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Conhecimento especializado do professor de matemática e conhecimento interpretativo: tecendo relações teóricas no âmbito da transformação geométrica isométrica rotação

Mathematics teacher's specialized knowledge and interpretative knowledge: weaving theoretical relations within the scope of isometric geometric transformation rotation

Conocimientos especializados del profesor de matemáticas y conocimiento interpretativo: tejiendo relaciones teóricas en el ámbito de la transformación geométrica isométrica de rotación

Connaissances spécialisées du professeur de mathématiques et connaissances interprétatives : tisser des relations théoriques dans le cadre des transformations géométriques isométriques de la rotation

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Abstract

To improve students' mathematical learning, it is necessary to do something different from what has been done, which implies a change in the focus of attention and prioritization on the specificities of the teacher's mathematical knowledge that underlies his/her practice. In this sense, two conceptualizations that support a specialized practice are assumed: the Mathematics Teacher's Specialized Knowledge and the Interpretative Knowledge. Considering these two conceptualizations in an intertwined way, it's possible to optimize assume as a starting point what students know and how they know it, listening to their mathematical thinking, to attribute meaning to the reasoning that supports their productions, in order to make more informed pedagogical decisions, enhancing students' mathematical understanding. Since the teacher's

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knowledge directly impacts students' understanding, a discussion focused on the most problematic mathematical topics is essential, and the isometric geometric transformation rotation is one of such topics. Thus, we held a discussion aiming to intertwine these conceptualizations, using the context of rotation, with the perspective of deepening and refining the understanding of the content of the specificities of the knowledge of the mathematics teacher and we ended with some propositions for research and formation.

Keywords: Mathematics teacher's specialised knowledge, Interpretative knowledge, Isometric geometric transformation rotation.

Resumen

Para mejorar el aprendizaje matemático de los alumnos es necesario hacer algo distinto de lo que se ha hecho, lo que implica un cambio en el foco de atención y priorización en las especificidades del conocimiento matemático del profesor que subyace a su práctica. En este sentido, se asumen dos conceptualizaciones que sustentan una práctica especializada: el Conocimientos especializados del profesor de matemáticas y el Conocimiento Interpretativo. Considerando estas dos conceptualizaciones de manera imbricada, es posible tomar como punto de partida lo que los alumnos conocen y cómo conocen, escuchando su pensamiento matemático para atribuir significado a los razonamientos que sustentan sus producciones, posibilitando prácticas matemáticas más informadas desarrollando y profundizando la comprensión matemática de los alumnos. Dado que el conocimiento del profesor impacta directamente en la comprensión de los alumnos, una discusión centrada en los temas matemáticos más problemáticos es esencial, y la transformación geométrica isométrica rotación es uno de ellos. Así, realizamos una discusión con el objetivo de entrelazar estas conceptualizaciones, utilizando el contexto de la rotación, con la perspectiva de profundizar y perfeccionar la comprensión del contenido de las especificidades del conocimiento de los profesores de matemáticas y finalizamos con algunas propuestas para la investigación y la formación.

Palabras clave: Conocimientos especializados del profesor de matemáticas, Conocimientos interpretativos, Transformación geométrica isométrica de rotación.

Résumé

Pour améliorer l'apprentissage mathématique des élèves, il est nécessaire de faire quelque chose de différent de ce qui a été fait, ce qui implique un changement d'attention et une priorisation sur les spécificités des connaissances mathématiques de l'enseignant qui sous-tendent sa pratique. En ce sens, deux conceptualisations sont supposées soutenir une pratique spécialisée: les connaissances spécialisées du professeur de mathématiques *e* et les connaissances interprétatives. En considérant ces deux conceptualisations de manière étroitement liée, il devient possible de prendre comme point de départ ce que les élèves savent et comment ils savent, en écoutant leur pensée mathématique, pour attribuer un sens aux raisonnements qui soutiennent leurs productions, afin de rendre une pédagogie plus éclairée. décisions, améliorant la compréhension mathématique des élèves. Puisque les connaissances de l'enseignant impactent directement la compréhension des élèves, une discussion centrée sur les sujets mathématiques les plus problématiques est essentielle, et la transformation géométrique isométrique de rotation en fait partie. Ainsi, nous menons une discussion entrelaçant ces conceptualisations, en utilisant le contexte de la rotation, afin d'approfondir et d'affiner la compréhension du contenu des spécificités des savoirs des enseignants de mathématiques et nous terminons par quelques propositions de recherche et de formation.

Mots-clés : Connaissances spécialisées du professeur de mathématiques, Connaissances interprétatives, Rotation de transformation géométrique isométrique.

Resumo

Para melhorar a aprendizagem matemática dos alunos é necessário fazer diferente do que tem sido feito, o que implica uma mudança de foco de atenção e priorização nas especificidades do conhecimento matemático do professor que fundamenta sua prática. Nesse sentido, assumem-se duas conceitualizações que sustentam uma prática especializada: o conhecimento especializado do professor de matemática e o Conhecimento Interpretativo. Considerando essas duas conceitualizações de forma imbricada, maximiza-se assumir como ponto de partida o que os alunos conhecem e como conhecem, escutando o seu Pensar matemático, para atribuir significado aos raciocínios que sustentam suas produções, de modo a tomar decisões pedagógicas mais informadas, potenciando o entendimento matemático dos alunos. Uma vez que o conhecimento do professor impacta diretamente o entendimento dos alunos, é fundamental uma discussão centrada nos tópicos matemáticos mais problemáticos, e a transformação geométrica isométrica rotação é um desses. Assim, efetuamos uma discussão objetivando entrelaçar essas conceitualizações, recorrendo ao contexto da rotação, tendo como perspectiva aprofundar e refinar o entendimento do conteúdo das especificidades do conhecimento do professor de matemática e terminamos com algumas proposições para a pesquisa e formação.

Palavras-chave: Conhecimentos especializados do professor de matemática, Conhecimento interpretativo, Transformação geométrica isométrica rotação.

Mathematics Teacher's Specialised Knowledge and Interpretative Knowledge: weaving theoretical relations within the scope of isometric geometric transformation rotation

Our knowledge as teachers is, among a set of controllable factors, the one that most impacts our professional practice and, consequently, students' learning and outcomes (Nye et al., 2004). For almost 40 years, a part of the research has focused on teacher's knowledge and expertise (e.g., Shulman, 1986; Tardif, 2002); however, this focus has not generated improvements neither in teacher's professional practice (understood here from the perspective of the focus on mathematical discussions developed), nor in students' outcomes, which continue to be below expectations. This is evident in the results revealed by the unsatisfactory mathematics learning indicators at any level of Basic Education (Brasil, 2024), in both of the assessment systems considered the national level (Basic Education Assessment System – SAEB) and international level (Program for International Student Assessment – PISA). This lack of satisfactory results is also related to the fact that the priority focus of the discussions that have been promoted is general and leaves aside both a mathematical discussion (Fiorentini & Crecci, 2017) specialised and the specificities of the teacher's knowledge for this practice, which are necessary for the transformation of learning and results in a sustained manner.

To change this scenario, it is essential to change the focus of research and the formation of generalities for the specificities (Ribeiro, 2018) of this teacher's knowledge, because if we seek different results, there is no point in continuing doing the same (Ribeiro & Silva, 2024). This specific knowledge for the teacher's professional practice can be understood according to different conceptualizations (e.g., Mathematical Knowledge for Teaching – MKT (Ball et al., 2008); Knowledge Quartet – KQ (Rowland, 2009); Mathematics Teacher's Specialised Knowledge – MTSK (Carrillo et al., 2018); Interpretative Knowledge – IK (Jakobsen et al., 2014). Here, we chose to focus on the last two conceptualizations (MTSK and IK) that complement each other and allow us to better understand the characteristics of the content of the mathematics teacher's knowledge so that we can later consider proactive ways[4] of developing this knowledge.

Mathematics Teacher's Specialised Knowledge (MTSK) is a theoretical conceptualization designed to deepen the understanding of the elements that make up the

[4] It is part of the CIEspMat research group's agenda to carry out specialised formation with this intention and within the scope of the CNPq-funded research project "Development of the Teacher's Interpretative and Specialised Knowledge and its Relations with the Tasks for Teacher Education in the scope of Measurement, and Algebraic, Geometric and Statistical Thinking" (404959/2021-0). We have developed several formation proposals aiming, ultimately, to improve the quality of students' mathematical learning of different mathematical topics.

teacher's specialised knowledge involved in the mathematical practice. It also serves as an analytical tool that allows investigating this knowledge, understanding it and, based on it, providing guidance for teacher formation (Carrillo et al., 2018). Interpretative Knowledge (IK) is also a theoretical conceptualization that unites the specialised mathematical knowledge involved and necessary for the mathematical practice of interpreting students' productions with the knowledge of how to approach errors and unusual reasoning, understanding them as learning opportunities (Di Martino et al., 2020). Thus, IK underpins the teacher's interpretative practice to interpret and attribute meaning to students' productions and propose constructive feedback that contributes to the development of students' understanding (Jakobsen et al., 2014).

Both conceptualizations refer to the specialised mathematical knowledge that supports the practice of teaching mathematics in each of the topics that students have the right to know, even if these topics are problematic, in terms of the difficulties that students may have. One of the topics that students (and teachers) have difficulties with is the isometric geometric transformation rotation, which is considered the most difficult among the isometries (e.g., Küchemann, 1980; Turgut et al., 2014; Xistouri & Pitta-Pantazi, 2011). Despite this, it is essential to understand that rotation corresponds to an operation that is performed with a figure following a set of specific and generalizable procedures (algorithm), obtaining an image as a result of this algorithm, since this understanding allows establishing and understanding the connections with other mathematical topics such as with the other isometric geometric transformations (translation and reflection), functions, symmetry and congruence (Hollebrands, 2003).

To overcome students' difficulties in rotation and other topics, it is essential to develop (and for this, to understand) the content of the teacher's specialised and interpretative mathematical knowledge. We assume the need and the importance of better understanding the relations between MTSK and IK, based on the specialisation they assume for the teacher's knowledge, because we believe that a broader and deeper understanding of these specificities will contribute to think about and operationalize educational focuses and contexts that contribute to do what has not yet been done and, thus, also enable students' learning and results to be within expectations. In this sense, we discuss the relations between these conceptualizations, focusing on the similarities and differences, to better understand the specificities of the content of the teacher's professional mathematical knowledge, and, for this, we resort to the context of the rotation topic.

Mathematics teacher's specialised knowledge

Mathematics Teacher’s Specialised Knowledge – MTSK (Carrillo et al., 2018) is a theoretical conceptualization that models the knowledge of mathematics teachers and allows characterizing and understanding the specificities of the content of their knowledge, understood as a specialised whole, and its constituent elements in both the mathematical and the pedagogical contexts, which makes it possible to investigate this knowledge with a focus on the teacher’s different professional practices.

In terms of operationalization, MTSK considers teachers’ knowledge as composed of two domains: Mathematical Knowledge – MK and Pedagogical Content Knowledge – PCK (Figure 1). MK corresponds to the mathematics teacher's knowledge regarding the scientific discipline within an educational context, while PCK refers to pedagogical knowledge specifically related to mathematics teaching and learning.

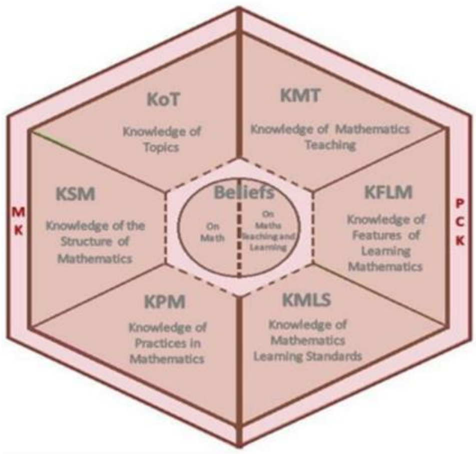


Figure 1.

Mathematics Teacher’s Specialised Knowledge (Carrillo et al., 2018, pp. 241)

Considering our objective, we focused our attention on the content of MK, since the specificities of its content support Interpretative Knowledge (Ribeiro, 2024). MK is subdivided into three subdomains: Knowledge of Topics (KoT), Knowledge of the Structure of Mathematics (KSM) and Knowledge of Practices in Mathematics (KPM). To provide examples of the content of this knowledge, we chose to focus on the topic of rotation because it is problematic concerning the difficulties faced by students and teachers (e.g. Gomes, 2012; Küchemann, 1980; Turgut et al., 2014; Xistouri & Pitta-Pantazi, 2011).

Knowledge of Topics (KoT) relates to the knowledge of the major mathematical ideas and the topics to be taught. It encompasses mathematical knowledge beyond the know-how of the student's level of knowledge (Carrillo et al., 2018). It consists of four categories: (i) procedures; (ii) definitions, properties and foundations; (iii) registers of representation; (iv) phenomenology and applications.

(i) procedures include the teacher's knowledge of different ways of doing things in mathematics, using algorithms (conventional and alternative) or other strategies. In the context of rotation, it involves knowing that, to determine the center of rotation in a task in which the figure and the image (rotated) are already available, we need to choose a point belonging to the figure and its corresponding point in the image, join them, draw the bisector, repeat this procedure by choosing another point and its corresponding point in the image to obtain the bisector between them, so that, at the intersection of these two bisectors, we have the center of rotation.

(ii) definitions refer to the knowledge of several definitions for the same topic and includes knowing a minimum set of properties of the topic that allow it to be uniquely identified. For example, it involves knowing that a possible definition of rotation is:

Let us fix a point O in the plane Π now oriented (as tradition recommends, the positive direction is counterclockwise). Given an angle α, the *rotation with center* O and *amplitude* α is the transformation that associates to each point A in the plane Π the point $A' = R_{\alpha}(A)$ so that we have AO' = AO, $\widehat{AOA'} = \alpha$ and the direction from A to A' (around O), positive (Wagner, 1993, pp. 75).

(ii) properties, include knowledge of the set of all mathematical attributes that are common to the topic. This includes knowing, for example, that the center of rotation is the only fixed point in the rotation (except in the identity rotation).

(ii) foundations, related to the knowledge of the set of mathematical attributes that "support" the topic and connect concepts (Camacho & Guerrero, 2019). It refers to knowing, for example, that the figure, the center of rotation, the angle of rotation (amplitude and direction) and the image are the foundations of rotation.

(iii) registers of representation, refer to the knowledge of the different ways of representing a topic, process or procedure and they can be arithmetic, concrete, graphic or pictorial records, involving verbal or symbolic language (Duval, 1996). In the context of rotation, it includes knowing that $R_o\ [(X, Y, Z), 60°]$ represents the rotation of a triangle with vertices X, Y and Z with the center of rotation at O from an angle of 60° in the counterclockwise direction.

(iv) phenomenology and applications, related to knowing the phenomena and concepts associated with each topic, attributing meaning to the possible externalizations of these phenomena. In the context of rotation, it involves knowing that the contexts that evoke it are associated with understanding it as a rigid movement, resulting from an operation (following the rotation procedures – generalizable algorithms) with an original figure, from a center (point) or axis (line), which includes the amplitude and direction of an angle (of rotation) and a bijective function (relation between points of two sets – figure and image).

In addition to this knowledge of each one of the topics, we, as teachers, must have the knowledge that allows us to understand mathematics as a network of concepts and, for this understanding, the connections between different mathematical topics are fundamental. Knowledge of the Structure of Mathematics (KSM) refers to the teacher's knowledge of the different connections between mathematical topics considering the temporal dimension of mathematical sequencing and the aspects of each topic. It is composed of the following categories (Montes & Climent, 2016): (i) connections based on increased complexity; (ii) connections based on simplification; (iii) transversal connections; (iv) auxiliary connections.

(i) connections based on increased complexity, involve a more complex perspective than the specific discussions required by the context. In the context of rotation, the connection between the complexification of rotation and the conversion[5] of trigonometric ratios in the context of the trigonometric circle refers to knowing that, by rotating the right triangle in the trigonometric circle, it is possible to convert the trigonometric ratios of the 3rd quadrant to the 1st quadrant, this conversion being in terms of the amplitude of the rotation angle and not of the elements of the figure.

(ii) connections based on simplification, refer to something simpler than the specific discussions required by the context. It includes knowing the connection of the angle of rotation (angle at the center of a circumference) and fraction (in the part and whole sense) because, for example, rotating a figure 90° from a point coinciding with the center of the circumference and a certain direction, corresponds to considering $\frac{1}{4}$ of the circumference.

[5] Although the term “reduction” of trigonometric ratios between quadrants is commonly used in curricular documents and even in textbooks, in terms of appropriate mathematical language what happens is a conversion.

(iii) transversal connections, include knowing that the nature of concepts is related when addressing different concepts throughout school mathematics. An example involves knowing the transversal connection between rotation and rotational symmetry, knowing that the entire image obtained, after the transformation, together with the figure, because they are congruent, have rotational symmetry.

(iv) auxiliary connections, include knowing different concepts or topics, which are not the focus of the discussion, adding a necessary element to contribute and support the mathematical discussion. In the context of rotation, it includes knowing the auxiliary connection between the procedures used to rotate a figure on the Cartesian plane and the procedures for locating points on this plane, since it is necessary to determine the coordinates of some points of the figure, as well as the center of rotation to obtain the coordinates of some points of the image.

In addition to the teacher's knowledge of the different connections between mathematical topics, we must know the central aspects of what the practice of producing mathematics means, in terms of its functioning as a science, knowledge that is included in the Knowledge of Practices in Mathematics – KPM (Carrillo et al., 2018; Rebolledo et al, 2022). The following categories are considered in this subdomain: (i) practice of demonstrating; (ii) practice of defining; (iii) practice of solving problems; (iv) the role of mathematical language.

(i) practice of demonstrating, related to knowing how demonstrations are developed, starting, for example, from particular cases and establishing regularities until reaching the level of generalization; or knowing the method by absurdity. It also includes knowledge of the use and role of examples and counterexamples, as well as justifying, making deductions and inductions. It involves knowing that since central reflection (in relation to a point or axis) is a particular case of rotations whose amplitude of the rotation angle is 180°, regardless of the direction, central reflection can be used to carry out demonstrations associated with rotation.

(ii) practice of defining, refer to the knowledge of the necessary and sufficient conditions to generate definitions, that is, the characteristics of hierarchy, non-circularity, non-ambiguity, non-contradiction, minimality, independence under change of representation, equivalence and elegance of a definition. It involves knowing that to define rotation one must specify its foundations (the figure, the center of rotation and angle of rotation – amplitude and direction – and the image), in addition to determine the plane (or space) in which it will be defined.

(iii) problem-solving practice, include the knowledge of strategies to simplify, reinterpret, decompose, systematize and introduce an auxiliary element to explore the solution to a problem, considering the use of graphics and drawings as heuristics. It also includes knowing that a simplification strategy when faced with the problem of determining the measurement of the generator of a right cone – obtained by rotating a right triangle –, corresponds to using the height of this triangle (which coincides with the height of the cone) as the axis of rotation and a rotation angle of amplitude of 360° (independent direction) and thus determining, using the Pythagorean theorem, the measurement of the hypotenuse that will coincide with the measurement of the generator of the cone.

(iv) the role of mathematical language, involve knowing the role of symbols to reduce and briefly express information and conventional symbols in validation contexts, in addition to knowing mathematical notation and the meaning of quantifiers. It also involves the knowledge of formal mathematical language. In the context of rotation, it includes knowing that in terms of formal mathematical language it is inappropriate to use the terms "turn", "spin" or "displace" to refer to this transformation, and it is necessary to specify that the movement is a rotation performed from an oriented angle (determined amplitude and direction).

This specialised mathematical knowledge substantiates the teacher's professional practice to enable students to understand what and why they do it at each moment, considering the different mathematical topics. By assuming this as a starting point for specialised practice, specialised knowledge associated with interpretation is required. It is called Interpretative Knowledge (Ribeiro et al., 2013; Di Martino et al., 2020; Mellone et al., 2020).

Interpretative knowledge

The professional practice of mathematics teachers when interpreting and assigning meaning to students' productions is supported by their specialised mathematical knowledge (Jakobsen et al., 2014; Ribeiro, 2024). Beyond knowing *how to do* something or merely applying a "rule", mathematical knowledge is essential for teachers to interpret students' reasoning and ways of thinking expressed in their productions (verbal and written), even if they are incorrect, unexpected or unusual – mathematically adequate, but different from what the teacher expected (Di Martino et al., 2020). To assign meaning to students' productions, it is necessary to know the mathematical topics to be taught, to develop connections between these topics and to recognize the mathematical potential of students' productions in a valid and meaningful way (Jakobsen et al., 2014) to, subsequently, make the best pedagogical decisions.

This teacher knowledge is called Interpretative Knowledge – IK (Ribeiro et al., 2013) and, according to the Springer Nature Encyclopedia of Mathematical Education, about this knowledge:

> It refers to a deep and wide mathematical knowledge that enables teachers to support students in building their mathematical knowledge starting from their own reasoning and productions, no matter how non-standard or incorrect they might be. IK completes the knowledge of typical errors or solution strategies, with the knowledge of possible source for typical or atypical error and the knowledge of possible use of errors (Di Martino et al., 2020, pp. 426).

IK is concerned with the teacher's interpretative practice that involves listening to students' mathematical thinking beyond the mere sensorial – hermeneutic listening (Davis, 1997; Di Martino et al., 2017) – paying attention to details to interpret deeply and attribute meaning to what is heard (Mellone, et al., 2020) and, thus, (re)design learning paths based on what students know in mathematics, so that they advance in the development of their mathematical knowledge (Di Martino et al., 2017) in order to explore the associated difficulties and errors and carry out more informed pedagogical guidance based on the attributed meaning.

These guidelines are understood as feedback, that is, the information that is provided by the teacher to a student about aspects of his/her performance or understanding (Hattie & Timperley, 2007). This information aims to minimize the distance between what is accomplished and what is expected (Sadler, 1989). The feedback is a form of communication between teacher and student (Black & William, 1998), so that, when proposing feedback, the teacher explains information, being able to indicate to the student how to proceed, helping him to “move”, to reduce the distance between the knowledge he already has and what has not been developed yet. Therefore, feedback is an essential element for the development of students’ learning (Santos & Pinto, 2009), so that they consider about their initial responses and think of new mathematical strategies (Dalto & Silva, 2020).

The *feedback* commonly proposed to students can be distributed into five categories (Galleguillos & Ribeiro, 2019; Santos & Pinto, 2009): (i) feedback on how to solve the problem; (ii) confusing feedback; (iii) counterexample as feedback; (iv) superficial feedback and (v) constructive feedback.

(i) feedback on how to solve the problem contains instructions indicating the procedures to be followed by students to solve a specific problem.

(ii) confusing feedback: although correct, is incomprehensible to the student, due to the complexity of its instructions.

(iii) counterexample as feedback presents an explanatory example of why the student's solution is incorrect.

(iv) superficial feedback is an insufficient or inconsistent guidance that does not help the student to understand his/her mistakes.

(v) constructive feedback contains clear guidelines proposed to students which contribute to the development of their mathematical knowledge.

Categories (i) and (ii) are associated with an instructive interpretative practice, guiding the student on how to proceed and does not require the teacher to attribute meaning to the student's productions nor to listen to their mathematical thinking. Categories (iii) and (iv) are associated with evaluative practices and focus on explaining why the student's production contains errors, but they demand from the teacher a correct interpretation of this production, requiring mathematical knowledge that allows the teacher to approach a problem in different ways and it requires him to know several examples to be able to explain why some ways of proceeding are incorrect. Category (v) relates to a practice that does not simply instruct the student on how to proceed or just evaluate his production, requiring a specialised look to understand it (even if it contains errors) and attribute mathematical meaning.

Thus, providing constructive feedback to students, that helps them effectively to understand mathematics, is broader than simply indicating whether the work is correct or not and may contain tips, examples, corrections, explanations or alternative strategies (Galleguillos & Ribeiro, 2019), always considered in the form of questions, that is, it needs to encourage students to re-analyze their answers, so that they can reformulate their ways of thinking and developing increasingly efficient strategies. This movement of providing feedback that helps students to develop their understanding is difficult and extremely complex (Santos & Pinto, 2009).

To propose constructive feedback, it is essential to know different strategies and representations for solving the same problem, or to develop this understanding during interpretation, making it possible to attribute meaning to all the students' productions, including the ones that differ from those of the teacher – which are outside of what is called the solution space.

The solution space refers to the set of multiple forms of reasoning, strategies and mathematical representations that each one (teacher and student) should conceive when having to solve a problem, even if it presents a single solution (Jakobsen et al., 2014). However, this teacher's solution space is generally composed of a single element, that is, the teacher knows, essentially, a single way to proceed to solve a problem (e.g., Jakobsen et al., 2014; Di Martino et al., 2017). Thus, it is necessary to expand the boundaries of the teacher's solution space and increase the cardinality of the elements of this space, through the development of its IK (Jakobsen et al., 2014), so that the feedback proposition not only considers the different mathematical approaches used in the search of a solution to the same problem, but can also understand and attribute meaning to the students' productions, assuming them as a starting point for subsequent mathematical discussions.

The level of the teacher's Interpretative Knowledge supports the development of a given interpretative practice (Mellone et al., 2017): (i) evaluative interpretation; (ii) interpretation for the educational design; (iii) interpretation as research.

(i) evaluative interpretation is based on the teacher's knowledge to establish a correspondence between his/her way of solving a problem and that of the student, taking into account his/her way as a parameter to obtain the correct answer and determining as incorrect the productions that do not match his/hers;

(ii) interpretation for the educational design is based on the knowledge that allows the teacher to (re)design the following steps based on the students' productions (interprets the production and rethinks the planning of the next discussions to be proposed, in order to outline a new path to achieve the objective of mathematical learning);

(iii) interpretation as research the teacher (re)analyzes his/her own mathematical formalization, reviewing his/her way of proceeding to solve a problem so that it is consistent with the students' productions, even if these seem to be in conflict with what is traditionally taught in school. In this sense, the teacher can research other ways of solving a problem, which may come from research results or by discussing the production with peers, which allows him/her to expand his/her solution space, getting to know new ways of proceeding.

If we seek a mathematical practice that enables students to understand what they do and why they do it at each moment, and in which the different ways of thinking mathematically that circulate in these moments are taken as a starting point, the interpretation of students' productions cannot be configured as a process that involves only evaluative listening (evaluative interpretation). This occurs when the teacher considers that there is only one “correct” answer and only makes a correspondence between the student's production and what he/she expects. To improve the teacher's professional practice, it is necessary to develop his/her IK, which will enable an interpretative practice as research (Mellone et al., 2020).

To understand these interpretative practices, we conceptualize in the CIEspMat group[6] the so-called Interpretative Tasks – IT (Mellone et al., 2020) that are associated with the formative objective of developing teachers' Interpretative Knowledge[7] and the investigative objective of better understanding the content of this knowledge and how it is developed. IT is a formative resource and an instrument for collecting information in contexts with this intentionality (understanding research and formation in an intertwined way), by situating discussions in the context of the teacher's mathematical practice associated with interpreting and attributing meaning to students' productions within the scope of a mathematical topic that proves to be problematic for students.

The IT is structured in three parts: (i) Preliminary Part; (ii) Part I; (iii) Part II. The Preliminary Part aims to access and, through discussion, develop the IK and for this purpose it contains questions about the topic being addressed in some of the dimensions of the teacher's specialised knowledge – mathematical and/or pedagogical. Part I is structured around a task for the student (typically indicated within a rectangle), followed by questions for the teacher that cover some of the categories of specialised knowledge of the different subdomains of the MTSK. Part II has the specific objective of developing the IK and is composed of a set of students' productions (written, on video, classroom discussions) selected for being mathematically powerful and either for focusing on the main difficulties of the students in the topic or for containing unusual strategies. Teachers are asked to interpret these productions and to propose constructive feedback for each of the different ways of thinking that can help students to understand and to expand their knowledge.

Let us consider an example of an Interpretative Task within the scope of rotation[8] in which the Preliminary Part contains two questions that seek to access and develop the teacher's specialised knowledge. The first one aims to discuss the phenomenology of rotation and the second one focuses on mathematically valid definitions of rotation that are understandable for the students.

[6] CIEspMat is a Research and Formation group that develops discussions focused on developing the Interpretative and Specialised Knowledge of teachers and future teachers of mathematics – from Early Childhood Education to High School. www.ciespmat.com.br

[7] There are also ITs designed for the formation of teachers' educator.

[8] They are part of one of the Interpretative Tasks within the scope of the rotation that have been conceptualized in the CIEspMat group.

Part I (Figure 2) includes a task for students (within a rectangle) whose mathematical learning objective is to develop students' understanding of rotation, in terms of identifying its constituent elements and procedures performed to make rotation, based on rotated images. This task for the student serves as a genesis for the formative discussion. Three questions for the teacher are also included. The first asks the teacher to solve the task for the students (knowledge associated with knowing how to do the task for the students); the second directs attention to identifying the greatest mathematical difficulties of the students when solving the task and to anticipate possible answers, considering them for the planning and implementation of mathematical discussions; the third focuses on what (and how) the students need to know to answer the task, in order to enable a discussion about topics associated with rotation.

Part II (Figure 2) includes four students' productions (from the student task included in Part I) and the teacher is asked to interpret and attribute meaning to the ways of thinking and proceeding in mathematics that support each of these productions and to provide constructive feedback for each production. In this Part II, the aim is to assess the level of Interpretative Knowledge of the participants and, through the associated discussions, to promote a change in the level of this knowledge.

Here, we focus on one of these productions (Camila) included in Part II, which allows us to discuss the difficulty and the problem in identifying the center of rotation as the only point that remains fixed when performing the rotation and which involves knowing the usual procedure for identifying the center of rotation. This production was selected since it contains one of the greatest difficulties for students in this topic (Küchemann, 1980), which is associated with not knowing one of the foundations of rotation (center), as well as not visualizing the movement.

Part I

Task: Rotating cards[9]

(You should always explain your reasoning by describing the process you use to answer the question. You can do it using diagrams, words, calculations, ...)

1. Look at the two situations below. In both situations we have a "queen of hearts" card:

[9] Adapted from Paques, O. T. W. Oferenda musical de Bach. Série Matemática na Escola. Guia do professor, versão para tela, (pp. 9). https://m3.ime.unicamp.br/arquivos/1143/oferendamusicaldebach-guia.pdf

situation 1

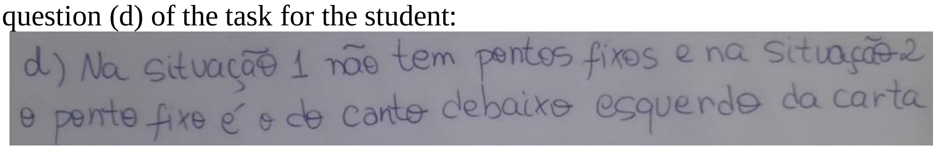
situation 2

[...]
(d) In each situation, can you identify a point that remains fixed when the movement is performed? If so, indicate which point it is and justify why.
[...]

[...]

Part II

1. After implementing this task with his 7th grade D students, teacher Mário got some answers and decided to also take them to discuss in the CIEspMat formation. See the production of student Camila regarding question (d) of the task for the student:



d) Na situação 1 não tem pontos fixos e na situação 2 o ponto fixo é o do canto debaixo esquerdo da carta

(*d) In situation 1 there are no fixed points and in situation 2 the fixed point is the one in the bottom left corner of the card.*)
Camila's production for question (d).

(a) For this production, indicate whether you consider it mathematically correct (adequate) or not, justifying the mathematical reasoning shown.

[...]

Figure 2.

Part of an Interpretative Task within the scope of rotation

Teachers need to interpret Camila's production and provide constructive feedback, which requires listening to the student's mathematical thinking beyond a "sensory listening", demanding a hermeneutic listening that, in fact, considers as a starting point what and how the student reveals to know and makes it possible to propose clear and objective discussions to help her to develop her mathematical understanding.

Based on this production (focusing on situation 1 of the task), examples of interpretative practices (Mellone et al., 2017) that can be carried out and that are associated with different levels of interpretative knowledge are presented, assuming as a starting point that the production is mathematically incorrect.

An evaluative interpretation involves interpreting the production as incorrect, showing the student how to identify the center of rotation, in the sense of "giving the rule" (feedback on how to solve the problem of the instructive type); an interpretation for the educational design involves interpreting the production as incorrect, because the student was unable to solve the problem (identify the center of rotation), telling him that the center of rotation is one of the points in the figure (superficial feedback of the evaluative type). In this type of practice, the teacher can think of other tasks to help students to understand the constituent elements of rotation; an interpretation as research would also consider the student's production as incorrect, however, in this practice, the teacher seeks to understand why the student did not identify the center of rotation and which mathematical knowledge needs to be developed for the student to solve the problem (know the constituent elements of rotation and the associated procedures). For the student to review his/her production and identify the center of rotation, the teacher guides him/her to draw some perpendicular bisectors, asks what happens to all these perpendicular bisectors and whether they intersect, so that the student perceives the intersection at a single common point which is the center of rotation (constructive feedback).

Relations between specialised mathematical knowledge and interpretative knowledge

Mathematics Teacher's Specialised Knowledge (MTSK) and Interpretative Knowledge (IK) conceive the teacher's knowledge as specialised[10], so it is essential to understand the relations and connections between these conceptualizations in order to contribute to refining and deepening the understanding of the content of the teacher's knowledge and its specificities and in order to complement what is already known (e.g., Piccoli and Souza, 2020; Almeida and Lopes, 2023; Jakobsen et al., 2022). Including knowledge on specific topics such as subtraction (Ribeiro, 2024) and measurements (Di Bernardo et al., 2018), in relation to the content of this knowledge and to consider that research is necessary for this deepening, as well as what to focus attention on formation to develop this knowledge and improve the quality of mathematical practice and students' learning and outcomes.

In terms of similarities between these conceptualizations, it is important to note that both consider the teacher's practice as being specialised and that this specialisation is considered

[10] It was decided to relate IK to MTSK, as this is the only conceptualization that conceives all the teacher's knowledge as specialised.

in the mathematical and pedagogical context. They also assume an intertwined relation between the content of the teacher's knowledge and the type and focus of their practice with students.

One difference between these conceptualizations is that MTSK focuses on teachers' knowledge in the mathematical and pedagogical dimensions, associated with different practices related to teaching, such as planning practice, classroom practice or assessment practice, while IK focuses on the involved and required specialised mathematical knowledge that supports an interpretative practice related to the explicit prioritization of assuming as a starting point for discussions and development of students' mathematical knowledge and competencies what they already know and how they know each topic. (This same principle occurs in formative contexts associated with the need for teachers to experience the same types of experiences that they are expected to provide to their students – just at a different level of professional formation.)

In the IK, different interpretative practices are considered and each of them is associated with levels of interpretation. These levels of interpretation are related to levels of specialised mathematical knowledge (in each category), which makes it possible to understand how much "we listen to students' mathematical thinking" (Mellone et al. 2023) and attribute meaning to what we hear and incorporate into our mathematical practice (feedback that we provide). However, even if the teacher has broad and deep mathematical knowledge (understood as associated with a higher level of knowledge) in each of the different categories of the MTSK, this does not necessarily guarantee that he/she will carry out an interpretation as research (higher level). More than having this specialised knowledge, it is essential to have "something more" that allows identifying and assigning meaning to students' errors or to unusual or unconventional strategies, making it necessary to support this interpretative practice in a teaching perspective that develops students' mathematical competencies.

Thus, when the student's production presents a way of proceeding similar to the teacher's way, the interpretation to be carried out requires only a direct correspondence between this way of proceeding of both – the teacher will consider it as correct. However, when faced with unusual productions (distinct from the teacher's ways of proceeding), it becomes necessary to have specialised mathematical knowledge, at a higher level, which makes it possible to make mathematical and pedagogical connections for an interpretative practice (direct relation) that is based on understanding the “hows” and the “whys” of these reasonings and ways of proceeding in mathematics, making it possible to provide feedback that helps students expand their understanding and competencies.

Besides understanding the mathematical “hows” and “whys” that support unusual reasoning, an interpretative practice at higher levels involves validating whether this reasoning is mathematically adequate or not and making a conscious decision with a view to future understanding. There is a similarity relation between the knowledge about the ways of producing mathematics that is evoked and allows the teacher to interpret the student’s production, which is associated with the knowledge related to the exploration and generation of mathematical knowledge.

In addition to identifying, the teachers’ professional practice involves making pedagogical decisions, which are based on their mathematical knowledge (related to their beliefs about mathematics, their teaching and learning, and previous experiences) to provide specific feedback and implement a specific mathematical practice. These pedagogical decisions need to ensure the use of appropriate mathematical language, regardless of the educational level in which the teacher teaches, including generalizable mathematical arguments and justifications, and the implementation of these practices is intertwined with the knowledge we have.

In order to elucidate and illustrate the operationalization of these relations between levels of knowledge, we provide an example within the scope of rotation (Table 1).

Table 1.

Relations between knowledge levels within the rotation

MK sub-domains and categories	**Examples of knowledge content**	**MK levels**	**Description of student production**	**IK levels**

KoT – Definitions, properties and foundations	Know that a possible definition of rotation is: Let us fix a point O in the plane Π now oriented (as tradition recommends, the positive direction is counterclockwise). Given an angle α, the *rotation with center* O and *amplitude* α is the transformation that associates to each point A in the plane Π the point $A' = R_\alpha(A)$ so that we have AO' = AO, $\widehat{AOA'} = \alpha$ and the direction from A to A' (around O), positive (Wagner, 1993, pp. 75). Know the property of rotation in which the center of rotation is the only fixed point in the rotation (except in identity rotation). Know that figure, center of rotation, angle of rotation (amplitude and direction) and image are the foundations of rotation.	L1: Know that there is a definition for rotation.	Production in which the student defines rotation considering the figure, the angle of rotation and the image, but does not specify the center of rotation, including as a fixed point (knowing a definition of rotation)[11].	Evaluative interpretation: it interprets the production as incorrect, without justifying what is missing in the production to be correct, as the teacher does not know the definition of rotation.
		L2: Know that there is a definition for rotation that must explicitly the figure, the image and the angle of rotation, but does not specify the direction and amplitude, and the center of rotation (fixed point).		Interpretation for the educational design: It interprets the production as incorrect, explaining that some element is missing from its definition, without saying which one. Furthermore, the teacher considers that in future tasks, he will need to discuss the fundamentals of rotation.
		L3: Know that there are different ways of defining rotation and that all its foundations must be explained.		Interpretation as research: It interprets the production as partially correct, since the student has shown that he/she knows some of the foundations of rotation. The teacher instructs him/her to check whether there is any ambiguity in his/her production and whether all rotations in the plane can be identified from it, so that the student realizes that the center of rotation is missing. From this, the teacher also makes questions about the role of the center of rotation and where it may be located, with the aim of having the student conclude that it is the only fixed point of rotation.
KSM –Transversal connections	Know the transversal connection of rotation and rotational symmetry, since the image obtained after the transformation	L1: Know that figures obtained by isometries have symmetry.	Production in which the student does not identify rotational symmetry	Evaluative interpretation: It interprets the production as incorrect, explaining that there is symmetry in its production (indicate the answer).

[11] One of the difficulties students faces is not considering the fundamentals of rotation, which is related to most of the associated errors (e.g., Turgut et al., 2014).

	together with the figure, being congruent, have rotational symmetry, symmetry being a property of all isometric transformations.	L2: Know that figures obtained by rotation-type isometries have rotational symmetry.	after rotating a figure and explains that there is no symmetry [12].	Interpretation for the educational design: It interprets the production as incorrect, explaining that the student should check again if there is no symmetry, since the figure and image are congruent. Consider that, in future tasks, he/she needs to discuss symmetry as a property of isometric transformations.
		L3: Know that all figures transformed by rotations necessarily have rotational symmetry as a property because the figure and image are congruent.		Interpretation as research: It interprets the production as incorrect, guiding the student to look at the whole (figure and image) and consider if there is a difference between them. Also, guide the student to rotate the image again according to the determined angle, but inverting the direction in a reversible perspective, so that he can verify that all the points of the figure and their respective points of the overlap image – coincide – so that he understands that there is congruence, which guarantees rotational symmetry.
KPM – The role of mathematical language	Know that in terms of formal mathematical language it is inappropriate to use the terms "turn", "spin" or "displace" to refer to this transformation, and it is necessary to specify that the movement is a rotation carried out from an oriented angle (determined amplitude and direction).	L1: Know that rotation is a type of displacement.	Production in which the student expresses rotation as a displacement, without specifying that this movement is a rotation carried out from an oriented angle[13].	Evaluative interpretation: It interprets the production as incorrect, stating that the correct term is rotation.
		L2: Know that rotation is a type of displacement, but that it is necessary to specify in relation to what – angle – this movement is performed.		Interpretation for the educational design: It interprets the production as incorrect, guiding him/her to specify what is considered in this displacement, saying that it is called rotation. The teacher begins to consider that, in future tasks, he/she needs to discuss and use the correct term to refer to this isometric transformation.
		L3: Know that rotation is an isometric transformation whose movement is a displacement		Interpretation as research: It interprets the production as partially correct, asking the student if the movement referred to is any type of displacement, what needs to be considered in this displacement and what is necessary to consider to

[12] This difficulty commonly occurs, since students do not differentiate isometric transformation from symmetry (Bulf, 2010) – a property of transformation.

[13] This inadequacy of referring to rotation as a displacement – a broad and imprecise term – is associated with not differentiating this isometric transformation from others (Thaqi et al., 2011), such as translation (unidirectional displacement in relation to a translation vector).

		in relation to an oriented angle.		perform the movement, so that the student understands that it is a specific movement associated with an oriented angle. Considering the last notion to be known by the student, the teacher explains what this transformation is called.

As a way of illustrating the theoretical relations that we have been discussing, we sought to synthesize all this information so that it is more "understandable" (Figure 3).

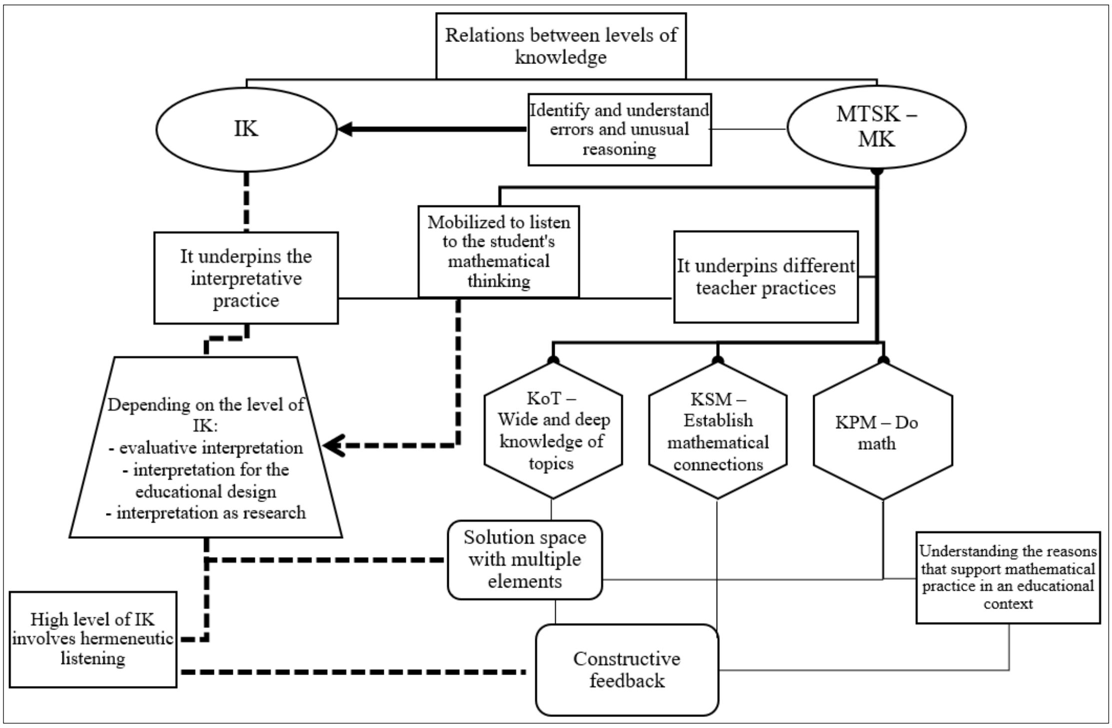


Figure 3.

Relations between Mathematics Teacher's Specialised Knowledge and Interpretative Knowledge

In this representation, each type of figure represents a component of MK and IK. For the theoretical conceptualizations involved, we used the oval representation; for the MK subdomains, we used the hexagonal representation (associated with the configuration of the MTSK representation); the notions related to IK – solution space and constructive feedback – are represented by a rounded rectangular shape; the IK levels are represented by a trapezoid. All other elements associated with the relations and complementarities between the conceptualizations were represented in rectangles.

Different types of lines are used to represent the relations between these elements: dashed lines connect elements that refer to the IK and the associated elements – notions of solution space and constructive feedback. Continuous lines are used to represent the elements of the MK, its subdomains and what it allows. The other relations are represented by thin lines and are associated with direct relations between the conceptualizations. In addition, a continuous line with an arrow was used to connect the element "identify and understand errors and unusual reasoning" to the elements of the IK. The main relation between the IK and the MK is represented by the dashed line with an arrow.

Final comments

The intertwined MK and IK are the intentional resources that enhance the teacher's professional practice to attribute meaning to students' productions. Thus, establishing possible theoretical relations between these conceptualizations is one of the ways to focus the attention of the research on the teacher's interpretative practice, since these conceptualizations are based on the premise that the mathematics teacher's knowledge is specialised – hence the assumption of MTSK – and required for this specialised practice, at higher levels of interpretation.

This search for a deeper understanding of the elements that constitute the teacher's professional knowledge, considering it specialised, is a starting point for conceptualizing and carrying out formation (developing research in an intertwined way) that aims to develop the teacher's specialised and interpretative knowledge, since this knowledge is not developed throughout their experience in the classroom, but requires formation contexts with this intentionality (Ribeiro et al., 2013). Knowing these relations and how they impact mathematical practice is fundamental for students' learning to be as expected. In this sense, understanding each one of these conceptualizations and their relations optimizes focusing on the discussions and formation bases that are essential to enable practices that develop knowledge and competencies, and contribute to eradicating students' difficulties, also because they are aligned with those of the teachers' – as it is the case of those revealed in the rotation (e.g., Turgut et al., 2014). The development of this knowledge in formative contexts, supported by what specialised research has already identified and serving as a context for refining this research, makes it possible to evaluate previous results and gives teachers the opportunity to do, in their professional practice, what has not yet been done, raising their level of knowledge, expanding their solution space and allowing them to provide constructive feedback, impacting their students' mathematical learning and, consequently, their results.

In order to broaden this understanding, validate these relations and their impact on teacher practice and student learning, it is necessary to envision a medium and long-term research agenda that seeks to better understand these relations between conceptualizations in different mathematical topics, expanding the range of examples of specialised mathematical knowledge content and the relations between the levels of knowledge of MK and IK. Thus, some open questions that can help guide this future research are:

(i) What relations are observed, and how are they transformed, between the specialised knowledge (MTSK) and the interpretative knowledge of the mathematics teacher within the scope of each specific theme (understood as consisting of a set of related topics), such as isometric geometric transformations?

(ii) What are the critical elements in the teacher's knowledge that optimize or limit, and how, an interpretative practice at each of the levels of interpretation and knowledge?

(iii) What specialised and interpretative knowledge do teacher' educators reveal (and how to develop) to develop the specificities of teacher knowledge to implement interpretative practices associated with the highest level of interpretation?

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