ISSN 1983-3156
Qualis A1
http://dx.doi.org/10.23925/1983-3156.2025v27i2p252-275

Mathematical knowledge for teaching flat and spatial figures in middle school: visualization in focus

Conocimientos matemáticos para la enseñanza de figuras planas y espaciales en los últimos años de la educación primaria: la visualización en foco

Connaissances mathématiques pour l'enseignement des figures plates et spatiales dans les dernières années du primaire : la visualisation en point de mire

Conhecimentos matemáticos para o ensino de figuras planas e espaciais nos anos finais do ensino fundamental: a visualização em foco

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Resumo

Neste artigo, discutimos sobre conhecimentos para o ensino de figuras planas e espaciais, com foco específico na visualização matemática, desvendados de episódios ocorridos em aulas de geometria em uma turma de 7º ano do ensino fundamental. A pesquisa foi realizada em uma escola pública da rede federal de ensino e os sujeitos da pesquisa foram o professor de matemática regente da turma, dois estagiários e os estudantes da turma. Os dados produzidos a partir dos episódios foram incluídos na literatura específica sobre a visualização no ensino de geometria na educação básica. A partir da análise de conteúdo de excertos dos episódios, foi possível identificar demandas de saberes que se mostraram específicos para o ensino de geometria, as quais dizem respeito ao tratamento da visualização no ensino de figuras planas e espaciais nos anos finais do ensino fundamental. A partir da análise dos dados, inferimos que as questões relativas à visualização a evidenciaram não somente como um recurso e/ou

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estratégia para o ensino de geometria, mas como um conhecimento específico para o ensino de geometria no ensino fundamental.

Palavras-chave: Conhecimento matemático para o ensino, Matemática escolar, Ensino de geometria, Visualização.

Abstract

In this paper, we discuss knowledge for teaching flat and spatial figures, with a specific focus on mathematical visualization, revealed from episodes that occurred in geometry classes in the 7th year middle school class. The research was carried out in a public school in the federal education network and the research subjects were the mathematics teacher in charge of the class, two interns and the students in the class. The data produced from the episodes were in the specific literature on visualization in teaching geometry in basic education. From the content analysis of excerpts from the episodes, it was possible to identify knowledge demands that were specific to the teaching of geometry, which concerns the treatment of visualization in the teaching of flat and spatial figures in the middle school. From the data analysis, we infer that the questions related to visualization highlighted it not only as a resource and/or strategy for teaching geometry, but as specific knowledge for teaching geometry in middle school.

Keywords: Mathematical knowledge for teaching, School mathematics, Geometry teaching, Visualization.

Resumen

En este artículo, discutimos conocimientos para la enseñanza de figuras planas y espaciales, con un enfoque específico en la visualización matemática, revelados a partir de episodios ocurridos en las clases de geometría en una clase de séptimo año de escuela primaria. La investigación se realizó en una escuela pública de la red educativa federal y los sujetos de investigación fueron el profesor de matemáticas a cargo de la clase, dos pasantes y los estudiantes de la clase. Los datos producidos a partir de los episodios fueron ubicados en la literatura específica sobre visualización en la enseñanza de la geometría en la educación básica. A partir del análisis de contenido de extractos de los episodios, fue posible identificar demandas de conocimiento específicas de la enseñanza de la geometría, que se refieren al tratamiento de la visualización en la enseñanza de figuras planas y espaciales en los últimos años de la escuela primaria. Del análisis de los datos, inferimos que las preguntas relacionadas con la visualización la destacaron no sólo como recurso y/o estrategia para la enseñanza de la geometría, sino como conocimiento específico para la enseñanza de la geometría en la escuela primaria.

Palabras clave: Conocimientos matemáticos para la enseñanza, Matemática escolar, Enseñanza de la geometría, Visualización.

Résumé

Dans cet article, nous discutons des connaissances pour l'enseignement des figures planes et spatiales, avec un accent particulier sur la visualisation mathématique, révélées à partir d'épisodes survenus dans les cours de géométrie d'une classe de 7e année de l´enseignement fondamental. La recherche a été réalisée dans une école publique du réseau éducatif fédéral et les sujets de recherche étaient le professeur de mathématiques responsable de la classe, deux stagiaires et les élèves de la classe. Les données produites à partir des épisodes ont été localisées dans la littérature spécifique sur la visualisation dans l'enseignement de la géométrie dans l'éducation de base. À partir de l'analyse du contenu d'extraits des épisodes, il a été possible d'identifier des exigences de connaissances spécifiques à l'enseignement de la géométrie, qui concernent le traitement de la visualisation dans l'enseignement des figures planes et spatiales dans les dernières années du fondamental. De l'analyse des données, nous déduisons que les questions liées à la visualisation l'ont mise en valeur non seulement comme ressource et/ou stratégie pour l'enseignement de la géométrie, mais aussi comme connaissance spécifique pour l'enseignement de la géométrie à l'école primaire.

***Mots**-clés* **:** Connaissance mathématique pour l’enseignement, Mathématique scolaire, Enseignement de géométrie, Visualisation**.**

Mathematical knowledge for teaching flat and spatial figures in middle school: visualization in focus

Discussions concerning mathematics teachers’ education have increasingly considered the knowledge related to teaching mathematics in basic education and the issues related to the processes of incorporating this knowledge[3] into the teaching degree (Moreira & David, 2005; Ball et al., 2008; Moreira & Ferreira, 2013). The literature presents some views regarding which knowledge is relevant to address in the teaching degree course, emphasizing what teachers need to know to teach. Understanding this knowledge, what it is, or how it should be characterized has challenged mathematics education.

One of the conceptions about the initial education of teachers who teach mathematics assumes that the teaching degree must be structured based on knowledge characterized as an amalgam of inseparable knowledge and whose nature is the demands of basic school, the foundations of mathematics itself, the relationships between knowledge and practice, the contexts in which teaching takes place, among others. This perspective aligns with what Moreira and David (2005) value when discussing the epistemological nature of school mathematics as both a “set of knowledge produced and mobilized by mathematics teachers in their pedagogical action in the school classroom, as well as results of research that refer to learning and school teaching of mathematical concepts, techniques, processes, etc.” (p. 20).

Based on our experiences –the authors of this article– when working as mathematics teachers’ educators at public universities, we realized that geometry is a mathematics area that requires discussions about what knowledge is related to its teaching in basic education. In line with the authors’ perception, studies such as those by Leivas (2009) and Procópio (2009) problematize that regarding knowledge of academic content, initial teacher education works on geometry dissociated from questions about teaching in basic education. This leads to gaps related to the concepts involved in treating geometry in teaching degree courses and the methodological aspects of its teaching. We can, therefore, consider the need to review the conception of geometry present in the teaching degree course that concerns knowledge of strictly academic content, dissociated from questions about the teaching of its content in basic education.

Understanding that teacher education can consist of a network of discussions about geometry based on approaches that involve issues closer to its teaching in basic education, we

[3] In this article, no epistemological distinction is made between the terms knowledge and expertise (knowing). Both mean the same, as we agree with the authors below, who are our theoretical references regarding knowledge specific to and of mathematics teachers' education.

understand that visualization, intuition, and imagination issues should gain prominence in debates about the teaching degree. Based on this idea, we understand that there will be a search to break with tradition in teacher education in which teaching in this area is exclusively axiomatic-deductive or refers to the conception that the teacher, knowing academic geometry, will be able to transpose it didactically to basic school students.

Given this problem, this article discusses part of the results of a doctoral work investigating what can be constituted as a body of knowledge related to geometry teaching in middle school. Based on two episodes that occurred in geometry classes in a 7th-grade middle school class, this article aims to discuss knowledge for teaching flat and spatial figures revealed in these episodes. The identified knowledge is delimited based on questions pertinent to mathematical visualization and its epistemological nature. The classes were taught by three research subjects: a tenured teacher from the school researched and two practicum students enrolled in a teaching degree course at a public university. In the supervised practices, the teacher and the interns worked with spatial and plane geometry content, more specifically, exploring elements of the characterization of polyhedrons and non-polyhedrons and the concept of a square.

The episodes indicated demands for knowledge of geometry, specifically regarding the mathematical visualization of two-dimensional and three-dimensional objects. We emphasize that the knowledge identified was mobilized, or not, by the teachers. Thus, considering the theoretical frameworks adopted in the research, this knowledge constitutes, in our view, mathematical knowledge specific to the teaching profession.

Theoretical inspirations: visualization in teaching geometry in basic education

One of the components of geometry teaching and learning defended by authors in psychology and education as an important element in the processes of developing geometric thinking at basic and higher levels of education is visualization (Presmeg, 1986; Kaleff, 2015). The National Council of Teachers of Mathematics (NCTM), through the publication of the Principles and Standards for School Mathematics (NCTM, 2000), considers that students' conceptual understanding is improved whenever visual images are used. Furthermore, it recommends using representations of geometric entities and two-dimensional and three-dimensional visualization as important skills to be worked on with students at all school levels so that they become experienced in using different types of representations of geometric shapes.

Several definitions and considerations in the research literature on visualization in teaching mathematics and learning address its importance as a support for intuition and learning of mathematical concepts. According to Zimmermann and Cunningham (1991, p. 3), mathematics educators understand that mental images and external (i.e., non-mental) representations should interact for students to understand concepts and solve mathematical problems. Thus, for the authors, visualization is the context in which this interaction occurs.

Arcavi (2003, p. 217) offers a characterization of visualization by approaching it both as a noun (the visual image) and as a verb (the process or activity). Besides considering the idea of visualization centered on the visual image-process dyad, he paraphrases and articulates the definitions of Zimmermann and Cunningham (1991, p. 3) and Hershkowitz et al. (1989, p. 75) to suggest that:

> Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings (Arcavi, 2003, p. 217).

Gutiérrez (2006) characterizes visualization in mathematics education by considering, firstly, that one of the main characteristics of elementary geometry teaching is the intensive use of objects, figures, diagrams, and schemes that, in the author's conception, can help students understand the concepts, properties, and relationships of what is being studied.

Presmeg (1986) defines visual image as a mental scheme that portrays visual or spatial information, with or without the need for the presence of an object or other external representation. Agreeing with Yakimanskaya's (1991) definition, the researcher above includes images with graphic support that are different from a mental image in the concept of visual image. Presmeg (1986) also postulates that generating conceptual understanding and solving mathematical problems involves logic and reasoning, and an individual may or may not consider the presence of visual images as an essential part of working on such activities.

Gonzato, Gondino, and Contreras (2010) conceive the visualization of three-dimensional objects in close relation with spatial reasoning. The authors emphasize that visualizing a spatial object involves the ability to see it, in a physiological sense and the ability to reflect on its possible representations, the relationships between its elements, structure, and characteristics. For these authors, visualizing also involves examining possible transformations of the object in focus. In our view, Yilmaz and Argun (2018) complement this defense by stating that visualization is a complex process of analyzing transformations, constructions, mental images, and representations and helps establish connections between unknown mathematical ideas and understandings that develop gradually.

In addition to what is understood by visualization in the teaching and learning of geometry in basic school and for teacher education, Fischbein (1993) points out that geometry, in particular, deals with mental entities that simultaneously have conceptual and figural characteristics. From the author's perspective, mental entities, also designated by him as geometric entities, reflect both spatial properties (shape, position, size) and conceptual qualities (ideality, abstraction, generality). Therefore, such entities have two components: a figural and a conceptual one.

In a theorization about a figural concept, the author initially explains that a geometric entity can be described by its conceptual properties. However, as they are also images, the conceptual description alone is insufficient. For example, when we reason in terms of composing, decomposing, and moving geometric entities, we are referring to images and not just concepts since they, although regulated by a conceptual nature, also have an intrinsic figural nature.

The figural concept (or component) is, therefore, "(...) the ideal limit of a process of fusion and integration between logic and figural facets" (Fischbein, 1993, p. 150, our translation[4]). In other words, the figural concept is visual in nature (size, shape, position) and is expressed through images directly or indirectly regulated by concepts. To a greater or lesser degree of formalism, the conceptual component expresses the definitions and properties associated with a particular class of geometric entities through written or spoken language.

In Fischbein's (1993) conception, it is part of the nature of geometry to understand the relationships between geometric entities through a symbiosis of their figural and conceptual components. As the figural component is related to the mental image associated by the student with

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the concept, the author advocates that the harmony between these two components (figural and conceptual) determines the adequate notion of the geometric entity and what characterizes geometric reasoning. Therefore, this "(...) can be interpreted in terms of a dialectical process between the figural and the conceptual aspects (…)" (Mariotti, 1995, p. 104).

In summary, experience-based research emphasizing teaching practices and the understandings that can develop from them discusses visualization and its nature, presenting it as a relevant issue for mathematics education. As relevant and enlightening as such research may be, its results should not mean that visualization is, in Arcavi's (2003, p. 238) words, "(...) a panacea for the problems of mathematics education". However, its understanding may advance the teaching of properties, concepts, and relationships present in the construction of knowledge about geometry in basic education (Presmeg, 1986; Zimmermann, Cunningham, 1991; Mariotti, 1995; Arcavi, 2003).

Geometry teaching practice as a field of research

In this article, we refer specifically to two episodes that occurred in classes observed in the 7th grade of middle school of an application school of the federal education network, which we identify in the following sections as **the case of the prism bases** and **a definition of a square**, respectively. This research was qualitative and participatory (Bogdan, Biklen, 1991). The data production instruments were participant observation of mathematics classes in middle school classes in which geometry content was covered, audio recordings of the observed classes, and research field notebooks containing various notes on the observed classes and the subjects involved in the research.

Therefore, the research subjects involved in the episodes discussed here are the mathematics teacher in charge of the 7th-grade class, codename Benjamin, the pre-service teachers attending Supervised Practicum who accompanied him in his practices when the observations were carried out, with codenames Hipátia and Hilbert (at the time, students on a mathematics teaching degree course at a public university) and the class students. It is worth explaining that when invited to participate in the research, Benjamin reported that many classes would be taught by both pre-service teachers under his supervision. Thus, the pre-service teachers were invited to participate in the research.

The data relating to the episodes were analyzed from the perspective of Bardin's content analysis (1977), in which we sought to identify and problematize, *a posteriori*, mathematical knowledge for teaching, as far as geometry in middle school is concerned, which we understand was demanded based on the teaching practices observed. In other words, the knowledge listed and discussed based on our analyses may or may not have been mobilized by the subjects. So, we analyzed that such knowledge emerged as related to the different types of questions asked in classes. We also emphasize that the data were analyzed and discussed in light of the specialized literature on mathematical visualization presented in the previous section.

The case of the prism bases

Together, the pre-service teachers and Benjamin planned that, in that school year, geometry classes on flat and spatial figures would start with activities for 7th-grade students to describe the geometric objects presented to them: polyhedrons and non-polyhedrons in acrylic and polygons and circles in rubber. This activity aimed to help students classify such objects according to their criteria and understand that non-planar objects have one or two of their faces on two distinct and parallel planes so that the other faces of these objects are outside these two planes.

In their planning, the teachers (Benjamin, Hipátia, and Hilbert) proposed gradually advancing from the students' initial conception to systematized geometric knowledge. They proposed starting from students' existing knowledge, relating it to objects in the social environment, and then moving towards a systematization of the knowledge constructed through this process.

Thus, in one of the observed classes, some students were invited to go to the front of the room, one at a time, to choose an acrylic object –which was a concrete representation of a spatial geometric shape– from among the objects that were arranged on two tables, and describe them according to what they saw and/or what they knew about them. In addition to the acrylic objects, there was a material made from a Styrofoam plate covered in fabric that Benjamin named *cloth* and represented, in classes, a plan.

One of the students –codenamed Otávio– chose the representation of a truncated pyramid with a square base for description. While exploring the material, the student drew attention to a part of the figure, indicating it as its base. This observation triggered a discussion around the basic conception of a spatial figure, as follows in the excerpts below:

> Otávio: This figure is a square here [referring to one of the bases of the pyramid trunk], but, here, um... well... it's... there's a big square for the base, which would be, and a small square to close it [..]
> Joana: Don't you think that, for example, the larger square part at the bottom, in this case, it is the base, but the small square at the top could also be a base?
> Hipátia: Do you think the figure can have only one base?
> Joana: No! I'm saying that this figure has more than one base. But, okay... But, when you look at it like this, right, from this angle [...] it can also have more than one base. Below is a larger square, which in this case is the base, but on the side and the smaller square can also be a base! (Dialogue between a pre-service teacher and students, 2018) (Pereira, 2020, pp. 78-79).

After some discussion, Hipátia asked another group member to come to the front of the room to examine a new object. The student chose a cone:

> Hipátia: Since it doesn't have what you called a base, what does it have? [...] That is why, Joana, in mathematics, I have to describe what I am considering the base. [...] If I place it like this [with the vertex of the cone resting on the cloth], can I say that this is its base? [...] We define what I consider to be the base of the figure. [...] I'm going to choose what I call the base.
> Melissa: I think it's kind of the fulcrum.
> Hipátia: If we say it is a fulcrum, then doesn't it have a support point here [placing an icosahedron supported on one of its triangular faces on the cloth]? [...] So, look. We cannot just remain at the base that provides support, that provides sustainment. We have to look at another condition to say: this is the basis of my figure. All right? Look at this [a prism with a triangular base with one of its lateral faces resting on the cloth]! It could stay like this and it is well supported. [...] So, how am I going to choose what the base for my figure is? [...] So, should I look at what characteristic I'm going to have to observe to say that that is the base of the figure? [...] These two objects here [showing a pyramid and an octahedron]. Do they have characteristics in common?
> Joana: Can I consider that it [pointing to the octahedron] has only one base, since all the parts are bases? [...] I just didn't understand what a base is...
> Hipátia: What is basis? It's a feature for, it's... How can I explain this part? Let's work with this figure here [he took a prism with a triangular base and a prism with a pentagonal base and placed them on the cloth]. [...] How can I explain this? When I take a family of figures, for example, they have, how could I name them, this figure here [prism with a triangular base]? […] And this one here [pentagonal base prism]? [...] as we explained to you just now, within this group of families, within this group of figures,

> within these families, the figure has some shared and some individual characteristics. This individual characteristic, whoever is going to bring me this information, is the basis of the figure. So, look at these two here [the triangular-based and pentagonal-based prisms], what do they have in common? Why can I classify them in the same family? [...] Because they are made of rectangles and... [...] So, I can group it following this idea. How many bases it has. It has these bases. And how do I differentiate them since they are part of the same family?
> Joana and other students reply: It's the basis!
> Hipátia: It's the base. […] The base is what differentiates the figure (Dialogue between the pre-service teacher and students, 2018) (Pereira, 2020, pp. 84-100).

The methodological approach used in the class was chosen because, according to the teacher and the pre-service teachers, the students had not had adequate contact with geometry concepts in the previous school grade. Despite this fact, we noticed the students had intuitive notions about spatial geometric figures and knowledge. However, they found it difficult to express some elements of these figures due to their characteristics and nomenclatures as treated in school geometry (edges, vertices, faces, bases, etc.).

The idea of the basics of a prism triggered several questions from the students, who, at times, were answered by the teachers, who were still not sure about the base as an element of some polyhedrons. In this class, we understand that the systematization of an idea on the base of the spatial figures was carried out from the visualization from Kaleff's (2013, p. 85) perspective: "[...] See the object (real, visual, or tactile image of the physical object) through the sensorial apparatus." When thinking that systematization is linked to the exploration of the figures, it can occur from the standpoint of the one who cares for "[...] seeing with the eyes of the mind."

From the reported episode and Arcavi's (2003) and Presmeg's (1986) remarks, we have evidence of a demand for knowledge for teaching geometry regarding understanding the use of visualization as a process where reason and intuition are privileged within a harmonious and coherent relationship with the level of schooling for which it is intended. Also, one can benefit from students' knowledge regarding the figural and conceptual components of the entities studied (Fischbein, 1993), thus favoring the construction and exploration of new and more advanced understandings of constructions and geometric characteristics of spatial figures.

In this way, we indicate the importance of knowledge of the analysis of possible obstacles and difficulties in using visualization, especially when it is linked to a pedagogical approach in which intuition is prioritized as the main level of construction of geometric thinking.

Teaching practices using visualization, in which students are presented with figural representations ready to manipulate, still require teachers to anticipate possible difficulties, doubts, and mistakes in this process. As examples of some questions that may permeate teachers' reflection on this, we can point out: What doubts may emerge when exploring ready-made concrete representations? What misunderstandings can be generated from the manipulation of these representations? What are the key concepts involved in exploring polyhedra and non-polyhedra? We are aware that the teacher cannot foresee all the obstacles facing the topic he will address. However, as stated by Moreira and David (2005), knowing some possibilities is part of the teacher's professional knowledge.

In this practice, the teacher and pre-service teachers' intentional use of the resource of visualizing objects to discuss their characteristics and bring them to the geometric figures they represent clearly shows knowledge for teaching. Through the visualization students proposed and carried out, intuitive characteristics were highlighted and transported to the geometric figures where mathematical properties could be systematized.

A definition of a square

In subsequent classes, Benjamin worked with drawings of two-dimensional and three-dimensional shapes, without the presence of the pre-service teachers, using geometric drawing techniques with a ruler and compass. The objective of each construction was to obtain and explore with the students a geometric figure and, from it, the construction of what he called **artmetry** (a mixture of art and geometry, according to Benjamin).

In one of these classes, Benjamin explained that the first construction would be an inclined square, measuring six centimeters on each side and that this figure should be constructed toward the center of an A4 sheet handed to the students. He guided them in constructing two adjacent sides of the square. As the students constructed a square on their papers, Benjamin continued to ask that the square be tilted and finished drawing the figure on the board, freehand, which resulted in distortions of the 90º angles, as illustrated in Figure 1:

Figure 1.

Tilted square drawn freehand on the board (Pereira, 2020, p. 123)

> Benjamin: [...] The idea is that the square is turned over, ok? It's still a square, okay? What are the elements and characteristics of a square? Do you know them?
> Melissa [raising her hand]: Me! It must have equal sides.
> Benjamin: The four sides are equal.
> Joana: Does the square have four 90º angles?
> Benjamin: It must have four angles of 90º. The four equal sides [writing this information on the board]. Okay? Then, we will look into this to see how these definitions work. Because I have a figure, look at this figure! It... Is it a square [she drew a rectangle on the board freehand]? [...] And it has four right angles, look [marking the right angles of the rectangle with little squares]!
> David: Oh, it's a rectangle, huh!
> Benjamin: Look at this figure here, look [he drew a diamond on the board, freehand]
> Student: It's a quadrilateral.
> Benjamin: It has four equal sides, but it doesn't have four right angles, see? So, we will have to study these elements, little by little, which is why we must be attentive. In the figure you are going to draw now, you need a four-sided figure. The four sides must be six centimeters, but the interior angles of this figure must be angles of... 90º. Come on, guys! [...] It's funny that when we turn the square in this position [the position the drawing was on the board], some students think that it's not a square. Then, they give it another name. Ah, it resembles that figure that looks like a sweet treat. What do you call that figure that is shaped like a treat? The one I drew there, half-baked, in the corner. What is this figure called here? (Dialogue between a pre-service teacher and students, 2018) (Pereira, 2020, p. 123).

While involving the students in constructing an inclined square, the teacher explored, based on visualization, some characteristics of quadrilaterals that classify them as rectangles or rhombuses. Benjamin returned to the construction of the square and told the students that, at that moment, they already knew how to build it or how to build a rhombus with 90º angles, as the following excerpts describe:

Benjamin: Look, guys. This geometric figure... so, look at this geometric figure. Does

it have four sides?
Students: Yes.
Benjamin: So, if it has four sides... what name do we give it in mathematics?
Some students: A quadrilateral.
Benjamin: The four sides... are they parallel? What does it mean? Look. If I take this side here [pointing to one side of the square drawn on the board] and stretch it infinitely [extended the side with dots in both directions] and this side infinitely [did the same thing with the opposite side] will they meet? [...] It has one pair of parallel sides... They won't meet, okay? Later, I will show you this better in GeoGebra. [...] So, what do I call a figure that has all parallel sides?
Some students: A parallelogram.
Benjamin: A parallelogram. Now we're going to look at the angles. Does it have four angles...?
Christopher: Of 90º.
Benjamin: Of 90º and which are called... right angles. If it has all right angles, what name do I give to this figure?
Christopher: A rectangle? Straight?
Benjamin: This figure is also a rectangle. Now, let's look at the sides. Does it have four equal sides?
Students [in chorus]: Yes.
Benjamin: A figure with four equal sides, what name do I give it?
Christopher: A diamond.
Benjamin: A diamond. Hey guys, check it out! A figure, which is a quadrilateral, a parallelogram, a rectangle, and a rhombus at the same time is called...?
Students [in chorus]: A square.
Benjamin: So what is a square? It is quadrilateral, because it has four sides. It's a parallelogram, because all sides are parallel, right? There will be two pairs of parallel sides. It is a rectangle, because it will have all angles of 90º, right angles. It is a rhombus, because all sides are equal. And then, because it is all of this, it receives a special name, which is square. So, for something to be square, it must go through all of this. And that's the care we should take. Ah, it has four equal sides but no right angles, will it be a square? (Dialogue between a pre-service teacher and students, 2018) (Pereira, 2020, pp. 124-125).

By proposing the construction of an inclined square, we consider it detached from the design in the prototypical position we generally find (Presmeg, 1986). The teacher explored with the students the idea that in that position, the drawn square would not interfere with its definition as such a figure, removing the positioning from the focus when thinking about its characterization. It is worth noting that Benjamin also made it clear to students that the non-standardization of the figures' configurations does not interfere with their characterization.

His discussion about the square, in our view, prioritized relevant elements of the study of a geometric figure since, in addition to creating a situation in which students could construct a square by handling geometric drawing instruments, it also led them to discuss the conditions of its characterization as such. Regarding the definition of a square, although the students already knew the figure through their school experience or social experience, based on the teaching method used, Benjamin reconstructed and systematized the concept related to this geometric figure so that visualization and intuition were in harmony in the development of geometric reasoning (Presmeg, 1986; Fischbein, 1993).

Thus, we understand that, in this class, the teacher worked from the perspective that a geometric figure has a conceptual nature (Fischbein, 1993). A square is not just an image drawn on a blackboard or a paper sheet. It is a form regulated by a definition, even if this definition is guided by a real object. In the episode, one can perceive the mobilization of knowledge of how to define a square, in which there is a characterization of this geometric figure, supported by a non-prototypical representation, which includes it in other classifications of flat figures, and which expresses to students both its figural and conceptual properties (Presmeg, 1986; Fischbein, 1993).

From this episode, knowledge mobilized by Benjamin about the use of non-prototypical figures also emerges, an important element of visualization that encourages the creation of mental images that are not necessarily standardized, which contribute to the production of meanings, favoring the understanding of concepts (Presmeg, 1986). Also related to this situation, in our view, is knowledge of a type of activity in which it is possible to work on both the non-standardized construction of the figure and the exploration of its characterization.

Discussions on mathematical knowledge for teaching geometric figures in middle school with a focus on visualization

Based on the reported episodes, when situating the data we consulted in the literature on visualization, we sought to bring to light the possibilities of mathematical knowledge for teaching geometry in elementary education that emerged from them. As already reported, we focus on knowledge demands related to visualization in geometry. Therefore, we identified some knowledge that we understand to be part of the constitution of a set of mathematical knowledge for teaching geometry in elementary education and, therefore, are specific to the mathematics teacher. We summarize some of them below.

Knowledge of prisms, pyramids, cones, and cylinders that help teachers construct, together with students, a characterization of these geometric shapes as those with predefined faces as their bases so that the role of these elements in this characterization is made explicit.

Knowledge of polyhedra and non-polyhedra that helps teachers in explaining a characterization of the former as spatial figures formed by a finite grouping of polygons, in which each side of each polygon is common to another polygon.

Knowledge of prisms, pyramids, cones, and cylinders that helps teachers and students together to construct a characterization of those geometric shapes as those with predefined faces as their bases so that the role of these elements in this characterization is made explicit.

Specialized knowledge of polyhedra and non-polyhedra that helps teachers explain the characterization of the former as spatial figures formed by a finite group of polygons, in which each side of each polygon is common to another polygon.

Knowledge of visualization as a component of geometry teaching and learning, mobilizing the knowledge that students already have as a process related to the creation and interpretation of mental images for the generation of information about geometric entities, the recognition of relationships and transformations between geometric objects, the characterization of geometric shapes, the construction and understanding of concepts and the symbiosis between the figural and conceptual components of geometric figures. This knowledge implies other knowledge related to the obstacles and difficulties associated with the use of visualization in geometry teaching in basic education and in knowledge of anticipating students' doubts, especially when working with the manipulation and visualization of concrete representations of geometric figures.

Knowledge of a diverse repertoire of representations of geometric figures that helps the teacher in guiding students' progress in creating mental images of geometric entities. Knowledge of a repertoire of representations of images of spatial and flat figures (drawings, concrete materials, dynamic images generated in software, among others) that are non-prototypical and encourage students to avoid excessive recourse to prototypical figures as the only mental images of geometric figures.

Knowledge of the choice and creation of didactic situations and tasks that help students develop spatial reasoning, especially regarding the transition from space to the plane, and provide them with opportunities for experimentation, construction, and analysis of diverse representations of geometric figures and the concepts related to them.

When analyzing the episode about the bases of a prism, we noticed that the idea about this element was established as a very fruitful, however controversial, discussion for the students and the teachers, who resorted to explanations about the bases and the characterization of prisms that did not satisfy or contradicted some elements of the characterization of the figures as abstract geometric entities at some points.

Regarding the idea of bases, the teachers proposed that they be defined as those that differentiate the figures from each other within the same group of figures. Regarding the characterization of prisms, the teachers developed the concept that their lateral faces are all rectangular. However, these systematizations triggered some students' difficulties in understanding, which we expose in some dialogues below.

From our standpoint, the approach to the subject of polyhedra and non-polyhedra revolved around the visualization of some types of representation of these figures, in the sense of "seeing to argue" (Santos, 2014). In other words, the arguments debated in class were formulated and discussed after the students had viewed the objects. In these cases, according to Gutiérrez (1996), visualization must reinforce the use of representations to obtain information that helps to create and use mental images as representations of geometric concepts.

Geometric concepts can be understood in the context of geometric taught in primary school as a set of properties, relationships, and definitions referring to geometric entities (Cifuentes, 2003). We consider that the concept of square mobilized by Benjamin exemplifies this notion. According to Cifuentes (2003, p. 60), given the visual nature of various designations in geometry (shape, congruence, similarity, etc.), geometric concepts must express "in addition to properties, transformations and correspondences inherent to the entities studied."

Again, regarding the process of constructing and learning geometric concepts, we can say that teachers must be aware of several contributions of encouragement to the creation of mental images. Dreyfus (1991), for example, argues that mental images of mathematical concepts are necessary to obtain the meanings associated with them. Gutiérrez (1996), citing Kosslyn (1980), states that this author identified four processes applicable to visualization in relation to mental images that are important in conceptual study and that teachers must recognize.

The first of these concerns the ability that students must develop to generate a mental image of a figure based on knowledge of information about it. The second refers to strengthening skills in exploring mental images so that they can observe the fundamental elements of the figure studied. The third corresponds to the development of the ability to transform the mental image of a figure into other mental images of a equal figure or others. In this way, finally, we have the development of students' ability to use mental images to support their answers to questions about the characteristics and properties of geometric figures.

Gutiérrez (1996) gives an example of what Kosslyn (1980) defends by inciting us to think that students who are learning concepts and relationships between quadrilaterals can be led, in the meantime, to imagine a rectangle shrinking to become a square and then transforming itself back into a rectangle. Bringing Gutiérrez's (1996) reflection to current teaching contexts, students' mental images while learning geometry concepts can be better developed with the help of dynamic images generated by software, for example.

About the characterizations of some of the three-dimensional shapes that could help clarify some misunderstandings during the observed classes –which, according to our understanding, are important knowledge for teachers when teaching spatial figures– we consider a brief discussion about the characterization of prisms.

From the point of view of academic mathematics, a prism can be constructed and defined from an $A_1A_2... A_n$ polygon contained in a plan α. Any point B_1 is chosen, not belonging to α and a plane is drawn through that point β parallel to α. We draw the line segment A_1B_1 and, by the other vertices $A_2...A_n$, the line segments parallel and congruent to A_1B_1 that cut β at points $B_2...B_n$. The reunion of line segments parallel to A_1B_1 that cut β at points $B_2... B_n$ is called a prism (or convex prism). Poligons $A_1A_2...A_n$ and $B_1B_2...B_n$, contained in the plans α and β, respectively, are called the bases of the prism. The quadrilaterals $A_1B_1A_2B_2$, $A_2B_2A_3B_3$, and so on, are parallelograms that make up the lateral faces of Figure 2 (Carvalho, 2005):

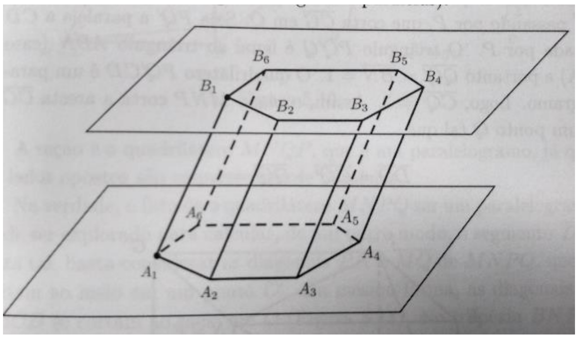

Figure 2.

Construction of a prism (Carvalho, 2005, p. 37)

This way of constructing and defining a prism clarifies that its bases are the two congruent and parallel polygons $A_1A_2...A_n$ and $B_1B_2...B_n$. The faces of the prism are parallelograms –not always rectangles– because of their construction from two parallel polygons. However, we argue that presenting students with a definition of these figures from this point of view in an already finalized manner is not appropriate from the perspective of teaching geometry in the 7th grade of middle school. During elementary school education, teachers must be aware of the complexity of geometry, which we consider, from the perspective set out by Battista (2007, p. 843), as "an interconnected network of concepts, forms of reasoning, and representation systems used to conceptualize and analyze physical and imaginary spatial environments. " Understanding geometry in this way, we understand that visualization and the use of representations can significantly contribute to the teaching and learning process as long as they complement and do not replace abstract thinking and/or cause a delay –almost unconsciously for those who use them in their practices– in its advancement.

In our analyses, the question about concepts in geometry proved to be very important. Yilmaz and Argun (2018) argue that in mathematics education, the conscious establishment of concepts and relationships between mathematical entities requires an organized construction of an abstraction process, and visualization plays a central role in this composition. Particularly regarding teaching geometry, knowledge of activities that provide students with opportunities to develop their visual skills and encourage visual and conceptual reasoning should have an important place in the knowledge domain of teachers.

According to the authors cited above, as students are encouraged to create mental images related to concepts, they become increasingly capable of evaluating the veracity of the information they receive and the inconsistencies of these images, which implies an increasingly precise conceptual elaboration. At stages where students do not yet have a clear mental image of the abstracted concept or have formed an incorrect image, the abstraction process can be supported with appropriate visual resources. Yilmaz and Argun (2018) also argue that knowledge about how to help students form image expressions through transformation processes from the visual to the symbolic should be considered as teachers' knowledge.

We also consider it crucial that teachers anticipate and recognize students' difficulties and doubts and identify when and how to deal with their conceptions and misunderstandings. In the analysis of the episodes, we could see examples of teaching supported by visualization processes revealing some students' difficulties, obstacles, and possibilities of inadequate internalization of concepts. Dreyfus (1991) argues that it is important for teachers to be aware of the difficulties students may present during visualization, such as:

> [...] [their] inability to see diagrams in different ways; difficulties in recognizing the transformations implied in the diagrams; [...] failure to distinguish between a geometric figure and the drawing that represents that figure; failure to link their visualizations to analytical thinking (Dreyfus, 1991 apud Costa, 2000, p. 177).

Linked to the treatment of representations and mental images is also the knowledge teachers must have of using prototypical figures as a component of geometry teaching (Presmeg, 1986). In the reported episodes, two distinct moments related to this issue occurred. In the classes on spatial figures, the acrylic objects were, to a certain extent, standardized representations of some figures. There were no objects among them, for example, representing prisms and oblique cones, nor polyhedrons in configurations that were more unusual for students' knowledge. In Benjamin's class on the square and the hexagon, the drawings constructed for analysis and conceptual construction were not standardized, and the teacher explored this.

Cifuentes (2005, 2010) indicates that it is inherent to visualization processes to make a spatial-temporal contextualization of the image, that is, a contextualization in which it makes sense to identify its standard representation in the initial moments of working with a specific shape. In a second moment of work with visualization, more devoted to abstraction, says the author, there is a sudden separation of the image from its contextual position. Even though this second moment seeks elements for the study of form in non-standardized representation, this phenomenon almost inevitably leads the student to conceive of a prototype of the geometric figure as its main identification (Hershkowitz, 1990; Presmeg, 1986).

Final considerations

Given the objective of this article, which seeks to discuss, based on two episodes that occurred in geometry classes in a 7th-grade middle school class, knowledge for teaching flat and spatial figures presented in these episodes with a specific focus on visualization, we highlight that the knowledge identified has a structure centered on aspects of geometry that are taught at school since they were characterized based on actual demands of teaching practices and elements indicated by the consulted specialized literature. In other words, we agree with the hypotheses raised by authors such as Moreira and David (2005).

Concluding, we emphasize that the issues related to visualization put it in evidence not only as a resource and/or strategy for teaching geometry. We could also notice it in our study within the scope of specific knowledge for teaching geometry in elementary education.

Concerning treating visualization as knowledge for teaching geometry, we infer, according to our study, that the fact that it does not occur may lead pre-service teachers to be unaware of the usefulness of visual reasoning and its different and relevant roles in mathematics education. Secondly, we realized that mathematics teachers, in their initial education, need opportunities to deepen and become familiar with knowledge of the role of visualization in teaching geometry and visual reasoning tasks, appreciating their educational potential, analyzing and also developing tools to apply them in their classrooms.

The findings of our study can contribute to the idea of structuring geometric education for pre-service teachers based on knowledge constituted according to demands of real teaching practices, as well as elements present in the research literature that consider discussions on teaching and learning geometry/mathematics. This understanding converges with Moreira's

(2012) statement that the perspective of structuring teacher education, according to a body of knowledge specific to this professional, has teaching practice as an essential element.

The teaching degree is still marked by a dichotomous education framework: academic content versus ways of teaching. According to several authors, the amalgamation of knowledge that characterizes specific knowledge for teaching has not yet been sufficiently investigated and understood by academic communities working in teacher education; therefore, more studies must be carried out. When these types of knowledge are investigated based on research such as this, questions and demands for teachers' knowledge are raised that deserve treatment at the level of knowledge of and for practice, without specific separation of this or that fraction of this knowledge as more or less relevant to the work of teaching.

References

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics, 52*, 215-241.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: what makes it special? *Journal of Teacher Education*, *59*(5), 389-407.

Bardin, L. (1977). *Análise de conteúdo*. Lisboa Edições.

Battista, M. T. (2007). The development of geometric and spatial thinking. *Second handbook of research on mathematics teaching and learning* (pp. 843-908).

Bogdan, R.; Biklen, S. (1994). *Investigação qualitativa em educação: uma introdução à teoria e aos métodos*. Porto Editora.

Carvalho, P. C. P. (2005). *Introdução à Geometria Espacial*. SBM.

Cifuentes, J. C. (2005). Uma Via Estética de Acesso ao Conhecimento Matemático. *Boletim GEPEM (USU)*, Rio de Janeiro, *46*, 55-72.

Cifuentes, J. C. (2010). Do conhecimento matemático à educação matemática: uma "odisseia espiritual". In: *Filosofia, Matemática e Educação Matemática*, Editora UFJF.

Cifuentes, J. C. (2011). O "Salto Arquimediano": um processo de ruptura epistemológica no pensamento matemático. *Scientiae Studia (USP)*, *9*, 645-667.

Dreyfus, T. (1991). On the status of visual reasoning in mathematics and mathematics education. *Proceedings of the 15th Conference of the International Group for the Psychology of Mathematics Education* (pp. 33-48).

Fischbein, E. (1993). The Theory of Figural Concepts. Educational Studies in Mathematics, *24*(2), *Springer*, 139-162.

Gutiérrez, A. (1996). Visualization in 3-dimensional geometry: In search of a framework. *Proceedings of the 20th PME Conference* (pp. 3-19).

Gutiérrez, A. (2006). La investigación sobre enseñanza y aprendizaje de la geometria. In Flores et al (org). *Geometría para el siglo XXI* (pp. 13-58). Federación Española de Sociedades de Professores de Matemática (FESPM). Sociedad Andaluza de Educación Matemática Thales.

Gonzato, M., Godino, J. D., & Contreras, J. M. (2010). Evaluación de conocimientos sobre la visualización de objetos tridimensionales en maestros en formación. Marín et al (org) *Investigación en Educación Matemática XV* (pp. 383-392). Ciudad Real: SEIEM.

Hershkowitz, R. (1990). Psychological aspects of learning geometry. P. Nesher & J. Kilpatrick (org.). *Mathematics and Cognition* (pp. 70-95). Cambridge University Press.

Kaleff, A. M. M. R. (2015). Formas, padrões, visualização e ilusão de ótica no ensino da geometria. *VIDYA*, *35*(2), 75-91.

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren*. University of Chicago Press.

Leivas, J. C. P. *Imaginação, intuição e visualização: a riqueza de possibilidades da abordagem geométrica no currículo de cursos de licenciatura em matemática*. (2009). [Tese de doutorado, Universidade Federal do Paraná].

Mariotti, M. A. (1995). Images and Concepts in geometrical reasoning. In: Exploiting Mental Imagery with Computers Mathematics Education. pp. 97-115, Springer.

Moreira, P.C., & DAVID, M.M.M.S. (2005). *A formação matemática do professor: licenciatura e prática docente escolar*. Autêntica.

Moreira, P. C., & Ferreira, A. C. (2013). O lugar da Matemática na Licenciatura em Matemática. *Bolema*, *27*(47), 981-1005.

National Concil of teachears of mathematics (2000). Principles and standards for school mathematics NCTM. Reston.

Pereira, A. N. (2020). *Conhecimentos matemáticos para o ensino de geometria na educação básica*. [Tese de Doutorado, Universidade Federal de Minas Gerais]. https://repositorio.ufmg.br/handle/1843/3614.

Presmeg, N.C. (1986). Visualization in high school mathematics. *For the Learning of Mathematics*, *3*(6), 42-46.

Procópio, R. B. (2011). *Geometria como um curso de serviço para a licenciatura em matemática: uma leitura da perspectiva do modelo dos campos semânticos.* [Dissertação de Mestrado, Universidade Federal de Juiz de Fora].

Roldão, M. do C. (2007). Função docente: natureza e construção do conhecimento profissional. *Revista Brasileira de Educação, 12*(34).

Santos, A. H. (2014). *Um estudo epistemológico da visualização matemática: o acesso ao conhecimento matemático no ensino por intermédio dos processos de visualização*. [Dissertação de mestrado em Educação em Ciências e Matemática, Universidade Federal do Paraná]. https://exatas.ufpr.br/ppgecm/wp-content/uploads/sites/27/2016/03/045_AlessandraHendidosSantos.pdf.

Shulman, L. S. (1986). Those who understand knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Yakimanskaya, I. S. (1991). The development of spatial thinking in schoolchildren. *Soviet Studies in Mathematics Education, 3.*

Yilmaz, R., & Argun, Z. (2018). Role of visualization in Mathematical Abstraction: The Case of Congruence Concept. *International Journal of Education in Mathematics, Science and Technology*, *6*(1), 41-57.

Wu, H. (2011). The miss education of mathematics teacher. *Notices of the MAS*, *58*(3).

Zimmermann, W., & Cuningham, S. (1991). Introduction: What is Mathematical Visualization? Zimmerman & Cunningham (org.). *Visualization in Teaching and Learning Mathematics* (pp. 1-7). MAA.

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