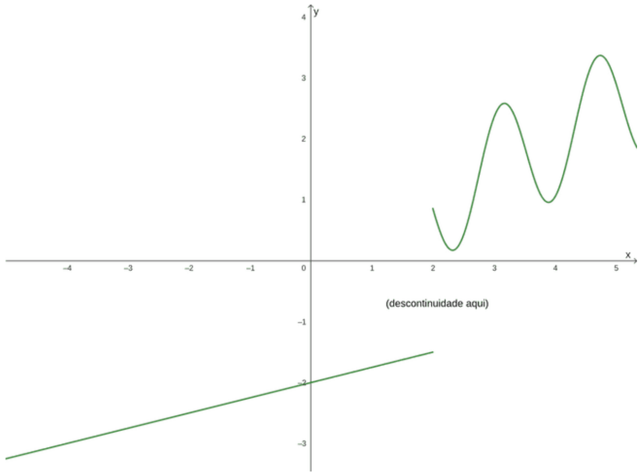


Figure 5.

*Graphing a discontinuous function.*

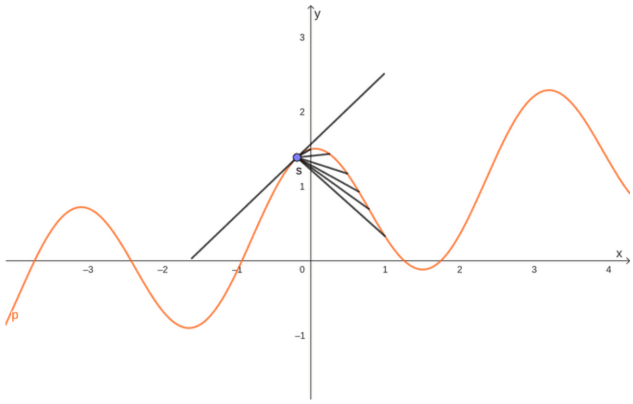

Figure 6.

*Graphical representation to motivate the definition of derivative via "secant lines that approach the (local) tangent line at a point on the graph".*

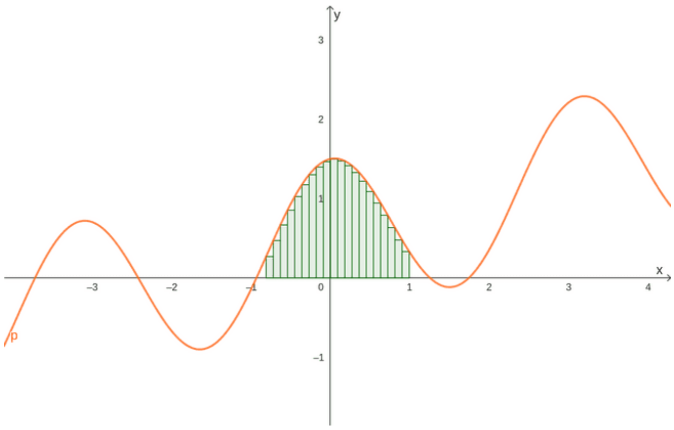

Figure 7.

*Graphical representation to motivate the definition of integral via area calculation by "approximations using rectangles to fill an area limited by the graph of a function that does not change sign over an interval".*

The question posed as central is broken down into several other questions, therefore highlighting the ecological imbalance regarding the theme “limits”:

- How to differentiate the study of a function from the study of the graph of a function?

- How to understand the limit on the graph of any function?

- Are the functions commonly covered in the Calculus course sufficient for the study of limits?

The last question illustrates how the Teaching of Calculus is not as general as it is intended to be (in the speech or in the syllabus): the functions adopted as examples or exercises are of the following types:

a) constant, that is, $f(x) = k$, where k is a fixed real number;

b) polynomial of degree n, that is, $f(x) = a_0 + a_1x + \ldots + a_nx^n$, where $a_0, a_1, \ldots, a_n$ are real numbers, $a_n \neq 0$ and $n > 0$ is a natural number;

c) power, that is, $f(x) = x^k$, where k is any fixed number and the domain of f can be changed depending on the value of k;

d) exponential, that is, $f(x) = a^x$, where a is a base for an exponentiation: a is a non-zero real number and different from 1;

e) logarithmic, that is, $f(x) = \log_a x$, where a is a base for a logarithm: a is a non-zero real number and different from 1;

f) trigonometric, for example, $f(x) = \sin(x)$, $f(x) = \cos(x)$, etc.;

g) hyperbolic trigonometric, $f(x) = \sinh(x)$, $f(x) = \cosh(x)$, etc.;

h) modular, that is, $f(x) = |x|$: $|x|$ is worth x, if $x \geq 0$ and $|x|$ is $-x$ if $x < 0$..

We note that the presentation of the types of functions listed above requires a more rigorous definition. Let's take the exponential as an example and consider the function f(x) = 2x, with x being any real number. If x = 3, for example, it is known from Basic Education that 23 = 2·2·2 = 8, that is, f(3) = 8. However, since x is any real number, it can assume the value x = π. What is f(π), that is, how to calculate 2π? This topic constitutes very fertile ground for future productions and will not be discussed in detail here. Furthermore, the functions mentioned, except for the modular one, are of class C∞, that is, for any natural m, the derivative of order m exists (they are “infinitely” differentiable). And, in the universe of real functions (to a real variable), the chance of randomly choosing a function of class $C\infty$ is zero, that is, the typical functions of a Calculus course are “artificial”, pointing out, among others, for the disparity between mathematical modeling and the empirical mechanism of data as “Nature offers”.

## A change of presentment

As mentioned, although they are not the same object from a conceptual point of view, “function” and “function graph” are constantly confused. This is an a priori element considered in REM: taking the graph as a starting point for studying functions makes it difficult to understand limits. Let us remember that a function f: Dom →CDom, in which $Dom \subset \mathbb{R}$ and $CDom \subset \mathbb{R}$ are the domain and range of f respectively, is governed by a law that “combines” elements of the domain with elements of the range. This way of “combining” elements from sets considered distinct is formalized through the notion of ordered pair. For example, given a function f, the pair $(a,b) \in Dom \times CDom$ is interpreted as follows: “b is in function of a” or “b = f(a)”. The graph of f, in turn, is the meeting of all pairs combined using this function:

$$graph(f) = \{(x,f(x)), \text{ where } x \in Dom(f)\}.$$

If we consider the definition of a function through ordered pairs (as in a Set Theory course), the distinction between a function and its graph is mathematically rigorous, as in practice the objects are indistinguishable. It is worth highlighting that $Dom \times CDom \subset \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, and it is for this reason that the graph of a function is represented in the Cartesian plane.

However, it is possible to graph a function before the “join” between domain and codomain.

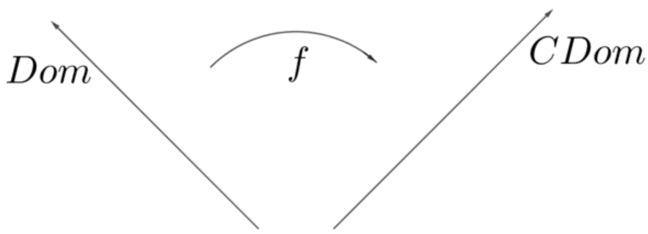


Figure 8.

*Graphical representation of a function without the graphical sketch on the plane.*

The set of real numbers ℝ is, justifiably, represented by an oriented Euclidean straight line; Therefore, this set is called a real line or, simply, a straight line. The sets Dom and CDom are subsets of ℝ; thus, the graphic representation adopted in Figure 8. The rounded line indicates the functional relationship between Dom and CDom, revealing a bottom-up resource regarding attention in relation to the teaching object.

In didactic experiments with a CDI class in which the first author works, observations were made that indicate that the non-joining between Dom and CDom gives a better idea of the behavior of the function when analyzing only the way in which it varies (graphically) in its codomain.

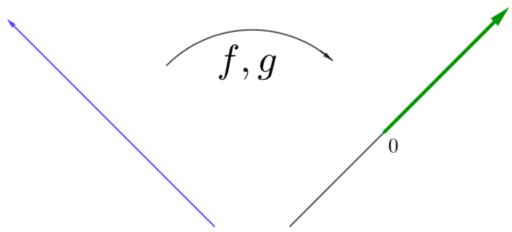


Figure 9.

*Graphical representation of domain, codomain and image of the functions f(x) =* $x^2$ *and g(x) = |x| by two oriented straight lines. Both have the same image: the range [0, +∞].*

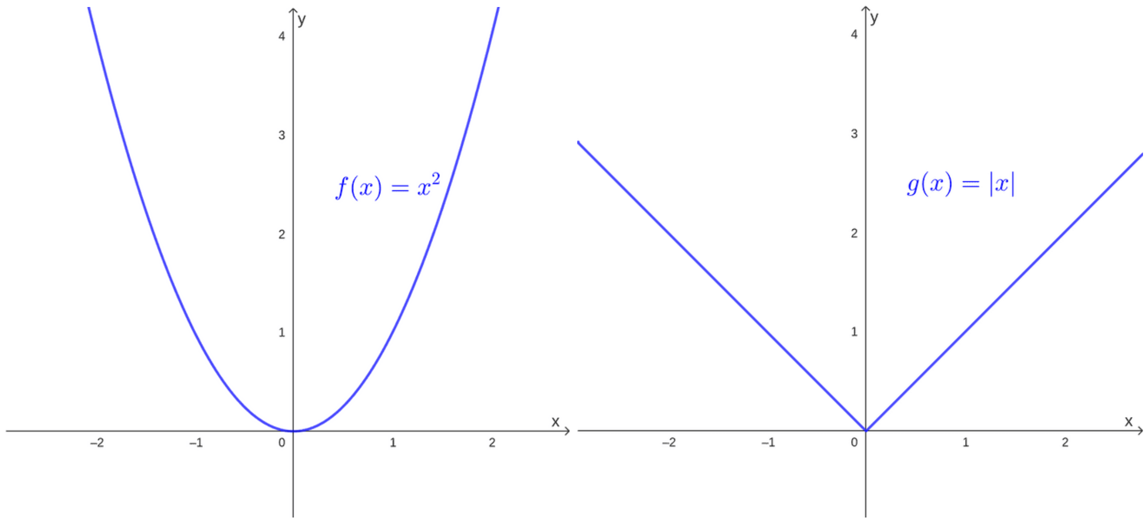


Figure 10.

*Representation of the respective graphs of the functions f(x) = x2 and g(x) = |x| in the Cartesian plane.*

We leave the following warning: the representation in Figure 10 is "myopic", as functions that are known to be distinct can be represented in an identical way.

**Praxeological organization**

Let us return to the concept of limit, in its formal definition: the limit of the function f as x tends to p is equal to L if, and only if, given arbitrary $\varepsilon > 0$, there exists from this fixed $\varepsilon$, $\delta > 0$ such that $0 < |x - p| < \delta \Rightarrow |f(x) - L| < \varepsilon$. Let us consider the following two subtasks.

*Subtask 1: Graph the limit, if any, of the function f(x) = x3 as x tends to -2.*

- Attention Mecanismo (*top-down*): Observe, with some degree of rigor, the logical rules that allow a mathematical conclusion and the necessary technologies, especially the prerequisites, with a view to understanding the subtask.
- Specific praxeology (task, technique, technology and theory): The subtask was given. The technique can be thought of in a few steps. The first AP is the algebraic manipulation, with geometric understanding, of the highlighted modules: $0 < |x - p| < \delta$ means "$p - \delta < x < p + \delta$ and $x \neq \delta$"; $|f(x) - L| < \varepsilon$ means "$L - \varepsilon < f(x) < L + \varepsilon$". Therefore, the organization of the modules, in the definition, presupposes the search for candidates in the intervals $]p – \delta, p + \delta[$ and $]L – \varepsilon, L + \varepsilon[$. The next step is to find suitable epsilon and delta values to satisfy the condition, so the epsilon-delta values change throughout the search. We note that this search for epsilons-deltas is finite for two crucial reasons: a) the computational limitation and b) the time limitation of the subjects who carry out the "epsilon-delta" search! Therefore, the proposed graphical representation does not constitute a demonstration of the existence of the limit. Therefore, a reasonable proposal is to compare the values of f(x), in the range, to a fixed point; Suppose without loss that this point is the origin, 0 (zero). Technology is "limits", and the associated theorems and theory are Real Analysis.
- Attention mechanism (bottom-up): use of colors for the different objects treated.

Theoretical considerations:

i. It makes sense to calculate the limit when x tends to -2, since -2 is an accumulation point of ℝ, the domain of the function $f(x) = x^3$.

ii. The existence of candidates for the limit will depend on the distance from f(] p – δ, p + δ [) to the zero point. More specifically, the length of the interval f(] p – δ, p + δ [) will be indicative of the existence of a candidate for the limit: if this length increases, we will consider that there are no candidates; if this decreases, then a candidate may exist (if the intervals were compact, for example, [p – δ, p + δ], the Embedded Interval Theorem would be able to guarantee the existence of the candidate to the limit).

iii. To choose the limit candidate, we will proceed in a completely intuitive way. Let's cause a “major” and a “minor” disturbance, and observe how the function behaves. The values in the middle column (from -15.525 to -8.0012) and those in the right column (from -3.375 to -7.9988) are the calculations for f(-2-δ) and f(-2+δ) respectively. By these calculations, a threshold candidate is -8.

iv.

Table 3

*Limit Candidate Calculation*

| δ | -2-δ | -2+δ |
|---|---|---|
| 0,5 | -15,625 | -3,375 |
| 0,4 | -13,824 | -4,096 |
| 0,3 | -12,167 | -4,913 |
| 0,2 | -10,648 | -5,832 |
| 0,1 | -9,261 | -6,859 |
| 0,01 | -8,120601 | -7,880599 |
| 0,001 | -8,012006001 | -7,988005999 |
| 0,0001 | -8,001200060001 | -7,998800059999 |

iv. In turn, the image adopted will be that of straight segments, representing real intervals; if a function is continuous on a real interval I, then its image f(I) will also be an interval; the concept of continuity usually appears after the concept of limit (based on the usual literature on this topic adopted in Brazil). So, let's start by "skipping over steps" in this graphical representation.

v. We adopted the heuristic procedure. Assuming that the limit is, in fact, equal to -8, we will create the domain-image representations on different lines, in the search for epsilon-delta. For each ε, the interval ]-8-ε, -8+ε[ will be called Jε; for each δ, the interval ]-2-δ, -2+ δ[ will be called $I_δ$.

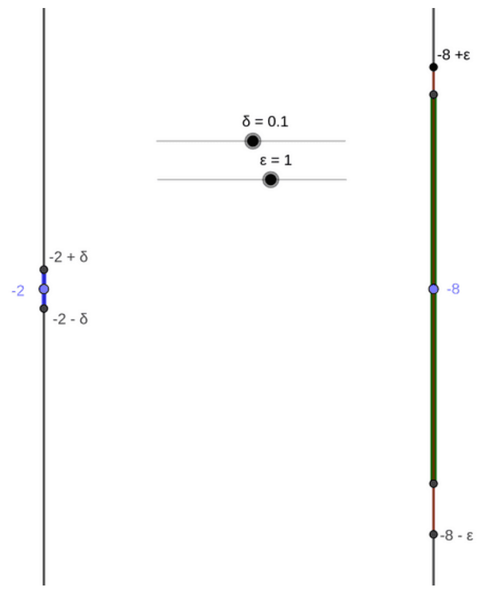


Figure 11.

*Geometric representation of ε and δ according to data in Table 3.*

For the function considered, fixed ε = 1, note that δ = 0.1 is such that f(Iδ)⊂ Jε. Therefore, fixed ε = 1, δ = 0.1 meets the condition for validating the search for a limit. Note that this behavior remains even if there are values of ε greater than 1, which makes this search inappropriate. Therefore, the search may become interesting with values less than 1 and greater than 0. In fact, the idea is to make ε tend to zero. In this work, the Geogebra software was used to generate this type of image. For ε = 0.5, we already found computational restrictions to make the graphical registration.

*Subtask 2: Graph the limit, if any, of the function f(x) = tg(x) as x tends to π/2-. Note: the notation used is lateral limit and tending to π/2- means approaching π/2 but always by values strictly smaller than π/2.*

The attention mechanisms and specific praxeology are the same as in Subtask 1.

Theoretical considerations:

i. It makes sense to calculate the limit when x tends to π/2, since π/2 is an accumulation point of the domain of the function f(x) = tg(x). It is important to note that π/2 is not part of the domain of the tangent function. Comment: this is a very important aspect with regard to attention mechanisms, because, in most cases involving student-teacher relationships in the classroom, we find that the calculation of the limit of a function f(x) when x tends to p is done without observing the fact that p is an accumulation point of the domain of f.

ii. The existence of limit candidates will depend on the distance from f(] $p - \delta_1$, $p - \delta_2$ [) to the zero point, with $0 < \delta_2 < \delta_1$, since it is a left-side limit. More specifically, the length of the interval f(] $p - \delta_1$, $p - \delta_2$ [) will be indicative of the existence of a limit candidate in conditions analogous to Subtask 1.

Table 4

*Limit Candidate Calculation*

| δ1 | π/2-δ1 | tg | δ2 | π/2-δ2 | tg | length_image |
|---|---|---|---|---|---|---|
| 0,500 | 1,071 | 1,830 | 0,250 | 1,321 | 3,916 | 2,086 |
| 0,400 | 1,171 | 2,365 | 0,200 | 1,371 | 4,933 | 2,568 |
| 0,300 | 1,271 | 3,233 | 0,150 | 1,421 | 6,617 | 3,384 |
| 0,200 | 1,371 | 4,933 | 0,100 | 1,471 | 9,967 | 5,033 |
| 0,100 | 1,471 | 9,967 | 0,050 | 1,521 | 19,983 | 10,017 |
| 0,010 | 1,561 | 99,997 | 0,005 | 1,566 | 199,998 | 100,002 |

The “length_image” column is the length of the interval ] f(p – δ1), f(p – δ2) [, with the specified deltas. These calculations point to the non-existence of a limit, but without presenting a formal demonstration of this fact.

Based on an *a priori* analysis and didactic experiments in a Differential Calculus class composed of 40 students from undergraduate and bachelor's Mathematics and Civil Construction Engineering courses, we infer that the REM presented in this section proved to be a valid teaching resource in teaching- limits learning, but it cannot be said that it is invested with the necessary rigor to be characterized as a mathematical demonstration.

## Final considerations

Teaching limits is a fundamental topic in Mathematics, being essential for understanding various concepts and applications. In this sense, the epistemological approach to teaching limits must consider students' attention mechanisms, seeking to create strategies that facilitate learning and promote a greater understanding of the concept.

By taking into account students' attention mechanisms, it is possible to identify which points arouse the most interest and engagement, allowing the teacher to adapt his teaching methodology and make classes more dynamic and participatory. Furthermore, by using resources that stimulate students' concentration and focus, such as practical activities and contextualized examples, it is possible to achieve more favorable conditions for understanding the objects studied and ensuring better absorption of the content.

Another important aspect to be considered in the epistemological approach to teaching limits is spaced repetition. Revisiting the same content at regular intervals helps with memorization and consolidation of information. Therefore, by creating a study plan that includes periodic reviews of limit concepts, it is possible to strengthen students' understanding and ensure more solid learning. Furthermore, it is essential to encourage students’ autonomy in learning, allowing them to develop skills such as self-assessment and self-regulation.

By providing tools for students to monitor their own progress and identify their difficulties, greater responsibility for learning is promoted and a collaborative environment among peers is encouraged. In short, adopting an epistemological approach that takes into account students' attention mechanisms when teaching limits allows creating an environment conducive to the construction of mathematical knowledge. Through the combination of strategies that stimulate students' interest, concentration and autonomy, it is possible to promote meaningful and lasting learning in this fundamental field of Mathematics.

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