ISSN 1983-3156
Qualis A1
http://dx.doi.org/10.23925/1983-3156.2024v26i3p243-279

# Epistemological reference model in teaching functions through its basic ideas: a praxeological conception

## Modelo epistemológico de referencia en la enseñanza de función en sus ideas básicas: una concepción praxeológica

## Modèle épistémologique de référence dans l'enseignement fonctionnel dans ses idées de base : un concept praxéologique

## Modelo epistemológico de referência no ensino de função em suas ideias básicas: uma concepção praxeológica

Karina de Oliveira Castro[1]
Universidade Federal de Goiás (UFG)
Doutorado em Educação Matemática
https://orcid.org/0000-0003-0315-2774

Antonio Sales[2]
Universidade Anhanguera (Uniderp)
Doutorado em Educação
https://orcid.org/0000-0001-5515-6625

**Abstract**

The aim of this paper is to develop a praxeological model that contributes to the analysis of knowledge based on the conception of an epistemological reference model (ERM). The mathematical content that served as the basis is the teaching of functions in its basic ideas. The starting point was a study of the elements that support the ERM through the Anthropological Theory of the Didactic (ATD). The justification lies in the field of calculus teaching and the attempt to contribute not only to the study of the mathematical content itself but also to provide an alternative analysis through a praxeological model built for this purpose. Through the theoretical frameworks of the dimensions of a didactic problem, an epistemological analysis of the concept's development was carried out for the purpose of constructing the ERM. The economic and ecological analyses allowed a reassessment of the theoretical foundations of the ATD, consequently leading to the creation of what we call the global praxeological model: a final product that serves as an analysis not only for the field of calculus but for institutional knowledge as a whole.

***Keywords:*** Dimensions of a didactic problem, Epistemological reference model, Anthropological theory of the didactic, Basic ideas of functions.

[1] karinadeoliveiracastro@gmail.com
[2] profesales@hotmail.com

**Resumen**

El objetivo de este trabajo es desarrollar un modelo praxeológico que contribuya para el análisis del conocimiento desde la concepción de un modelo epistemológico referencia (MER). El contenido matemático que sirvió de base es la enseñanza de la función en sus ideas básicas. El punto de partida fue un estudio de los elementos que sostienen el MER por medio de la Teoría Antropológica del Didáctico (TAD). La justificativa se encuentra en el campo de la enseñanza del cálculo y del intento de contribuir no solo en el estudio del contenido matemático en sí, pero, sobre todo em la fornecer una alternativa de análisis a través de un modelo praxeológico construido para este propósito. Por los modelos teóricos de las dimensiones de un problema didáctico, se realizó un análisis epistemológico del desarrollo del concepto pára efectos de construcción del MER. Los análisis económicos y ecológicos permitieron una reanudación de las bases teóricas de la TAD y, en consecuencia, la elaboración de lo que llamamos de modelo praxeológico global: producto final que sirve de análisis no para el campo del cálculo, sino para el conocimiento institucional en su conjunto.



***Palabras clave***: Dimensiones de un problema didáctico, Modelo epistemológico referencia, Teoría antropológica de lo didáctico, Ideas básicas de función.



**Résumé**

L’objectif de ce travail est de développer un modèle praxéologique contribuant à l’analyse des connaissances basé sur la conception d’un Modèle Épistémologique de Référence (MER). Le contenu mathématique qui a servi de principe est l’enseignement de la fonction dans ses idées de base. Le point de départ a été une étude des éléments qui font partie du MER à travers la Théorie Anthropologique de la Didactique (TAD). La justification a lieu dans le domaine de l’enseignement du calcul et dans la tentative de contribuer non seulement à l’étude du contenu mathématique lui-même, mais surtout de fournir une analyse alternative à travers un modèle praxéologique construit à ce but. A partir des modèles théoriques des dimensions d’un problème didactique, une analyse épistémologique du développement du concept a été réalisée en vue de construire le MER. Les analyses économiques et écologiques ont permis une reprise des bases théoriques de la TAD et par conséquent l’élaboration de ce que nous appelons un modèle praxéologique global : un produit final qui sert à l’analyse pas pour le domaine du calcul tout seul, mais pour la connaissance institutionnelle efficacement



***Mots-clés*** : Dimensions d’un problème didactique , Modèle épistémologique de Référence ; Théorie anthropologique du didactique, Idées fondamentales de fonction.

**Resumo**

O objetivo deste trabalho é desenvolver um modelo praxeológico que contribua na análise do conhecimento a partir da concepção de um modelo epistemológico de referência (MER). O conteúdo matemático que serviu de base é o ensino de função em suas ideias básicas. O ponto de partida foi um estudo dos elementos que embasam o MER por meio da Teoria Antropológica do Didático (TAD). A justificativa se encontra no campo do ensino do cálculo e na tentativa de contribuir não só no estudo do conteúdo matemático em si, mas, principalmente, fornecer uma alternativa de análise por meio de um modelo praxeológico construído para esse fim. Pelos moldes teóricos das dimensões de um problema didático, foi feita uma análise epistemológica do desenvolvimento do conceito para efeitos de construção do MER. As análises econômica e ecológica permitiram uma retomada das bases teóricas da TAD e, consequentemente, a elaboração do que chamamos de modelo praxeológico global: produto final que serve de análise não só para o campo do cálculo, mas para o conhecimento institucional como um todo.

***Palavras-chave:*** Dimensões de um problema didático, Modelo epistemológico de referência, Teoria antropológica do didático, Ideias básicas de função.

## Epistemological reference model in teaching functions through its basic ideas: a praxeological conception

The purpose of this work is to contribute to deepening the ideas that make up the theoretical framework supporting the conception of an epistemological reference model (ERM). Starting with the justification regarding calculus teaching, we chose the concept of function as the guiding thread of the analyses. The intention is to theorize the development of the ERM through the theoretical bases that underpin the didactic problem, as seen by Gascón (2011) and the basic constructs of the Anthropological Theory of the Didactic, especially the notion of praxeology, developed by Chevallard in 1998.

At the end of this study, we will not only have elements for studying the teaching of the concept of functions but also an expansion of the praxeological model built upon the ERM used.

## Brief notes on the praxeological model in the anthropological theory of the didactic

The Anthropological Theory of the Didactic developed by Chevallard (1999), has three main elements: objects, individuals, and institutions. Objects are material and immaterial entities, represented by the letter o, that exist for the individual x. If there is a set of objects o and an individual x, there is an institution I. These elements are mutually related. The individual x relates to the institution I through what the ATD calls the institutional relationship, and the individual x also relates to the object or through what the author calls the personal relationship. Therefore, object, individual, and institution are the foundation of this theory, which interprets reality through the relationship between its elements.

The concept of the institution is fundamental in ATD, and according to Michèle Artigue, it is this notion that reveals and understands the individual: "For ATD, the basic objects are the institutions. The individual emerges from their various institutional subjections, becoming visible through their effects. The individual primarily serves to reveal and understand them" (Artigue, 2010, p. 42, our translation). Institutional practices come to life through the relationship between their basic entities (rapport). Chevallard (1998, 1999) sought to develop a model that could describe and analyze these institutional practices. The author thus developed the praxeological model, or praxeology: "[...] it is indeed assumed that all regularly performed human activity can be summarized in a single model, which here is summarized by the word praxeology" (Chevallard, 1998, p. 1, our translation). The praxeological model consists of four elements that, together, can model human activity, social practices, and mathematical activity, according to Chevallard (1999). Let's explore these elements further.

The notion of praxeology is built upon the concept of tasks and types of tasks. The idea is that every institutional practice can be analyzed through a system of tasks. When considering the methodological universe of mathematics in a classroom, for example, we refer to study tasks. Using a symbolic and axiomatic approach, Chevallard (1998) shows that there are tasks $\tau$ and types of tasks T. The types of tasks T are expressed through precise verbs and objects. For instance, "calculate the root of the equation" is a type of task, but "calculate" on its own is not. The notion of task $\tau$ is the broader and more encompassing concept. From this, we deduce the idea of types of tasks T. Furthermore, Chevallard (1998) also discusses task genres, which are tasks composed of a verb without specification, such as "calculate." Therefore, we have a hierarchical structure: tasks $\tau$, task genres $\tau$, and types of tasks T. In addition, Chevallard (1998) emphasizes that the essential object of didactics is the reconstruction of these artifacts in an institutional setting. We can affirm that tasks aim at reconstructing a specific institutional work that is, to some extent, validated.

The second element of the praxeological model is what explains how to perform the types of tasks T: it is what the author calls technique $\tau$. The word "technique" comes from the Greek word *tekhnê*, meaning "know-how." Bosh and Chevallard (1999) clarify that the term "technique" in this theoretical approach carries a broad meaning, referring to a way of solving a given situation. Therefore, in the praxeological model, technique does not primarily mean a specific algorithm or method. It is a broader concept that includes, for example, techniques to solve an equation, to install an application, or to open a door, etc.

The concept of technique $\tau$ forms the technical-practical block in the praxeological model, symbolized by [T / $\tau$]. This is what is usually referred to as "know-how." Thus, "we have a certain type of task, T, and a certain technique, $\tau$, to carry out tasks of this type" (Chevallard, 1998, p. 2, our translation). There are three important considerations regarding the concept of technique in the praxeological approach that we will explore below.

First, a technique $\tau$ may not be sufficient to fully accomplish a task of type T. This means that there may be techniques that address part of T, its entirety, or techniques that are superior to others. Chevallard (1998) illustrates, for example, that any calculation technique in $\mathbb{N}$ may not be sufficient for numbers of certain sizes. Second, not every technique $\tau$ can be classified as algorithmic. For instance, painting a drawing and explaining the axioms of a particular mathematical field are techniques that do not depend on algorithms or step-by-step processes. However, Chevallard (1998) argues that there seems to be a certain algorithmic tendency in task execution. Finally, Chevallard (1998) postulates that generally, there are only a small number of techniques τ associated with a type of task T. Institutionally speaking, only a few are recognized, which means that some alternative techniques are excluded or might be adopted by other institutions. In some cases, we may witness different techniques coexisting to solve certain equations. For example, in chemistry and physics, students may be introduced to a cancellation technique for similar terms that differs from the technique demonstrated by a mathematics teacher.

This first pair of elements in the praxeological model, [T / τ], the technical-practical block, establishes a connection with the notion of relation, which is the objective of praxeology. In other words, Bosh and Chevallard (1999) model institutional and personal relationships through the concept of tasks and techniques. It is through performing techniques and tasks that the individual establishes a relationship with the institution. This gives rise to a phenomenon. Bosh and Chevallard (1999) explain that this relationship emerges through what we call "knowledge" and "knowing," which is almost a reductionist description. What actually appears is a kind of naturalization of the task/technique pair. This results in the institution operating with routine, naturalized tasks, almost always unquestioned.

Most institutional tasks are, in fact, routine tasks: the technique used to carry them out, though once constructed, has been routinized to the point of no longer appearing as such—using this technique to accomplish a task is now self-applicable, evident, and no longer shows any problem (Bosh; Chevallard, 1999, p. 6, our translation).

Through these postulates of Bosh and Chevallard (1999), we can assert that institutional knowledge is reached through tasks and techniques that are often routine. Analyzing the types of tasks in a particular institution can lead us to understand the type of knowledge being produced. It is also important to highlight the thoughts of Casabò (1994), who argued in her thesis that it is only possible to partially describe a personal or institutional relationship, as there will always be a kind of evolution where techniques intertwine, forming a higher-level technique. Moreover, the author explains that a partial technique can be described as a "moment" of a broader technique (Casabò, 1994).

There is a set of postulates dictating the conditions and restrictions for the tasks and techniques of an institution. The goal of these postulates is to make the technique understandable and justified. Thus, praxeology, as a model of human activity, not only states and demonstrates an institution's techniques and tasks but also presents a discourse capable of describing them. Bosh and Chevallard (1999) indicate that this descriptive discourse is the technology of the technique. The technology is symbolized by $\theta$. It is the postulate that validates the technique.

Some relevant considerations about technology (Chevallard, 1998): every technique $\tau$ always brings some trace of a technology $\theta$. In some cases, the technology is integrated into the technique. There are cases in where multiplication can be the technological discourse that validates the technique, and it can even be the technique itself. The author also notes that while the primary function of technology is to justify the technique, the secondary function is to explain that justification. However, according to him, this last step leads to a demonstration. This means that, in mathematics, the most commonly used technologies are those of a demonstrative nature, which use the field and language of mathematics to explain the given technique. As an example, the technology of complex numbers justifies why we do not work with negative square roots in the field of real numbers when solving a quadratic equation. It is interesting to note that techniques used by basic education students may only be justified in high school.

Finally, there is a final element in praxeology whose function is to explain and justify the technology (just as technology justifies the technique). This element is theory, $\Theta$. Chevallard (1998, p. 4) describes theory as a kind of "[...] higher level of justification-explanation-production [...]". An interesting point the author raises is when he questions whether we could think of an infinite movement where there would be a theory of the theory, and so on. He explains that the three levels—technique/technology/theory—seem sufficient to analyze the intended institutional activity. Chevallard (1998) also comments that theory—the highest level—seems to be disappearing, as some justifications are treated as references to other institutions.

> [...] The justification of a given technology is, in many institutions, treated by simply referring to another institution, real or supposed, which is supposedly the bearer of such justification. This is the meaning of the classic 'We demonstrate in mathematics...' of the physics teacher or 'We saw it in geometry...' of the former mathematics teacher (Chevallard, 1998, p. 4, our translation).

At this point, it is worth questioning whether the study of functions in higher education is conducted as a reference to basic education—that is, whether the theory is built praxeologically and merely revisited through brief recollections of content. Another important element to consider is Chevallard's (1998) reminder that there is a spectator who watches the theoretical presentation but does not participate in its exposition. It is inferred that such moments are constructed for this purpose. In the school environment, we can ask: Are there moments for theoretical exposition? Does the student watch or participate in this construction? Is all theory presented in school done this way? What are these theoretical moments? In the author's words:

> In Greek, theôria took on from Plato the modern meaning of 'abstract speculation'. But originally, it simply referred to the idea of contemplating a performance, with the theôros being the spectator who watches the action without participating in it. In fact, theoretical statements often seem abstract, far removed from the concerns of simple 'technologists' and technicians. This effect of abstraction is correlated with what underpins the great generality of theoretical statements—their capacity to justify, explain, produce (Chevallard, 1998, p. 5, our translation and emphasis).

The second assertion by the author that draws attention is the fact that he shows that theoretical statements possess the characteristic of great generality. Let us ask: if theory embraces properties of generalization, is it possible to affirm that when a student generalizes, they are constructing their theoretical environment? And, if they construct a theoretical environment, can we verify which technical elements from the know-how block they rely on? Moreover, if we shift the focus to the mathematics curriculum in calculus, which topics are treated as theory and which as technique?

Next, we will briefly review the three dimensions of a didactic problem according to Gascón (1999, 2011), seeking to relate these ideas to what has already been presented.

## The three dimensions of a didactic problem – praxeological implications

Gascón (1999) explains that teachers' concerns regarding the object of teaching, the ways and reasons for teaching, give rise to a didactic problem. This research problem is made explicit through three basic dimensions: epistemological, economic, and ecological.

The epistemological dimension aims to describe and interpret the mathematical organization being studied. Almouloud and Silva (2021) point out that epistemological studies analyze the reasoning behind mathematical content and that these studies can provoke teachers to construct new know-how. According to the authors, this is done through a historical study that analyzes its development. Gascón (2011) explains that the Epistemological Reference Model (ERM) is the tool that didactics use to analyze these didactic and mathematical facts in the epistemological dimension. Thus, through the ERM, it is possible to deconstruct and reconstruct the praxeologies being analyzed.

From the theoretical discussion in the previous section, we highlight that there is a constant process of revisiting in the praxeological model: techniques are revisited to assist in constructing other technologies, and mathematical organizations are continuously expanded. In other words, there are moments of construction and moments of revisiting, where it is assumed that a mathematical organization serves as the basis for another construction. Using praxeological terms, we see that what was considered a technological-theoretical environment at one moment can become the technical-practical block at another. This praxeological conception is relevant to our study. We raise the hypothesis that a student may view the theoretical environment through generalization and may justify certain techniques in a way not anticipated by the teacher (or curriculum). When a student revisits certain concepts, they reveal how they construct their theoretical environment.

The ERM must always be taken as a working hypothesis and constantly revisited. An important point regarding this model is that it is necessary to study mathematical knowledge before it is transported into teaching (Gascón, 2011). According to the author, the ERM allows us to understand and interpret what is taught and shows why certain objects are included in a mathematical curriculum while others are not. Farras, Bosch, and Gascón (2013) explain that the ERM enables us to question how institutions interpret mathematical knowledge.

Returning to the praxeological model, its objective is to describe the institutional relationship. There are two fundamental blocks: a technical block and a theoretical block. Thus: $[T, \tau]$ and $[\theta, \Theta]$. Chevallard (1998) brings further developments from here and states that around a single type of task T, there is what is called a "punctual praxeology," with the praxeological set denoted by $[T, \tau, \theta, \Theta]$.

One implication of this approach is that the set $[T, \tau, \theta, \Theta]$ is identified as knowledge, in a broad sense. That is, what is called knowledge seems to disregard its technical part, the know-how block. In fact, in this model, theory is understood as a discourse capable of validating the technical part. There is, therefore, the impression that knowledge is everything abstract. As a result, according to Chevallard (1998), institutions end up eliminating specific praxeologies, where theory corresponds to a single type of task. Thus, what we most often see are praxeological organizations: a technology serving several types of techniques (local organizations) or a theory covering various technologies (regional organizations). There are also global organizations: a praxeological complex formed by several regional organizations.

An important consequence that arises from this idea of local, global, and regional levels of organization is that when a punctual praxeology shifts to a local praxeology, technology comes into focus. Similarly, when it shifts to regional praxeology, theory comes into focus. In both cases, theoretical knowledge increases, at the expense of the know-how block. Furthermore, according to the author, theory also allows the generation of technique, which is why there is a tendency to view the know-how block as a mere application of theory.

There are differences in the approach to the entities [practice: T, $\tau$] and [theory: $\theta$, $\Theta$] considering the praxeological approach. Here, we also see a dialectical existence between the technical and theoretical blocks. It is clear that in mathematics education, it is possible to guide students to construct a theoretical environment through the exploration of tasks and techniques, and moreover, some theoretical approaches can be justified through technical approaches. Regarding this, Chevallard (1998) states that some topics in mathematics are presented in theoretical discourse, while others appear in the form of tasks.

Gáscon (2011) presents some important considerations regarding the ERM used by the educator. It is necessary that the chosen model has the appropriate breadth for the mathematical field of the mathematical problem to be studied; to highlight the didactic phenomena that become visible through the interpreted ERM; to propose research problems that are coherent with the conceptual position of the analysis and the researcher; and to produce provisional explanations related to the study conducted. The author explains that the epistemological dimension of the didactic problem "[...] is a nuclear dimension, as [...] it permeates and strongly conditions the other dimensions" (Gáscon, 2011, p. 210, our translation).

In this work, we aim to propose a praxeological model for the study of the basic ideas of Functions. Some aspects need to be highlighted so that the constructed Educational Reference Model (ERM) aligns with the research objectives. In this sense, Gáscon (2011) advises that it is necessary to formulate questions that seek to analyze: the breadth of the mathematical field of the content of functions to be adopted; the way knowledge has been incorporated in schools through the scale of levels of co-determination; the conceptual position regarding the type of problem analyzed, among others.

In order for the epistemological dimension of a didactic problem to be analyzed, it is necessary to construct ERM, of a local or regional scope, formulated in terms of praxeological organizations, following the model of ATD. "This means that the ERM must be written according to the genesis and development [...] of certain mathematical praxeologies" (Gáscon, 2011, p. 212, our translation). According to the author, the analysis of the epistemological dimension allows for the integration of the genesis, development, and institutional transposition of mathematical knowledge.

The economic dimension of a didactic problem includes questions regarding the functioning of the mathematical organization (OM) and didactic (OD) involved. This encompasses the set of institutional norms that govern the system and a detailed analysis of these existing OM and OD, as well as the experimentation and evaluation of new organizations and didactic facts—always through the adopted ERM (Gáscon, 2011). It is important to consider that the didactic problem must contain a broad praxeology that references all stages of didactic transposition (Chevallard, 2005), meaning it should encompass everything from scholarly knowledge to learned knowledge. Thus, it is necessary to analyze how the issue of the didactic problem is interpreted within the educational institution.

Some questions that can be raised in the economic dimension include: What is the institutional sphere to be considered in the study of the problem? What are the general characteristics of OM and OD in the institution? What difficulties arise if the OD are modified?

In the ecological dimension, the conditions and restrictions imposed on praxeologies at each level of the co-determination scale developed by Chevallard (2001) are analyzed. In this case, according to Gáscon (2011), there is a concern with the study of the ecology of mathematical praxeologies from an institutional perspective. The author explains that at the higher levels of the scale (pedagogy, school, society), there are issues that are considered not belonging to mathematics but that impact the classroom. Here, we infer that the knowledge required in areas such as Chemistry, Economics, and Biology can be analyzed from the perspective of society and civilization, possessing characteristics of breadth and generalization. Thus, there are questions that society proposes to be studied in schools, in a kind of cultural legitimacy (Gáscon, 2011).

In a certain way, the structure of praxeologies is more detailed; however, one must consider the diffusion of knowledge from the perspective of the levels of didactic co-determination. Thus, the ecological dimension of a didactic problem investigates which restrictions—and from which levels they originate—affect the ecology of mathematical praxeologies (Gáscon, 2011). One of the questions that can be addressed through the ecological dimension is whether any dominant mathematical activity conditions the entire way of organizing teaching. An important consideration made by Gáscon (2011) is that the restrictions imposed at each level are not definitive but can be modified by an agent of the institution, such as the teacher, for example. We argue that the first step is to become aware of these restrictions. The analysis of the epistemological, economic, and ecological dimensions contributes to the identification of dominant phenomena and the formulation of proposals for didactic emancipation. This work flows in this direction.

## Dominant epistemological model in the teaching of calculus and the initial didactic problem

We broadly focus this study on the understanding of the concept of Function. It is not difficult to find works that address this theme. We have chosen the study by Rodrigues et al.

(2021) as a benchmark to explore other research that also deals with this topic. In this regard, Gáscon (2011) emphasizes that scientific problems end up integrating with others, as it is not possible for questions to be isolated and independent. In the author's words:

> Scientific problems are not developed in isolation and independently; rather, they integrate different types of problems. Throughout their development, didactic problems relate not only to problems of the same nature but also with others that are apparently very distant (Gáscon, 2011, p. 225, emphasis added by the author, our translation).

In this way, we share the same general questions when dealing with the issue of "learning in mathematics," which is a very broad approach. Here, we do not focus on relating the didactic problems that surround this issue—we believe that this topic could be the focus of more in-depth research. However, it is possible to gather some very relevant points regarding this in Rodrigues et al. (2021). This work shows that the most recurring themes are linked to the superficiality of the approach to the concept of function; difficulties in relating to various representations; lack of understanding in dependence on variables; and difficulty in relating the concept to everyday problems.

What is observed is that the themes are linked to issues ranging from the basic understanding of the concept to symbolic manipulation. This leads us to suppose that the dominant model in the teaching of functions fails to solidify foundational ideas and ultimately favors a symbolic approach, as there is difficulty in transferring the concept to everyday life.

We now quote the work of Rezende (2003), who mapped epistemological difficulties in the teaching of calculus. The author is emphatic in his conclusions.

> From the mapping conducted, it was observed, essentially, a single **matrix location** of the epistemological learning difficulties in the teaching of Calculus: **the omission/avoidance of the basic ideas and the foundational problems of Calculus in the teaching of Mathematics in a general sense**. (Rezende, 2003, p. 402, emphasis added by the author).

The research concludes that the lack of focus on the basic ideas of calculus in primary education contributes to the failure in learning it. This study aligns with my investment and interest as a researcher in the field of mathematics education. My master's (Castro, 2012) and doctoral (Castro, 2022) studies aimed to defend and analyze the work with basic ideas of functions. In this sense, we found elements that allow us, to a certain extent, to broaden our investigative focus and contribute to the ongoing discussions in the field. To achieve this objective, we will present the conception of the initial didactic problem of this study.

Gáscon (2011) announced the development of a didactic problem schematically as indicated below:

$$\{[(P_0 \oplus P_1) \subsetneq P_2] \subsetneq P_3\} \subsetneq P_\delta$$

Here, $P_1$, $P_2$ e $P_3$ represent the three fundamental dimensions of the problem. What is interesting here is that $P_0$ is the initial didactic problem and must be added to $P_1$, the first dimension. There is, then, a kind of inclusion into $P_2$ e $P_3$, which are the other dimensions. Therefore, $P_\delta$ represents the didactic problem already incorporated by the three fundamental dimensions. Thus, we formulated the initial question of this study in the following terms:

- $P_0$ : In what way can the basic ideas of functions contribute to the development of the concept?

The initial problem is defined as **the development of the concept of function through its basic ideas**. It carries characteristics of generality and breadth, as we believe this questioning should be formulated. In the following sections, we will study each of the fundamental dimensions so that the problem can be analyzed through relative historical breadth, characteristics, and institutional restrictions. At the end of this analysis, we will have our didactic problem delineated.

## Epistemological dimension: an educational reference model for the development of basic ideas of functions

The epistemological dimension allows for the construction of ERMs through the analysis of didactic and historical phenomena. Here, we will briefly present the historical development of the concept of function, highlighting its basic ideas. The aim is to analyze which concepts were part of its construction.

It is important to emphasize that the ERM is always a relative model of didactic phenomena and must be in accordance with the epistemological model of mathematical activity (Gáscon, 2011). Thus, we seek to construct a specific ERM for the phenomenon we intend to analyze, according to the initial problem identified: the development of the concept of function through its basic ideas. Furthermore, we will propose a model that takes into account the praxeological dimension of already institutionalized knowledge, that is, after it has been transported into the school environment. Gáscon (2014) reminds us that the specific ERM serves not only to interpret the dominant epistemological models but also to determine which research problems can be formulated.

Let's return to the initial question of this study:

- $P_0$ : In what way can the basic ideas of functions contribute to the development of the concept?

By the conception of a didactic problem, the question $P_0$ is added to the first dimension, $P_1$. Let's create the following epistemological questions based on Gáscon (2011). The objective is to construct an analytical pathway that can ultimately generate a praxeological model of the didactic problem $P_\delta$ which is already incorporated by the three fundamental dimensions.

- What is the purpose of the concept?
- How has it been interpreted according to its development?
- What is the scope of the development of the concept?

We chose to base our discussion on the work of Caraça (2010) to address the evolution of the concept of function. The author indicates that humans, since the dawn of their existence, have felt the need to make predictions about the natural phenomena surrounding them, as it is through prediction that one can take precautions and attempt to master nature. However, two fundamental aspects of the reality in which humans are embedded must be considered: interdependence and fluidity.

**Interdependence** refers to the idea that all things in the world are interconnected, meaning they depend on one another. Caraça (2010) cites the analysis of a plant in a specific region as an example. It is observed that various aspects and characteristics affect it. For instance, one can mention the type of soil that promotes its growth, the animals that feed on it, and those that depend on it for survival. Additionally, one must consider the aspects of the region in which it is situated, how humans take advantage of its presence, its position in a chain of other crops, and other factors.

**Fluency**, on the other hand, is related to a cycle of evolution and development and can be found in the plant and animal kingdoms, with the exception of minerals. Thus, we observe that phenomena occur in a sequence of birth, growth, and death.

The importance of these two aspects becomes evident when a researcher proposes to study a specific phenomenon in reality. Caraça (2010) explains that the interdependence and fluidity of things complicate the analysis of just one specific aspect. For this reason, one must adopt the notion of the isolated. The isolated is a segment of reality, as it is unfeasible to study everything at once. However, due to the fluidity and development of things, even the isolated is not free from changes and evolution. This evolution is referred to as a Natural Phenomenon.

What is observed, however, is that there are regular phenomena, that is, phenomena that behave identically when their initial conditions are preserved. This characteristic is of utmost importance, as regularity implies repetition, which in turn allows for prediction; that is, “[...] **to repeat and predict** is fundamental for humans in their essential task of mastering nature. [...] As a result, one of the most important tasks in the investigation of Nature is **the search for regularities** in natural phenomena.” (Caraça, 2010, p. 112, emphasis in the original). Therefore, we encounter one of the fundamental ideas of Functions here: the ideia of **regularity.**

Caraça (2010) discusses the long process behind the step taken from identifying a natural law to its **generalization** in mathematical language, in this case, the very formalization of the concept of function. To illustrate this, he presents the description of Natural Law. This conception is related to the idea of regularity. In other words, regular natural phenomena are referred to natural laws by Caraça (2010).

There are qualitative laws and quantitative laws. The former are related to variations in quality, while the latter pertain to variations in quantity. Let’s consider an example: if we take two circles, A and B, with radii of 3 cm and 5 cm, respectively, can we say that circle A is more circular than circle B? Another question: can we say that circle A is larger than circle B? What allows for a discussion about quality and quantity is that the characteristic of curvature is relative to the judgment of quality. There are qualities that do not allow us to admit different degrees of intensity. Others, however, allow us to make judgments of greater than, less than, larger than, and smaller than. Thus, it is said that these cases allow for variation according to quantity.

Other examples of qualitative and quantitative laws in Physics include: “each planet describes an ellipse around the Sun, of which the Sun occupies one of the foci (1st law of Kepler)” (Caraça, 2010, p. 113). This is a qualitative law. “For every gas, there exists a temperature, called the critical temperature, above which it cannot be liquefied; as soon as the temperature falls below the critical temperature, the gas can be liquefied by applying a suitable pressure” (Caraça, 2010, p. 113). Here we have an example of a qualitative-quantitative law. “For anybody in free fall in a vacuum, the heights of fall are directly proportional to the squares of the times of fall” (Caraça, 2010, p. 113). This represents a quantitative law.

Caraça (2010) draws attention to the fact that the development of Science is directly linked to the focus given to quantitative laws. According to him, for a time, people were bound to qualitative explanations of phenomena. However, starting from the Renaissance, scholars abandoned qualitative explanations and gave “[...] a new direction to the ship of Science, dedicating themselves to observation and experimentation, seeking to measure, attempting to explain through variations in quantity, weaving a web of quantitative laws.” (Caraça, 2010, p.

117, emphasis in the original). The mathematization of these laws is the formally established concept of Function that we now know. Caraça (2010) explains that:

> "[...] the new direction of Science [...] is the direction of a mathematical ordering of the Universe. [...] Therefore, see, the reader, how, after 20 centuries, that ideal of quantitative mathematical ordering, which we saw emerge with the Pythagoreans, is reborn from the ashes, where it seemed buried forever." (Caraça, 2010, p. 190, emphasis in the original).

The new direction of Science has brought countless benefits. There is now, everywhere, a tendency toward the quantitative, toward measurement. Thus, we can affirm that the truly scientific state of each branch only begins when measurement and the study of quantitative variations are introduced as explanations for qualitative variation (Caraça, 2010). In this way, **The Modern Science adopts the quantitative law as a means of explaining reality.**

It is noticeable that there was a long period in history when qualitative laws prevailed in a "[...] tendency to avoid anything that was linked to quantitative and dynamic conceptions [...]" (Caraça, 2010, p. 185). From the 11th century onward, Europe witnessed a great transformation, mainly driven by the development of the first cities. This event gave rise to a new society, with the creation of a new class of individuals, which imposed a new direction on Philosophy and Science.

> The needs of Commerce and Industry demand a study of the external world as it presents itself to us. [...] The problems of navigation, for example, lead to an increasingly careful investigation of the movements of the stars and, in general, demand a more rigorous study of motion, a **quantitative** study that allows for **measurement and prediction**. (Caraça, 2010, p. 187, emphasis in the original).

There is, consequently, a need for the existence of an appropriate mathematical tool for studying quantitative laws. However, according to the author, this achievement did not occur in a linear fashion. In other words, "it was a slow gestation in which necessity and instrument interacted, mutually aiding and enlightening each other" (Caraça, 2010, p. 118). We can affirm that the historical conditions of the time did not favor the development of the basic ideas of function related to its formalization.

The idea of a variable appears as foundational for the formalization of the concept of function. Before its introduction, we encounter rudimentary aspects, that is, elements that comprise the notion of function but are not sufficient to formalize it mathematically. Thus, we could say that the "instrument" was still, in a sense, incomplete, as the grand tools of the concept were not yet available. It is only from the 11th century onwards that society began to show signs of a significant transformation. Therefore, the introduction of this idea—the variable—was fundamental to the development of the notion of Function.

Indeed, according to Caraça (2010), there is a need to create a mathematical instrument sufficient to study the variations of quantity described in quantitative laws. To this end, the author emphasizes how these laws can be translated: by the way magnitudes correspond to each other. He cites the example of free-falling bodies in a vacuum and presents a table that, in his view, provides only an illustration of this law (Table 1).

Table 1.

*Values for the law of free fall in a vacuum (Caraça, 2010, p.118)*

| **Time (in seconds)** | **0** | **1** | **2** | **3** | **4** | **5** |
|---|---|---|---|---|---|---|
| **Space (in meters)** | **0** | **4,9** | **19,6** | **44,1** | **78,4** | **122,5** |

Caraça (2010) shows us, in Table 1, two sets placed in correspondence. Thus, the law of the fall of bodies in a vacuum corresponds between the time set and the space set. The appropriate mathematical instrument for studying quantitative laws must essentially include the correspondence between two sets.

Thus, another basic idea associated with the concept of Function emerges: **the correspondence**. It is important to highlight that the idea of correspondence, linked to the concept of Function, relates to the correspondence of variables. That is, to formalize the concept, it is still necessary to develop another basic notion: that of variables. However, we believe in the importance of developing the notion of correspondence, even if it is not directly linked to the variables.

The idea of a variable emerges as a tool for enhancing the formalization of the concept. Thus, it is necessary to create a symbolic representation for the sets that will be corresponded; otherwise, "[...] we would always have to rely on tables of particular results and would not achieve the desired generality" (Caraça, 2010, p. 119). The variable is, therefore, the symbol that represents any of the elements of a set; it is thus an entity of a higher nature, that is, it is the symbol of the collective life of the set.

The idea of a variable formalizes the concept of a function through a symbolic generalization, which proves to be the appropriate instrument for studying quantitative laws. The previous example of the law of falling bodies consists of the correspondence between the set of times and the set of spaces. If $t$ is the variable of the set of times and $e$ is the variable of the set of the spaces, the law consists of a correspondence between $t$ and e. In this way, the variable $e$ is the function of the variable $t$. Symbolically $e = f(t)$, being $t$, the independent variable*;* and $e$, the dependent one.

In table 1, previously presented, we observe only a few pairs of values from the law of falling bodies, while $e = f(t)$ implies that any value of $t$ corresponds to one (and only one) value of $e$.

Here is the definition of a function given by Caraça (2010, p.121): "We have x and y as the two variables representing sets of numbers; y is said to be a function of x and it is written as **y = f(x)** if there exists a unique correspondence between the two variables in the sense of x → y."

In this context, **x** is referred to as the independent variable and **y** as the dependent variable.

The author highlights that the mathematical expression of the concept (in this case, $e=f(t)$) allows the following statement: for any value of $t$ corresponding one (and only one) of $e$, while the tables present only a few pairs of the corresponding values. This notion, according to Caraça (2010), exemplifies the power inherent in the concept of Function.

According to the author, the law of association between the two variables completes the chain: **quantitative law → function → analytical definition**. However, the author also emphasizes that this is not the only way to establish the correspondence between the two variables, as the concept of function is often inappropriately confused with that of analytical expression. He stresses that this is merely the ground in which the function takes root. In this context, it becomes clear that the mathematical definition of a function, through the notion of correspondence between variables, implies the idea of **dependence**, as one of the variables depends on the other.

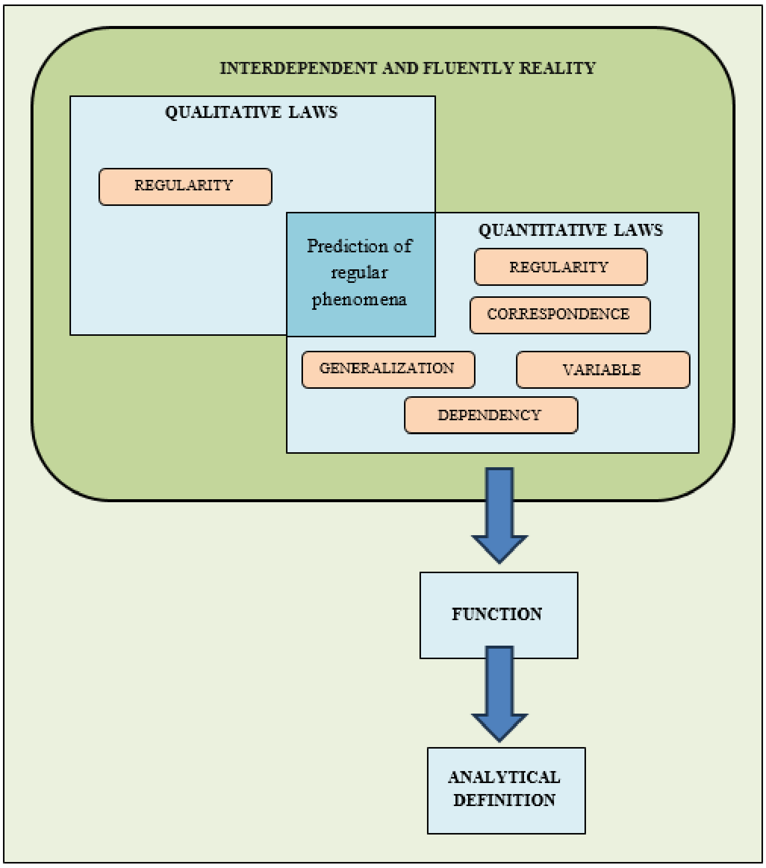
Caraça (2010) points out that associating a function with its analytical expression prevailed for a long time and still persists in the language of mathematics today. However, the same author demonstrates that there was a need to refine the concept in order to highlight what he interprets as essential to the notion of a function: the correspondence between the two variables.

Eves (2008) highlights that the concept of function has undergone significant evolutions. According to the author, "the history of the term **function** provides [...] an interesting example of mathematicians' tendency to generalize and expand concepts" (Eves, 2008, p.660, author's emphasis). From the moment the concept was formalized, there has been a continuous refinement of the mathematical symbolism that defines it in the way it is presented to students. This represents only the **final chapter** of a story that has accompanied the evolution of both the Sciences and mathematical thought itself.

It is possible to trace the evolution of the concept from antiquity to modern times, and several studies present these developments, such as Youschkevitch (1976), Boyer (1986), Rogalski (2013), and Eves (2008). These works generally show that the symbolism eventually leads to formalization, as demonstrated below.

- Galileu Galilei (1564-1642) gives a quantitative treatment to mathematical laws.
- René Descartes (1596-1650) establishes a dependency relationship between the variables.
- Isaac Newton (1643-1727) introduces the term independent variable.
- Leibniz (1646-1716) uses the word "function".
- Johan Bernoulli (1667-1748) e Leonhard Euler (1707-1783) use analytical expressions to define function.
- Refinement of the concept, starting from the 20th century, through the relationship between sets.

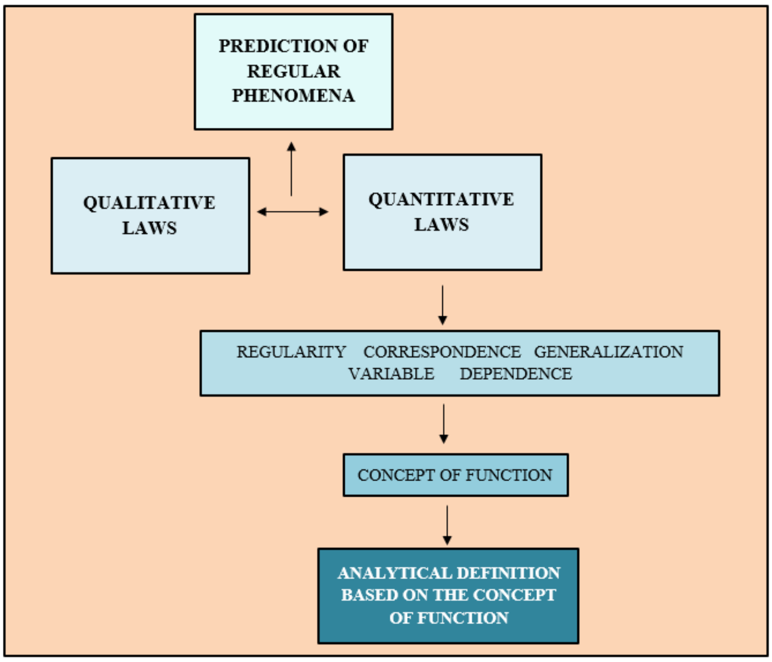
In conclusion, it is evident that the basic human need that gave rise to the concept was the **prediction** and description of **regular phenomena**. Its mathematization is a tool that enables the study of **quantitative laws**. These laws manifest through the **correspondence** between quantified entities—the **variables**—such that these variables become dependent on one another. In picture 1, we present a diagram of what has been discussed so far.




Picture 1.

*Diagram of the Evolution of the Concept of Function*

Gáscon (2011) explains that the composition of the epistemological reference model (ERM) allows for an analysis of knowledge before it is transformed into teaching and why certain mathematical objects are found in schools while others are not. Therefore, this study of the epistemological dimension enabled the identification of the notions that underlie the concept of function before it is formalized or institutionalized. This developed ERM considers the expansion of ideas in a crescendo until it reaches formalization (picture 2).




Picture 2.

*ERM for the Development of the Concept of Function Through Its Basic Ideas*

This model considers the evolution of the concept before its institutional formalization by the school. We revisit the questions raised:

- What is the purpose of the concept?
- How has it been interpreted according to its development?
- What is the scope of the development of the concept?

It is interesting to observe the importance of the idea of natural law in the genesis of the concept. However, we argue that the reason for the concept's existence was the prediction of regular phenomena. The constructed ERM (Picture 2) demonstrated this evolution before the didactic transposition was made. It is clear that there was a very slow genesis, and indeed, the symbolic—algebraic—language was fundamental in the evolution of science as a whole.

The basic ideas of regularity, correspondence, variable, generalization, and dependence interpret this knowledge, but symbolic language emerges as a powerful tool for its effective application. In terms of breadth, one can observe an evolution that stretches from the dawn of humanity and, in some way, accompanies the development of science up to the present day.

As we stated earlier, Farras, Bosch, and Gáscon (2013) assert that it is possible to question how institutions interpret knowledge through the ERM. Here, it is already possible to see that in higher education, the calculus discipline starts from the already defined analytical definition of function. The analysis of the ERM allows us to identify the following "major moments" in the development of the concept: genesis, development, language, and application.

After analyzing the other dimensions of the didactic problem, we will have elements to examine the diffusion of the ideas of function from a praxeological perspective, that is, after the concept has been institutionalized.

### Economic Dimension: Analysis Possibilities through the Praxeological Approach

The analysis of the economic dimension seeks to interpret phenomena related to mathematical organization (OM) and didactic organization (OD). Here, we are addressing an institutionalized concept, the knowledge to be taught. We have developed the following questions to guide the analysis in this dimension:

- What is the purpose of the concept within the school institution?
- How are the mathematical organization (OM) and didactic organization (OD) structured according to the evolution of the concept?

We will now study the characteristics of this dimension considering the theoretical constructs of ATD (Anthropological Theory of the Didactic). The idea is that in the end, we will have elements that allow us to construct a praxeological model describing the development of the concept of function.

As shown, the notion of institution is the backbone of ATD. Institutional practices are visible through the relationships between its basic entities. The praxeological model describes these practices. Furthermore, the themes traverse the institution in two ways: through a mathematical reality that derives from wise knowledge – the mathematical organization (OM) – and through a way of conducting this knowledge already within the institution – the didactic organization (OD).

In this section, we will analyze how praxeological tools can participate in the development of the economic dimension of the didactic problem. Thus, we will study the structure of OM (mathematical organization) and OD (didactic organization) from a theoretical perspective based on the fundamental points of ATD (Anthropological Theory of the Didactic).

Chevallard (1998) makes some considerations about how praxeologies manifest in institutions. He explains that there are two types of questions developed in social reality: “What is this?” and “How to determine this?” According to the author, answers to the first question are in a weak sense. The second question, however, leads to a strong answer. For him, it is the strong questions that demands **the construction of a praxeology**. Thus, praxeologies are constructed. Studying a question is, therefore, to develop a new praxeological organization.

What happens in school, however, is that studying the question means recreating an answer that has already been produced in another institution.

Indeed, constructing new praxeologies requires relationships that are different from those where it is necessary to recreate what has already been produced. If the mathematics curriculum stipulates that the topic of algebra, for example, should be consolidated at certain stages for a given grade, the praxeology to be constructed may include in its practical block notions that the teacher assumes have already been established. Conversely, if the praxeology is of the type that seeks student-driven construction, the teacher can explore themes that might not be anticipated in a model aimed at recreating previously covered techniques, for instance. Recreating an existing answer, according to Chevallard (1998), can lead to a reversal in the question-answer relationship in certain cases: the answer may come before the question.

An example is provided by the author himself: in regional organizations controlled by the same theory, there is a tendency to "push to the periphery, under the name of applications, the types of tasks that are in principle generators of the work [...] in the face of a potentially productive technology for new techniques that cannot be limited to some predefined applications" (Chevallard, 1998, p. 15, our translation). In other words, the teacher may choose to first work with a theory and then derive tasks from it, rather than constructing that theory with their students based on those tasks. Thus, within a Didactic Organization (DO), there is no guarantee that tasks will precede theory, praxeologically speaking. Therefore, this represents a possibility in the way knowledge is conducted in classes and/or institutions. It becomes clear that the teacher faces two tasks: determining the Mathematical Organization (MO) based on official documents and leading a construction or reconstruction through a Didactic Organization (DO).

When we talk about official documents, it is important to consider that the object, from the perspective of the ATD (Anthropological Theory of the Didactic), is an entity that needs to be recognized—both by the institution and by the individual. According to Chevallard (2003), there are various ways to fulfill this script, meaning it is practically impossible to guarantee that there will always be a good relationship between the institution and the individual. This is the concept of institutional relativity. The school, through its official curriculum, prescribes certain behaviors through the objects it recognizes. From this conception, we arrive at the concept of learning in the ATD.

The subject X relates to an institutionally recognized object, validated by the clauses of what we can call the institutional contract . If subject X appropriates the object in the way that the institution prescribes and recognizes, it is said that X is a "good subject" of I (Chevallard, 1992). We note, then, that through learning, a change occurs in the subject, in the person (the one who "lives" in the institution) and not in the individual. Therefore, learning in the ATD is a change that occurs under the jurisdiction of the institution. If it does not occur, it is because the personal relationship does not align with its postulates. Thus, it is important to discuss how praxeologies are constructed and reconstructed.

According to it, we mean that the OM are structured according to what is outlined by the institution through its official documents. Therefore, an analysis in this regard should investigate how the genesis, development, language, and application of the concept of function appear in both basic education and higher education in the manner expected by the institution.

There are specific praxeologies for each OM, and the way to develop them occurs through gestures that comprise the didactic moments. These didactic moments, according to Chevallard (1997, 1998), are surprisingly situations that will always be present in the teacher's activity. There are six didactic moments. In summary, we can say that the first of these is the moment of encountering the OM to be studied. Next, the tasks T are explored through techniques $\tau$. That is, the technical-practical environment of praxeology, $[T, \tau]$, it is built here. The third moment, which composes the theoretical-technological pair is: $[\theta, \Theta]$. This environment is related to the technique $\tau$. However, the author emphasizes that this third moment ends up becoming the first! That is, the problems and tasks become an application of technology and theory, as we mentioned earlier.

> For reasons of general didactic economy, however, traditional study management strategies often make this third moment the first stage of study, a stage that is then common to the study of various types of problems Ti – All those, among the types of problems to be studied, that seem to respond to the same theoretical-technological environment $[\theta / \Theta]$. The study of this type of problem thus arises, classically, as a series of applications of the theoretical-technological block thus constituted (Chevallard, 1998, p. 21, our translation).

For example, in the ninth year of Elementary School, the following skill is found in the National Common Curricular Base – BNCC (Brazil, 2018, p. 315), for the thematic unit Algebra: (EF09MA06B) – 'Understand functions as univocal dependency relationships between two variables and their numerical, algebraic, and graphical representations, and use this concept to analyze situations involving functional relationships between two variables.' In this case, the teacher may choose to start the study of functions with its theoretical approach and then present the class with problems that have an application character. Thus, there would be a kind of inversion in the frameworks of praxeology. The theoretical-technological block, $[\theta, \Theta]$, would be the first. If the planned didactic objective for this skill in the BNCC first explores tasks and techniques, the concept of function would emerge as a technology-theory constructed by the student. We note that the didactic organization adopted by the teacher allows for a kind of alternation in the juxtaposition of praxeological blocks. It is worth asking what the consequences of such a movement are on the student's personal relationship with the studied objects.

There is a fourth didactic moment: this is when other techniques are explored, but it is expected that the student utilizes a theoretical environment in their development. At this moment, therefore, it can be said that the class needs to use a mathematical reference in the application of the technique. For example, a skill outlined in the BNCC for the eighth year (Brazil, 2018, p. 310): EF08MA07 – 'Associate a first-degree linear equation with two unknowns to a line on the Cartesian plane.' For this skill, the student needs to have constructed their theoretical environment regarding first-degree equations. There is clearly a progression in theoretical conceptions in the management of didactic moments."

The moment of institutionalization anticipates the absorption of mathematical elements necessary for mathematical organization (OM) within its proper mathematical discourse. It is noteworthy that there are auxiliary objects that do not participate in the mathematical elaboration of this moment.

The sixth and final didactic moment is evaluation. It, in some way, articulates with institutionalization. Chevallard (1998) explains that there is a broad sense in evaluation: it should be seen as an analysis not only of the individual's personal relationship with the object but also of the evaluation of institutional relationships. In our view, this perspective relieves the student of an assumed responsibility for a particular difficulty in any mathematical organization.

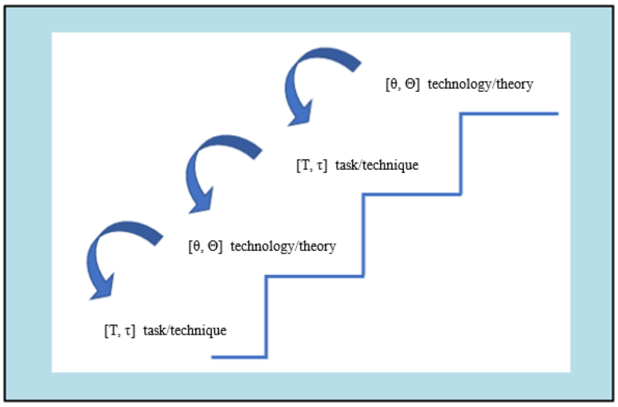
Now, returning to the questions formulated for the economic dimension:

- What is the reason of the concept within the school institution?
- How are the OM and OD structured according to the evolution of the concept?

Through a brief theoretical analysis of the elements of ATD, we perceive that the purpose of the concept must be identified from its **institutional perspective**. The epistemological analysis showed that it was the prediction of regular phenomena that gave rise to the notion of function. When the concept is institutionalized in education, we argue that its purpose can be determined after a thorough analysis of the OM that comprise algebra teaching. Given the limitations of this study, we propose that this analysis can be conducted from a longitudinal perspective, with the necessary depth. Furthermore, we believe that this purpose can only be determined after analyzing the OM and OD.

Regarding the structuring of OM and OD Chevallard (1997) explains that an OM can always be modified and expanded. There are various types of organization in praxeology: punctual, local, regional, and global organizations. With each of these expansions, the theoretical environment becomes more prominent. Thus, theoretical knowledge expands in relation to the technical block, allowing theory to generate new techniques. We see a direction from practice → theory. This is not a measured, precise movement, but an approach whose initial (and final) objective is to build, together with the student, an institutional relationship within the desired mathematical organization. To achieve this, the didactic organization anticipates a revisiting of techniques that become increasingly infused with a developed mathematical discourse.

We can see a kind of expansion of praxeological discourses accompanied by a revisiting of concepts that become the foundation for future constructions. Thus, the theoretical element that justifies the technique will serve as the basis for new techniques, which in turn require a new theoretical discourse (Picture 3).




Picture 3.

*Expansion of the praxeological discourse privileging the revisiting of theoretical and technical elements*

In a large-scale analysis of the ERM we constructed, one can think of the basic ideas of the concept of function composing the task/technique block, while the analytical and symbolic definition represents technology/theory. This would be a perspective for the final construction of the concept. However, these ideas are addressed throughout basic education. What happens is that, in our view, the student does not perceive this growing movement and always sees each content as an end in itself. This consideration takes into account that the relationship between the student and the mathematical object may stem from how the OMs and ODs were presented to him.

## Ecological Dimension: Levels of Didactic Codetermination and the Institutional Diffusion of Knowledge

In the same way as the previous section, we will analyze the development of the ecological dimension from a more theoretical perspective. We have formulated a question to be analyzed.

- How is knowledge disseminated in the school institution?

To analyze a mathematical object (OM), Chevallard (2002) introduces the notion of levels of didactic codetermination. This scale allows for the analysis of how a didactic object (OD) can permeate a mathematical object (OM). For example, a specific topic should be related to a theme, which belongs to a specific domain and sector. Following the order presented by the author, Chevallard (2001), there is a movement that goes from outside the school to inside: from humanity to the topic level, already in the classroom. Here is the scale: Humanity ⇔ Civilization ⇔ Society ⇔ School ⇔ Pedagogy ⇔ Discipline ⇔ Domain ⇔ Sector ⇔ Theme ⇔ Topic.

It is important to say that in relation to praxeologies, Chevallard (2002) explains that there are practically no specific organizations. – those which technology $\theta$ refers to only one type of task T. On the contrary, according to him, **the student tends to see each type of task as a unique subject, practically independent of other topics**. The teacher, however, sees a broader unity: tasks imbued with certain technologies. What Chevallard (2002) argues is that the student engages in a kind of specific reconstruction where the teacher sees a regional organization. In our view, this fact leads the student to perceive the topics as unique and often disconnected from each other. **The teacher, with the curriculum in hand, visualizes the global organization. The student does not**.

Regarding this reconstruction, the author explains: 'In the movement of deconstruction-reconstruction of the works to be studied, we only reconstruct fragments of a puzzle that will never be reconstructed as a whole' (Chevallard, 2002, p. 3, our translation). Furthermore, the fact that the teacher does not situate the themes in their respective sectors and domains causes him to present them as a 'single block.' This is due to the fact that the teacher often focuses his work on the levels of greater specificity. Like in a puzzle, we could say that the teacher (the school, the institution) works on the whole, on the set, while the student sees the pieces separately, the units, and often does not have the chance to visualize the complete picture.

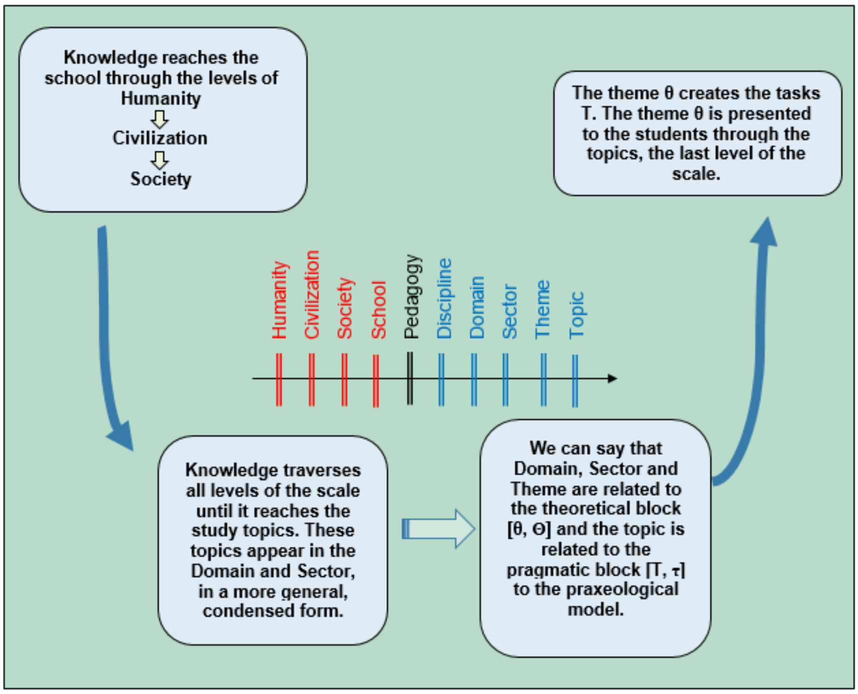
Chevallard (2011) shows that the use of this scale allows for the recognition of the conditions and restrictions of the considered praxeologies, and the actors involved. He places pedagogues at the school level and sociologists at the societal level. For him, the attitudes of these characters ultimately affect the classroom. In the case of algebra, for example, he shows that its diffusion occurred due to a constraint at the civilization level. What happened was that there was a Western preference for words, for rhetoric, and a repulsion for the symbolic, syncopated. However, it is symbolic algebra that modern society finds in schools. This illustrates how conditions are imposed and absorbed level by level. Nevertheless, it is necessary to take into account the consequences arising from these restrictions and conditions.

Chevallard (2007) explains that the topic (worked on by the student through task T) reaches the discipline (mathematics, for example) through the theme θ. What happens is that the sector and domain levels take on a role more related to labels and curriculum specificities and are not used as a source for the development of the lower levels (theme, topic).

There are two possibilities for "reading" the scale of levels of didactic codetermination: Humanity ⇔ Civilization ⇔ Society ⇔ School ⇔ Pedagogy ⇔ Discipline ⇔ Domain ⇔ Sector ⇔ Theme ⇔ Topic. One that goes from top to bottom, showing how the contents were conceived and constructed to model the reality that should reach the classroom; and another that starts from the bottom, from the topic level, revealing the teacher's work. In this case, one can consider how the teacher adapts (or constructs) the questions that should (or ought to) take the reverse path, that is, reach back to society. However, Chevallard (2001) states that, in teaching work, there is a belief that the higher levels of didactic organization—the upper levels —"[...] do not matter in the fate of the knowledge whose dissemination they are supposed to assume" (Chevallard, 2001, p. 6).

What we have observed so far is that there are different ways to fulfill this roadmap. The school, by modeling the study of questions of humanity, ends up fragmenting these issues in such a way that, through the hierarchy of levels of codetermination, it can be stated that the knowledge that reaches the student (through the teacher's work) is disconnected from the broader reality that generated it. Meanwhile, the teacher's work, which ends up being very distant from the higher levels, becomes confined to the specificity of their discipline, completely detached from a broader context. This leads to what Chevallard (2001) refers to as the decoupling of content.

A small diagram can be seen in picture 4, which does not intend to exhaust the topic or structure the concepts in a rigid manner. The idea is to create an intertwining between the praxeological model and the levels of didactic codetermination.



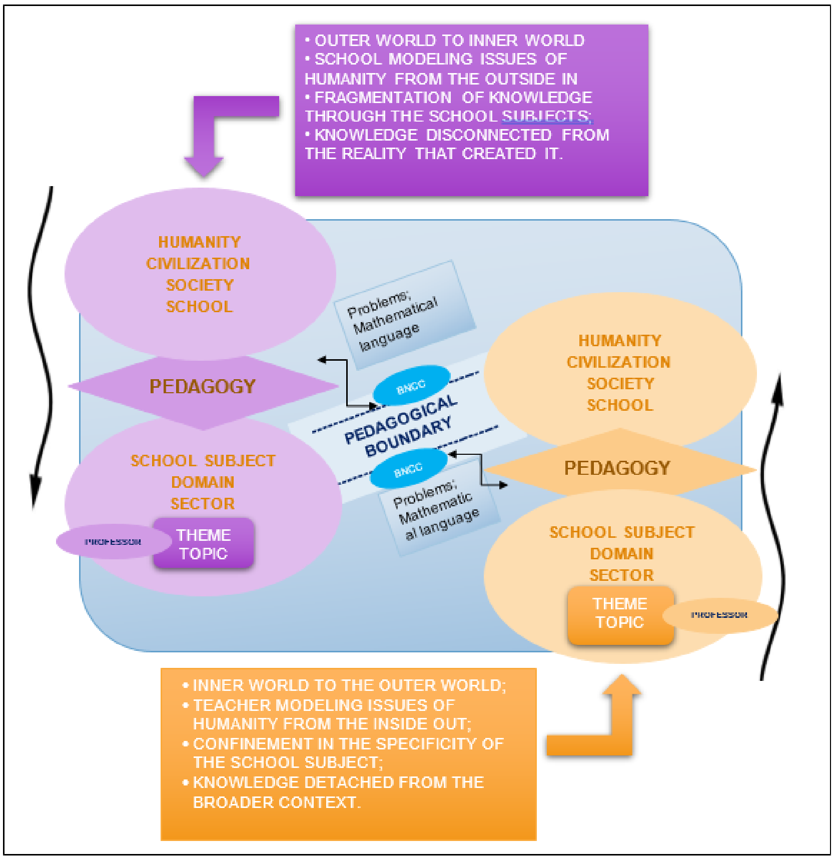

Picture 4.

*Levels of codetermination and the praxeological mode*

In this microanalysis, we can affirm that the pedagogical level, in basic education, is represented by the principles of the BNCC, which establishes general competencies for basic education. This is the boundary that marks the limit between the school and the external world. From there, what exists are prerogatives related to the discipline of mathematics.

Picture 5 provides a diagram regarding our ideas about the possibilities of 'reading' the scale of didactic codetermination (Chevallard, 2001) and sheds some light on our questioning:

- In what ways is knowledge disseminated in the school institution?




Picture 5.

*Dissemination of knowledge according to the scale of didactic codetermination*

According to picture 5, it can be inferred that the teacher's work, confined to themes and topics, remains, in a way, limited to its own world. Furthermore, it is evident that it is necessary to cross the pedagogical boundary – here represented by the BNCC – in both directions we are considering: from the classroom to outside the school and in the reverse movement. In our view, there are two pathways for the dissemination of knowledge in schools. One that comes from "outside to inside," which is the official and, why not, the ideal for the institution, and the one that goes from inside to outside. We believe that this is the real path, the one that actually occurs. Therefore, the OM describes this official route, while the OD reflects the real one, the one that students engage with through themes and topics.

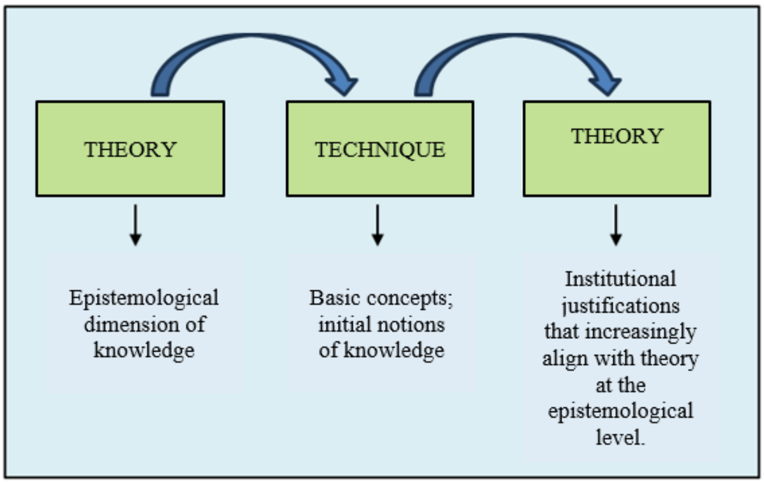
## Paths to the Construction of a Praxeological Model

Through the previous analyses, we argue that the epistemological reference model can be interpreted through two paths: from the outside world to the inside and vice versa. It is known that the Epistemological Reference Model (ERM) describes a knowledge that has not yet been institutionalized (the knowledge that will compose the OM), but at the same time, it serves as an instrument to describe didactic organizations, that is, the knowledge that has already been transferred to the institution.

Indeed, Gáscon (2011, p. 211, our translation) explains that “[...] the issues that are part of the epistemological dimension suggest that it not only occupies a privileged position in all didactic problems but is also inseparable from the other dimensions (the economic-institutional and the ecological) [...].” In this sense, our assertion is that analyzing the institutional path of knowledge, considering its two directions of diffusion, can identify points where a certain concept “stalls” or is even not addressed at all.

Moreover, even though the epistemological model is a provisional dimension that needs to be revisited, it is clear that the institutional path is, to some extent, closed and predefined. Therefore, this wise knowledge must fit within an institutional parameter. What we mean by this is that there are differences between the conceptual rationale of mathematical knowledge and its institutional rationale. Beyond these observations, there is also the student's perception, which is confined to their interaction with the topics. Thus, the student engages with the teacher's praxeology, which may differ from the institutional praxeological model due to the "freedom" present in those didactic moments, as previously discussed.

At this point, we argue that the conception of the ERM should be accompanied by a praxeological analysis that describes a broader institutional diffusion of knowledge, reaching different levels and sectors: thus, we will have a **global praxeological model** that allows for a study in terms of conceptual expansion in the sense of theory → technique → theory. Therefore, the first level, theory, comes from the genesis of knowledge analyzed at the epistemological level. The second level addresses the techniques that derive from this wise knowledge (emphasizing that we are discussing techniques in the sense of ATD), and in the third level, we address the institutional theories that justify and confer application to the techniques. The following scheme allows us to visualize this idea (picture 6).



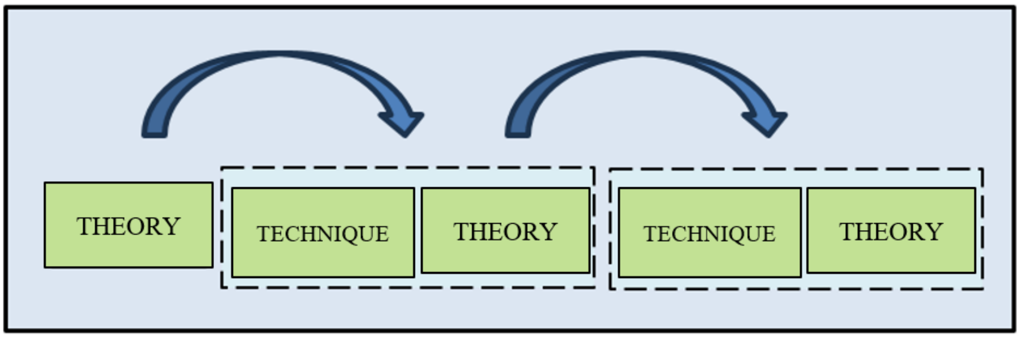

Picture 6.

*Global Praxeological Model*

The scheme shows a first application of this model in the sense of utilizing the primitive concepts of the concept. The idea is that technique and theory should be analyzed in an increasingly broader expansion. Thus, we have a praxeological model that takes knowledge in its genesis as a principle and seeks the ideas of this genesis in a theoretical approach. In the parameters of this study, we have as the first level the basic ideas of the concept of function, assuming a role as theory so that the techniques, from there, can be justified.

The institutional possibilities, in the field of function, suggest that theory can be approached through the formalization of the concept, which ranges from natural language (rhetoric) to the algebraic symbolization of ideas. The technical level, therefore, will account for technical praxeologies: those in which the theoretical block appears afterward; and the theoretical level is where praxeologies assume a previously formalized knowledge as a basis in the technical-theory sense. It is worth noting that Chevallard (1997) warns that the teacher should initiate an OD through a task as an application of a theory. In the model we propose, there is a more careful observation in the sense of theory → technique → theory.

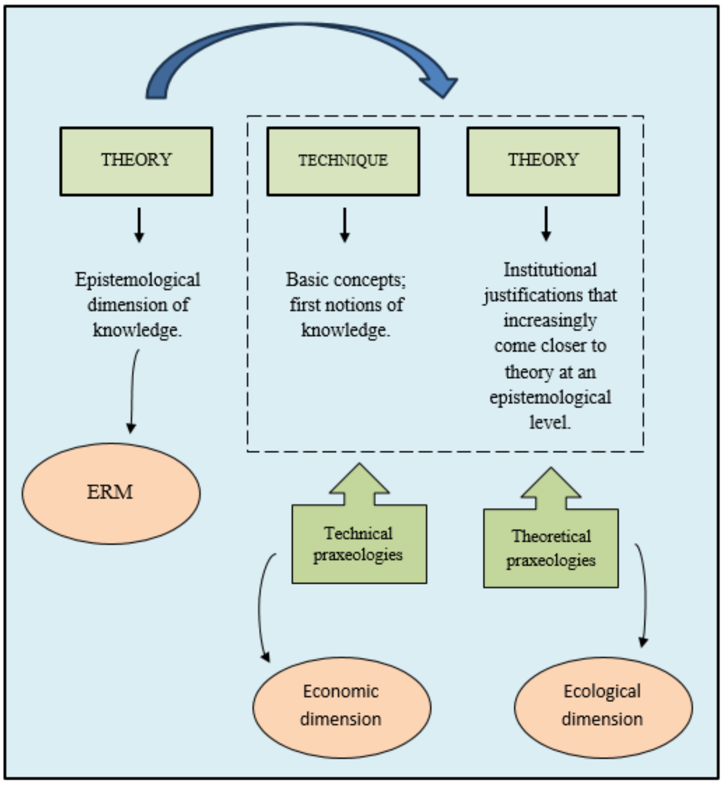
An expansion of this model is one that allows for the diffusion of institutional knowledge through an extension in the movement that always goes from theory to praxeology in its original forms: technical [T, $\tau$] and theoretical [$\theta$, $\Theta$] (picture 7):




Picture 7.

*Diffusion of Institutional Knowledge in the Global Praxeological Model*

In conclusion, we assert that the global praxeological model addresses the analysis of the three dimensions of a didactic problem (Gáscon, 2011). Technical praxeologies can be studied from an economic perspective, emphasizing OM and OD; while theoretical praxeologies allow for an ecological analysis within the framework of the didactic codetermination scale (picture 8).




Picture 8.

*Global Praxeological Model and the Three Dimensions of a Didactic Problem*

**Didactic Problem, ERM, and Global Praxeological Model: Final Considerations**

Outlined, the three dimensions of a didactic problem start from an initial question that is expanded through the epistemological, economic, and ecological dimensions, each with its own inquiries. To reiterate, the development of the didactic problem that we present here to guide this study is framed as follows:

Question $P_0$ or the Problem: Development of the Concept of Function through Its Basic Ideas.

How can the basic ideas of a function contribute to the development of the concept?

Questions $P_1$ of an epistemological dimension:

- What is the rationale behind the concept?
- How has it been interpreted throughout its development?
- What is the scope of the development of the concept?

Questions $P_2$ of economy dimension:

- What is the rationale behind the concept in the school institution?
- How are the Objectives and Outcomes structured according to the evolution of the concept?

Question $P_3$ of ecological dimension:

- In what way is knowledge disseminated within the school institution?

Seeking support in the ERM we developed for analyzing the evolution of the concept of function, which considers the expansion of ideas in a crescendo until it reaches formalization (picture 2), the didactic problem $P_\delta$ in general, it is defined as: how do the basic ideas of function traverse the institutional path from a technical and theoretical perspective, considering the global praxeological model for the dissemination of knowledge? Based on the ideas developed in the global praxeological model, the challenge will be to determine which theoretical elements will be transposed to the knowledge-doing block in this movement of expansion and constant revisits.

**Future considerations**

The general objective of this study was to develop a praxeological model that contributes to the analysis of knowledge based on the conception of a ERM for teaching functions in their basic ideas.

We will assert that the ERM is the slice of reality that the researcher takes as their object of analysis. Given that reality is fluid and interdependent (Caraça, 2010), this slice possesses all the characteristics of a fragment that has been cut out and directed for analysis: it also has limitations of an epistemological and theoretical nature. With this, we mean that the researcher/teacher must adopt a stance of epistemological vigilance and seek to submit the product of their analysis back to this reality, in a perspective of reanalysis and confrontation of their ideas with the initial fragment. mathematical organization.

We present the ideas of the ERM through the conception of the didactic problem and use the teaching of functions as a pathway for analysis, as we believe that this concept is foundational for the teaching of calculus in all its applications. Therefore, we utilize theoretical tools, aiming to explore the basic ideas of ATD.

In this work, we have a first approximation of the ideas we intend to develop regarding the global praxeological model. The next step is, therefore, to submit our conclusions to the institutional reality, which is also fluid and interdependent.

## References

Almouloud, S. A. & Silva, M. J. F. (2021). Estudo das três dimensões do problema didático de números racionais na forma fracionária. *Educação Matemática Sem Fronteiras: Pesquisas Em Educação Matemática*, *3*(2), 114 - 151. https://doi.org/10.36661/2596-318X.2021v3n2.12660

Artigue, M (2010). La théorie anthropologique du didactique: rapports et articulations possibles avec d'autres approaches. In: Bronner, A.; Larguier, M.; Artaud, M.; Bosch, M.; Chevallard, Y.; Cirade, G.; Ladage, C. (org.), *Apports de la théorie anthropologique du didactique: diffuser les mathématiques (et les autres savoirs) comme outils de connaissance et d'action.* (p. 29-39). IUFM de l'académie de Montpellier.

Bosch, M. & Chevallard, Y (1999). La sensibilité de l'activité mathématique aux ostensifs. *Recherches en didactique des mathématiques.* 19(1), p. 77-124. http://yves.chevallard.free.fr/spip/spip/IMG/pdf/Sensibilite_aux_ostensifs.pdf

Boyer, C. B (1986). *História da Matemática*. Trad. Elza Gomide. Edgard Blucher.

Brasil (2018). *Base Nacional Comum Curricular*. Brasília: Ministério da Educação - MEC; Secretaria de Educação Básica – SEB http://basenacionalcomum.mec.gov.br/images/BNCC_EI_EF_110518_versaofinal_site.pdf

Caraça, B. J (2010). *Conceitos fundamentais da matemática*. Gradiva.

Casabò, M. B (1994). *La dimensión ostensiva em la actividad matemática: el caso de la proporcionalidad*. [Tesis Doctorado en Ciencias – Matemáticas – Facultad de Ciencias, Universidad Autónoma de Barcelona], Barcelona.

Castro, K. O. (2012). *Ideias e conceitos básicos de função no 7º ano do ensino fundamental: possibilidades e desafios.* [Dissertação de mestrado em Educação Matemática, Universidade Severino Sombra], Vassouras, RJ.

Castro, K.O. (2022). *Generalização algébrica: uma abordagem praxeológica*. [Tese de doutorado em Educação Matemática, Universidade Anhanguera de São Paulo]. https://repositorio.pgsscogna.com.br/handle/123456789/48261

Chevallard, Y (1982). Pourquoi la Transposition Didactique? Séminaire de didactique et de pédagogie des mathématiques de l'imag, Grenoble. *Actes de l'année*. Grenoble: Université scientifique et médicale de Grenoble http://yves.chevallard.free.fr/spip/spip/IMG/pdf/Pourquoi_la_transposition_didactique.pdf .

Chevallard, Y (1992). Concepts fondamentaux de la didactique: perspectives apportées par une approche anthropologique. *Recherches en didactique des mathématiques.* v.12, n. 1, p. 73-112. https://revue-rdm.com/1992/concepts-fondamentaux-de-la-didactique/

Chevallard, Y (1997). Familière et problématique, la figure du professeur. *Recherches en didactique des mathématiques.* v.17, n. 3 http://yves.chevallard.free.fr/spip/spip/IMG/pdf/Familiere_et_problematique.pdf

Chevallard, Y (1998). Analyse des pratiques enseignantes et didactique des mathématiques: l'approche anthropologique. *Recherches en didactique des mathématiques*. 19(2), http://yves.chevallard.free.fr/spip/spip/article.php3?id_article=27 .

Chevallard, Y (1999). La recherche en didactique et la formation des professeurs: problématiques, concepts, problèmes. In: Xe école d'été de didactique des mathématiques, 18-25 août , Houlgate. *Actes [...]*. Houlgate, 1999, p. 98-112. http://yves.chevallard.free.fr/spip/spip/IMG/pdf/ete.pdf .

Chevallard, Y (2001). Aspectos problemáticos de la formación docente. *Conférence donnée le 1er avril 2001 dans le cadre des XVI Jornadas del Seminario Interuniversitario de Investigación en Didáctica de las Matemáticas (SI-IDM) tenues à l'Escuela de Magisterio de Huesca (Université de Saragosse*). http://yves.chevallard.free.fr/spip/spip/IMG/pdf/YC_2001_-_Osca.pdf

Chevallard, Y (2002). Organiser l'étude. 3. Écologie & régulation, In: XIe école d'été de didactique des mathématiques, 21-30 août , Corps. *Actes* [...].La Pensée Sauvage, p. 41-56. http://yves.chevallard.free.fr/spip/spip/IMG/pdf/Organiser_l_etude_3.pdf

Chevallard, Y (2003). Approche anthropologique du rapport au savoir et didactique des mathématiques. In: Maury, S. S.; caillot, M. (ed.). *Rapport au savoir et didactiques*. p. (p. 81-104). Éditions Fabert.

Chevallard, Y (2005). *La transposición didáctica: del saber sábio al saber enseñado*. Buenos Aires.

Chevallard, Y (2007). Passé et présent de la Théorie Anthropologique de Didactique. In: A. Estepa, L. Ruiz, F. J. García (Eds.), *Sociedad, escuela y matemáticas. Aportaciones de la Teoría Antropológica de lo Didáctico (TAD*) (pp.705-746). Jaén: Publicaciones de la Universidad de Jaén.

Chevallard, Y (2011). Les problématiques de la recherche en didactique à la lumière de la TAD. *Texte d'un exposé réalisé le 28 janvier 2011 dans le cadre du Séminaire de l'ACADIS* (ADEF, Marseille). http://yves.chevallard.free.fr/spip/spip/IMG/pdf/YC_Acadis_28-01-2011_Notes1.pdf

Eves, H (2008). *Introdução à história da matemática*. Trad. Hygino H. Domingues. Unicamp.

Farras, B. B., Bosch & M. & Gáscon, J. (2013). Las tres dimensiones del problema didáctico de la modelización matemática. *Educ. Matem. Pesq.*, v.15, n.1, p. 1-28.

Gascón, J (1999). Fenómenos y problemas en didáctica de las matemáticas. In *Actas del III Simposio de la SEIEM*, (p. 129-150), Valladolid.

Gascón, J (2011). Las tres dimensiones fundamentales de un problema didáctico. El caso del álgebra elemental. *Revista Latinoamericana de Investigación en Matemática Educativa, RELIME*, v.14, n.2, p. 203-231.

Gascón, J. (2014). Los modelos epistemológicos de referencia como instrumentos de emancipación de la didáctica y la historia de las matemáticas. *Educación Matemática*, p. 99-12. ISSN: 0187-8298. https://www.redalyc.org/articulo.oa?id=40540854006

Rezende, W. M. (2003). *O ensino de cálculo: dificuldades de natureza epistemológica* [Tese de Doutorado, Faculdade de Educação, Universidade de São Paulo]. doi:10.11606/T.48.2003.tde-27022014-121106.

Rodrigues, R. F., Lucas, C. O., Castro, S. L., Menezes, M. B. & Santos, M. C (2021). Elaboração, Análise e Aplicação de um Modelo Epistemológico de Referência para o Ensino do Conceito de Função na Licenciatura em Matemática. *Bolema: Boletim de Educação Matemática* [online]. v. 35, n. 70, pp. 614-636. https://doi.org/10.1590/1980-4415v35n70a04

Rogalski, M (2013). Quelques points sur l'histoire et l'épistémologie des fonctions, pouvant claire certain es questions didactiques sur leur enseignement. *JIEM; IJSME*. v.6(1),

Youschkevitch, A. P (1976). The concept of function up to the middle of the 19th century. *Arch. Hist. Exact Sci.* 16, 37–85 https://doi.org/10.1007/BF00348305