ISSN 1983-3156

Qualis A1

http://dx.doi.org/10.23925/1983-3156.2025v27i2p063-090

Mathematical knowledge for teaching: limits and possibilities for brazilian research

Conocimiento matemático para la enseñanza: límites y posibilidades de la investigación brasileña

Connaissances mathématiques pour l'enseignement : limites et possibilités de la recherche brésilienne

Conhecimento matemático para o ensino: limites e possibilidades para pesquisas brasileiras

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Abstract

The mathematical knowledge for teaching (MKT) proposed by Deborah Ball and collaborators is a widespread theoretical framework in Brazil. Therefore, it is necessary to investigate and discuss how it is being used in research. This article aims to carry out a critical analysis of the MKT framework presented by Ball et al. (2008), focusing on how it has been used in research in Brazil. To achieve such goal, a theoretical and speculative study was carried out, the corpus of which consisted of (i) Ball et al.'s (2008) article, which is the main text used in Brazilian research to present the MKT; (ii) articles published in Qualis A1 and A2 journals and which used MKT in their analyses; (iii) articles by authors who produced other theoretical models about teachers' mathematical knowledge after MKT; and (iv) recent works by researcher Deborah Ball and researchers linked to her, which indicate current perspectives emerging from her group for teacher knowledge. Based on the interpretation and argumentation based on these

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materials, it was possible to conclude, among other aspects, that research needs to overcome the mere use of MKT subdomains as a priori assumed categories and needs to start to consider them as categories to be investigated, questioned, expanded and reformulated considering the Brazilian context. Thus, we advocate including the Brazilian context in the set of inquiries on the MKT model to prevent research carried out in the country from being merely a reproduction of theoretical references elaborated in such different sociocultural contexts.

Keywords: Mathematical knowledge for teaching, Theoretical and speculative research, Limits and possibilities, Brazilian context.

Resumen

El modelo teórico *Mathematical Knowledge for Teaching* (MKT) propuesto por Deborah Ball y colaboradores es un marco teórico muy difundido en Brasil. Teniendo esto en cuenta, es necesario investigar y discutir cómo las investigaciones lo están utilizando. El artículo tiene como objetivo realizar un análisis crítico del modelo MKT, centrándose en el uso que se ha hecho de este modelo en la investigación en Brasil. Para ello, se llevó a cabo una investigación teórica y especulativa cuyo *corpus* consistió en: (i) artículo *Content Knowledge for Teaching: What Makes It Special?* de Ball y su equipo, el principal texto utilizado en la investigación brasileña para presentar el MKT; (ii) artículos publicados en los periódicos Qualis A1 y A2 (cuadrienio 2017-2020) y que usaron el MKT en sus análisis; (iii) artículos de autores que produjeron otros modelos teóricos sobre el conocimiento matemático del profesor después del MKT; y (iv) trabajos recientes de la investigadora Deborah Ball e investigadores vinculados a ella, indicando perspectivas actuales emergentes de su grupo para el conocimiento del profesor. A partir de la interpretación y argumentación, basada en estos materiales, fue posible concluir, entre otros aspectos, que las investigaciones necesitan superar el mero uso de los subdominios de MKT como categorías asumidas *a priori*, considerándolas como categorías a investigar, cuestionar, ampliar y reformular, teniendo en cuenta el contexto brasileño. Por lo tanto, buscamos incluir el contexto brasileño en el conjunto de investigaciones sobre el modelo MKT, evitando que las investigaciones realizadas en el país sean solo una reproducción de referencias teóricas elaboradas en contextos socioculturales tan diferentes.

Palabras clave: Conocimiento matemático para la enseñanza, Investigación teórica y especulativa, Límites y posibilidades, Contexto brasileño.

Résumé

Le modèle théorique *Mathematical Knowledge for Teaching* (MKT) proposé par Deborah Ball et collègues est un référentiel théorique beaucoup répandu au Brésil. En considérant cela, il est nécessaire d'examiner et de discuter la manière dont les recherches scientifiques l'utilisent. Cet article a pour but une analyse critique du modèle MKT, visant sur son utilisation courante par les recherches menées au Brésil. Pour ce faire, une enquête théorique et spéculative fut menée auprès d'un *corpus* composé par : (i) l'article *Content Knowledge for Teaching: What Makes It Special?* de Ball et son équipe, le texte le plus utilisés par les recherches Brésiliennes pour présenter le MKT; (ii) articles publiés par des revues Qualis A1 et A2 (période 2017-2020) et qui ont utilisé le MKT dans leurs analyses; (iii) articles dont les auteurs ont créé d'autres modèles théoriques sur les connaissances mathématiques des enseignants post-MKT; et (iv) les travaux plus récents de la chercheuse Deborah Ball et des chercheurs liés à elle, indiquant les perspectives émergentes actuelles du groupe pour les connaissances des enseignants. En interprétant ces données et en discutant après elles, il est possible de conclure que, parmi d'autres aspects, les recherches scientifiques doivent surmonter la simple utilisation des sous-domaines du MKT autant que catégories assumées *a priori*, désormais les considérant comme des catégories à étudier, contester, amplifier et reformuler, en tenant compte du contexte Brésilien. Ainsi, l'intention est d'insérer ce contexte Brésilien dans l'ensemble d'enquêtes à propos du modèle MKT, en évitant que les recherches menées dans le pays ne soient qu'une reproduction de références théoriques élaborées chez d'autres contextes socioculturels si divers.

Mots-clés: Connaissances mathématiques pour l'enseignement, Recherche théorique et spéculative, Limites et possibilités, Contexte brésilien.

Resumo

O modelo teórico *Mathematical Knowledge for Teaching* (MKT) proposto por Deborah Ball e colaboradores é um referencial teórico bastante difundido no Brasil. Assim, faz-se necessário investigar e discutir como as pesquisas o estão utilizando. O artigo objetiva realizar uma análise crítica a respeito do modelo MKT, com foco no uso que tem sido feito desse modelo em pesquisas no Brasil. Para tanto, foi realizada uma pesquisa de natureza teórica e especulativa cujo *corpus* foi constituído por: (i) artigo *Content Knowledge for Teaching: What Makes It Special?* de Ball e sua equipe, principal texto utilizado em pesquisas brasileiras para apresentar o MKT; (ii) artigos publicados em periódicos Qualis A1 e A2 (quadriênio 2017-2020) e que fizeram uso do MKT em suas análises; (iii) artigos de autores que produziram outros modelos teóricos sobre o conhecimento matemático do professor após o MKT; e (iv) trabalhos recentes da pesquisadora Deborah Ball e pesquisadores ligados a ela, indicando perspectivas atuais emergentes do seu grupo para o conhecimento do professor. Com base na interpretação e na argumentação, a partir desses materiais, foi possível concluir, entre outros aspectos, que pesquisas precisam superar o mero uso dos subdomínios do MKT como categorias assumidas *a priori*, passando a considerá-las como categorias a serem investigadas, questionadas, ampliadas e reformuladas, levando em consideração o contexto brasileiro. Desse modo, busca-se incluir o contexto brasileiro no conjunto de investigações sobre o modelo MKT, evitando que as pesquisas feitas no país sejam somente uma reprodução de referenciais teóricos elaborados em contextos socioculturais tão distintos.

Palavras-chave: Conhecimento matemático para o ensino, Pesquisa teórica e especulativa, Limites e possibilidades, Contexto brasileiro.

Mathematical knowledge for teaching: Limits and possibilities for Brazilian research

This research is part of a broader investigation that has been developed by members of the Working Group (GT) 07 – The Education of Teachers who Teach Mathematics, of the Brazilian Society of Mathematics Education (Sociedade Brasileira de Educação Matemática - SBEM), dedicated to investigating the specific mathematics of/in teachers' action[4]. Aiming to contribute to the special issue entitled "Theoretical frameworks for discussing mathematical knowledge and expertise mobilized and produced by teachers who teach mathematics in the final years of elementary school and high school," we seek to present a discussion about a theoretical framework disseminated in Brazil, *mathematical knowledge for teaching* (MKT) by Deborah Ball and collaborators.

The relevance of the MKT theoretical model for discussions about teachers' mathematical knowledge is undeniable, as evidenced by the number of theses and dissertations in Patrono and Ferreira's (2021) survey. Considering the works defended after 2008, the year of publication of Ball et al.'s (2008) article, until 2019, the authors found 33 studies, 19 master's dissertations, and 14 doctoral theses that used the MKT model. Similarly, Sousa and Oliveira (2023) surveyed scientific articles in Brazilian journals between 2008 and 2022, computing a total of 49 articles related to the MKT. The number of publications on the MKT in Brazil indicates its relevance for national research and, mainly, the need to investigate and discuss this theoretical model.

Thus, this article aims to carry out a critical analysis regarding the MKT model, focusing on the use of this model in research in Brazil. This analysis is carried out from theoretical and speculative research perspectives, seeking to understand how they can be explored in research in Brazil and the limitations of the model that must be adapted or complemented.

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[4] Currently, researchers from different universities in Brazil are part of this initiative. The research is coordinated by Dr. Flávia Cristina de Macêdo Santana (State University of Feira de Santana, Uefs), Dr. Roberta D'Angela Menduni Bortoloti (University of Southwest Bahia - Uesb) and Dr. Victor Augusto Giraldo (Federal University of Rio de Janeiro, UFRJ). The team includes: Dr. Eliana Matesco Cristovao (Federal University of Itajubá, Unifei), Dr. Enio Fernandes de Paula (Federal Institute of São Paulo, IFSP), Dr. Henrique Rizek Elias (Federal University of Technology of Paraná, UTFPR), Dr. Lya Raquel Oliveira dos Santos (Federal University of Piauí, UFPI), Marta Élid Amorim (Federal University of Sergipe, UFS), Ms. Mayara de Miranda Santos (Federal Institute of Piauí, IFPI), Dr. Sabrina Bobsin Salazar (Federal University of Pelotas, UFPel), Ms. Dr. Silvânia da Silva Costa (UFS), Dr. Vania Batista Flose Jardim (IFSP), Ms. Lana Thaís Santos Silva (Federal University of Sergipe, UFS, Municipal School Dr. Lourival Baptista) and Dr. Vânia Cristina da Silva Rodrigues (Federal University of Triângulo Mineiro, UFTM).

Methodology

This text discusses the MKT based on Van Der Maren's (2004) and Martineau et al.'s (2001) concepts of theoretical and speculative research. Speculative research aims to theorize without using empirical data, as with empirical research (Van Der Maren, 2004). This type of research involves mental work of producing theoretical statements rooted in other theoretical statements, in which the researcher, based on the study and reflection on the chosen theme, aims to construct, through writing, theoretical statements (Van Der Maren, 2004).

Passos (2015) states,

> One of the central issues of speculative narrative research is the ability to produce a new question to propose a new analysis based on the interpretation of previous texts and rigorous arguments. In a way, it is a way of telling a story about the topic being researched. (Passos, 2015, p. 32)

Constructing theoretical statements based on other theoretical statements requires not only actions such as interpreting and arguing but also telling a story about the topic researched (Martineau et al., 2001). These are the three main axes on which this type of research is based: interpreting, arguing, and telling.

Thus, theorizing cannot be done from scratch; that is, speculative research requires a resource, other theoretical statements produced previously (Van Der Maren, 2004). The corpus of theoretical statements produced previously can be constituted in three ways: *single corpus*, *intertextual corpus*, and *contrast corpus*. The *single corpus* is used when the purpose is to discuss a theme or concept from the same author and the same perspective. The *intertextual corpus* is used when the purpose is to discuss a theme or concept from different authors whose interaction of information is convergent or from the same author who produces information on the same theme but from different perspectives (for example, approaching the same theme from the perspective of a student and a teacher). The *contrast corpus*, in turn, comprises statements from authors with different viewpoints on a concept or event, with divergences in the information (Van Der Maren, 2004).

This research uses the contrast corpus, as the focus is on interpreting different sources (authors) who used MKT or produced different theoretical statements based on MKT, indicating possible discrepancies in the information. The *corpus of contrast* to be considered in this research is made up of (i) Ball et al.'s (2008) article, the main text used in Brazilian research to present the MKT; (ii) articles published in Qualis A1 and A2 journals that used MKT in their analyses; (iii) articles by authors who presented other models and types of knowledge subsequent to the MKT; and (iv) recent works by researcher Deborah Ball and researchers linked to her, indicating current perspectives emerging from her group for knowledge of the teacher. Therefore, this research aims *to carry out a critical analysis of the MKT model presented by Debora Ball and her team, focusing on how the model has been used in research in Brazil*. This critical analysis is configured in a text constituted from the interpretations and arguments made throughout the investigation, telling a story (Martineau et al., 2001) about the MKT model in research published in Brazil. In the next section, we discuss the first source of the *contrast corpus*, i.e., the MKT model according to Ball, Thames, and Phelps (2008).

The model for the mathematical knowledge for teaching

The MKT model for describing the knowledge needed to teach mathematics emerges from the notion of *pedagogical content knowledge* (PCK), one of the seven categories5 proposed by Shulman (1987) for the basic knowledge for teaching. The PCK, as presented by Shulman (1987), is a kind of amalgam between content and pedagogy that is exclusive to the teacher, their specific way of knowing the subject, different from content knowledge (CK).

Lee Shulman was a professor at Michigan State University, where his research in teaching, pedagogy, and teacher education became quite influential in the world, partly due to the work of his colleagues and his advisees, future researchers in the area. Deborah Ball comes from this research group. As a professor at the University of Michigan, Deborah Ball founded a research team that developed and refined the PCK specifically for mathematics teaching.

In the seminal text that defines the MKT model, Ball et al. (2008) recognize two important contributions from Shulman’s work. The first was to relate the work of teachers, the teaching task, and the content to teach since, at the time, research on teaching and teachers’ practice was quite generalist. The second and most important contribution was to realize that it was necessary to have a type of knowledge of the content to teach that differed from the usual, common knowledge of the content. In other words, from Shulman’s studies, teaching work began to be investigated in connection with the content to teach and that to teach this content, it is not enough to know it in depth; it is necessary to know it in a unique way that is related to the practical activity of teaching.

5 The seven categories or domains of basic knowledge for teaching, according to Shulman, are content knowledge, general pedagogical knowledge, knowledge of the curriculum, pedagogical content knowledge, Knowledge of learners and their characteristics, knowledge of educational contexts, and knowledge of the educational ends, purposes, values, and philosophical and historical grounds

Despite these important contributions, the PCK, as theorized by Shulman and appropriated by the scientific community, was still too broad and lacked clarity in its definition, allowing its use with a wide range of meanings. In this way, Deborah Ball's group, when refining the model for the area of mathematics, presented more precise categories and definitions and sought to validate the model empirically.

From the perspective of the MKT model, teachers must have both CK and PCK to teach mathematics. Although Shulman and colleagues recognized different types of knowledge needed for teaching practice, the focus of the PCK model was pedagogical content knowledge. And when Ball and colleagues refine Shulman's model, uniting CK and PCK becomes relevant and meaningful to the teacher's work.

Within the CK domain, Ball and colleagues defined the following subdomains: *common content knowledge* (CCK), such as mathematical knowledge and practices used in areas other than their teaching; *specialized content knowledge* (SCK), which consists of knowledge of a mathematical nature that is only necessary for the task of teaching mathematics; and *horizon content knowledge* (HCK), which consists of knowledge of a mathematical nature that allows the teacher to be aware of how themes are related throughout the mathematics included in the curriculum. Within the PCK domain, Ball and colleagues brought Shulman's original idea as an "amalgam of content and pedagogy," being divided into the subdomains: *knowledge of content and students* (KCS), which is the knowledge that combines knowledge about students and knowledge about mathematics; *knowledge of content and teaching* (KCT), which is the knowledge that combines knowledge about teaching and knowledge about mathematics; and *knowledge of content and curriculum* (KCC), which consists of knowledge about how mathematics is organized throughout the curriculum.

Figure 1 presents the MKT model and its subdomains according to Ball et al. (2008):

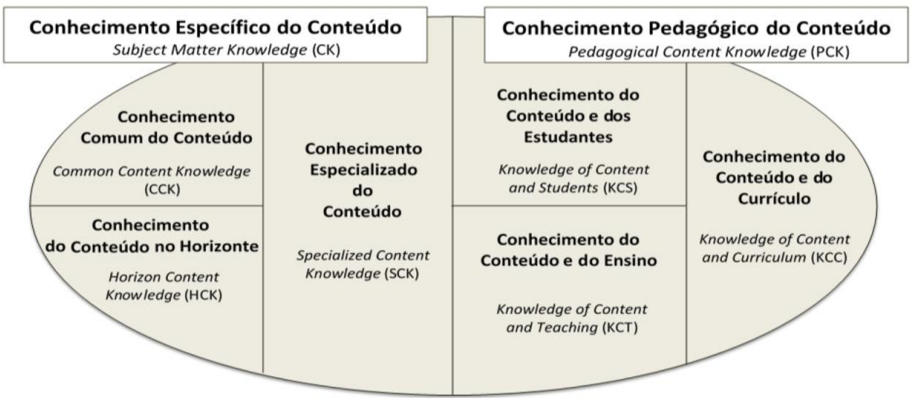

Figure 1.

Adapted from Ball et al. (2008)

Since its initial work, Deborah Ball's group has invested in validating the theoretical model, whether from a more precise definition (Ball et al., 2008) or empirical studies. In the next section, we discuss the second source of the *contrast corpus*, research published in Brazil that uses MKT.

How scientific research published in Brazil uses the MKT model

The following journals were included in the survey: *Boletim de Educação Matemática* (Bolema), *Educação Matemática Pesquisa* (EMP), *Acta Scientiae*, *Perspectivas da Educação Matemática*, *Revista Internacional de Pesquisa em Educação Matemática* (RIPEM), *Zetetiké* and *Educação Matemática em Revista* (EMR). These journals were selected because they are Qualis A1 or A2 in the Capes teaching area, indicating each one's relevance. As the purpose of the survey was to obtain an overview of how research uses the mathematical knowledge for teaching model by Ball et al. (2008), the choice of such journals was sufficient since these are significant spaces for the production and dissemination of theoretical perspectives in the area of mathematics education in Brazil.

We chose a time frame restricting the survey to articles published between 2019 and 2023. In the search field of the journals, we considered the following triggers: Knowledge AND Mathematics; Knowledge AND Professional AND Mathematics; Concept Study AND Mathematics; and Knowledge AND Mathematics. The articles obtained in the search were then evaluated according to the following criteria:

1) The articles must focus on teaching (and not exclusively on learning by basic school students);

2) The research theme or subjects must be middle school or high school teachers.

3) The articles must address, in some way, knowledge/knowings related to mathematical content for teaching (with their different names, depending on the theoretical framework).

From this analysis, we selected 63 articles that met the criteria above. Then, we refined again, adding a fourth criterion: 4) The article must address the MKT model. With this new criterion, 12 articles met our requirements, and are the focus of analysis in this section.

Table 1.

Pre-selected articles (Research data)

Article	Title	Objective
Armas (2020)	Evaluación de la Faceta Epistémica del Conocimiento Didáctico Matemático de Futuros Profesores de Matemáticas en el Desarrollo de una Clase Utilizando Funciones [Evaluation of the Epistemic Facet of the Mathematical Didactic Knowledge of Prospective Mathematics Teachers in the Development of a Class Using Functions]	Evaluating the epistemic facet of pre-service teachers' didactic-mathematical knowledge by transforming the representations of a function.
Amorim, Pietropaolo, Galvão, & Silva (2020)	Uma Sequência de Atividades para o Ensino de Equações Diofantinas: Possibilidade para Ampliar a Base de Conhecimentos de Futuros Professores de Matemática [A Sequence of Activities for Teaching Diophantine Equations: Possibility to Expand the Knowledge Base of Prospective Mathematics Teachers]	To analyze, under the assumptions established by Ball, Thames, and Phelps, the knowledge base for the teaching of diophantine equations of a group of prospective mathematics teachers participating in a formative process.
Wanderley & Souza (2020)	*Lesson Study* como Processo de Desenvolvimento Profissional de Professores de Matemática sobre o Conceito de Volume [Lesson Study as a Professional Development Process for Mathematics Teachers on the Concept of Volume]	To describe, analyze, and discuss the results of a participant observation investigation on volume and its teaching, as defended by the scientific community of mathematics education, mobilized in teacher education from the lesson study perspective.
Moreira & Ferreira (2021)	A Formação Matemática do Professor da Educação Básica: das Concepções Historicamente Dominantes às Possibilidades Alternativas Atuais [Mathematical	To tell an abbreviated history of the ideas that influenced the curriculum designs of mathematics teaching degree courses in Brazil since 1930 and those that influenced the

	Education for Elementary School Teachers: From Historically Dominant Conceptions to Current Alternative Possibilities]	alternative theoretical views developed throughout this period.
Silva, Silva, & Julio (2021)	Contribuições para a Formação Inicial de Professores de Matemática a partir de seu Envolvimento em um Projeto Extensionista Direcionado ao Público Idoso [Contributions to the Initial Education of Mathematics Teachers from their Involvement in an Extension Project Aimed at Older People]	Understand the contributions to mathematics teachers' initial education when involved in a university extension project aimed at older people.
Aguiar, Ponte, & Ribeiro (2021)	Conhecimento Matemático e Didático de Professores da Escola Básica acerca de Padrões e Regularidades em um Processo Formativo Ancorado na Prática [Mathematical and Didactic Knowledge of Elementary School Teachers about Patterns and Regularities in an Education Process Anchored in Practice]	To investigate the mathematical and didactic knowledge mobilized and deepened by mathematics teachers when collectively preparing, developing, and analyzing a class on patterns and regularities in a high school class.
Marins, Teixeira, & Savioli (2021)	Práticas de Ensino Exploratório de Matemática e a Mobilização/Desenvolvimento do Conhecimento Matemático para o Ensino por Participantes do PIBID [Mathematics Inquiry-Based Teaching Practices and the Mobilization/Development of Mathematical Knowledge for Teaching by PIBID Participants]	Investigate professional knowledge that is mobilized/developed by participants of the Programa Institucional de Bolsa de Iniciação à Docência (PIBID) [Institutional Teaching Initiation Grant Program] when inserted in an education process supported by the exploratory teaching perspective.
Biza, Kayali, Moustapha-Corrêa, Nardi, &	Afinando o Foco em Matemática: Desenho, Implementação e	Introduce some mathtasks and present the general principles of design and use of

Thoma (2021)	Avaliação de Atividades MathTASK para a Formação de Professores de Matemática [Sharpening the Focus on Mathematics: Designing, Implementing, and Evaluating MathTASK Activities for Mathematics Teacher Education]	mathtasks for research and teacher education purposes.
Martins & Curi (2022)	Conhecimentos e crenças manifestados por professores que ensinam Matemática e fazem uso de materiais manipuláveis em suas práticas [Knowledge and beliefs expressed by teachers who teach mathematics and use manipulative materials in their practices]	To highlight the knowledge and beliefs expressed by teachers who teach mathematics in the interdisciplinary cycle when using manipulative resources in their interactions with the documents and curriculum materials of the municipal schools of São Paulo.
Fonçatti & Morelatti (2023)	*Lesson Study* no Programa de Residência Pedagógica: desenvolvimento do conhecimento pedagógico do conteúdo de futuros professores de matemática [Lesson Study in the Pedagogical Residency Program: development of pedagogical content knowledge of prospective mathematics teachers]	Analyze the implications of using the lesson study, an education process context based on teachers' collaboration and reflection to promote the development of pedagogical knowledge of the content of prospective mathematics teachers, scholarship holders of the CAPES Pedagogical Residency Program linked to FCT/UNESP.
Gross, Trevisan, Araman, & Trevisolli (2023)	Uma Proposta para Elaboração e Análise de Tarefas de Aprendizagem Profissional [A Proposal for the Design and Analysis of Professional Learning Tasks]	Present a proposal for the development and analysis of professional learning tasks to support identifying mathematical and pedagogical knowledge that are revealed in the reflections that occur in a formative context.
Garden, Aguiar, & Ribeiro (2023)	Tarefas de aprendizagem profissional e o conhecimento matemático envolvendo a estrutura algébrica de Grupos: uma experiência na licenciatura em	Understanding how professional learning tasks are carried out in Algebra classes in the mathematics teaching degree.

	Matemática [Professional learning tasks and mathematical knowledge involving the algebraic structure of groups: an experience in the mathematics teaching degree]		

It is worth mentioning that we excluded from the selection (Table 1) articles that, despite being included in the search and mentioning works by researcher Deborah Ball and collaborators, do not present the MKT model[6]. Of the 12 works that remained in the survey (Table 1), there are different mentions of the MKT model. Moreira and Ferreira (2021) present a theoretical article discussing different approaches to mathematical knowledge specific to school education, including MKT. Biza et al. (2021) mention MKT, especially horizon content knowledge (HCK), but do not delve into this topic or develop data analysis from this perspective, focusing on the so-called *mathtasks* (mathematical tasks). Likewise, Aguiar et al. (2021) also mention the MKT model but do not use it in the analyses, using instead the notions of didactic knowledge and mathematical knowledge proposed by Ponte (1999). Therefore, we constructed Table 2 with the remaining nine articles from Table 1.

Table 2.

Selected articles (research data)

Article	Context	Mathematical content covered	MKT subdomains considered/highlighted in data analysis
Armas (2020)	Mathematics teachers' initial education	Function	Common content knowledge, horizon content knowledge, and specialized content knowledge.
Amorim, Pietropaolo, Galvão, & Silva (2020)	Mathematics teachers' initial education	Diophantine equations	Common content knowledge, specialized content knowledge, horizon content knowledge; knowledge of content and students; knowledge of content and teaching; and knowledge of content and curriculum

6 For example, Molina and Samper's (2019) work.

Wanderley & Souza (2020)	Continuing education for mathematics teachers	Volume	Specialized content knowledge; knowledge of content and students; knowledge of content and curriculum; knowledge of content and teaching.
Silva, Silva, & Julio (2021)	Mathematics teachers' initial education	It did not have specific content	Knowledge of content and students, knowledge of content and teaching.
Marins, Teixeira, & Savioli (2021)	Mathematics teachers' initial education	Probability	Specialized content knowledge; knowledge of content and students; and knowledge of content and teaching.
Martins & Curi (2022)	Continuing education of teachers who teach mathematics	Numbers (for 4th grade), Geometry (for 5th grade), Numbers (for 6th grade)	Knowledge of content and teaching.
Fonçatti & Morelatti (2023)	Mathematics teachers' initial education	Linear equations with two unknowns; the study of functions	Knowledge of content and teaching.
Gross, Trevisan, Araman, & Trevisolli (2023)	Continuing education of teachers who teach mathematics	Functional thinking	Knowledge of content and curriculum; knowledge of content and teaching.
Garden, Aguiar, & Ribeiro (2023)	Mathematics teachers' initial education	Rational numbers, Matrices, and Functions	Common content knowledge; specialized content knowledge.

Table 2 shows that six studies were conducted in the context of mathematics teachers' initial education and three in continuing education. No research was carried out outside the context of teacher education, for example, in the daily routine of a classroom, without being linked to a formal formative space. Among the formative approaches, it is worth noting that three articles (Wanderley & Souza, 2020; Martins & Curi, 2022; and Morelatti, 2023) used lesson study as a collaborative process of professional development, while two articles (Gross et al., 2023; Jardim et al., 2023) used professional learning tasks as artifacts for mathematics teachers' education. There are various topics regarding the mathematical content covered, with functions being the most explored topic. In Silva et al. (2021), because their research was carried out in the context of an extension project aimed at the elderly public, no specific mathematical content was being investigated[7].

Regarding the MKT subdomains, Table 2 only considered those subdomains that (i) were explicitly selected by the authors as those to be considered for data analysis - for example, Armas (2020) used only the subdomains related to specific content knowledge, namely, common content knowledge (CCK), horizon content knowledge (HCK), and specialized content knowledge (SCK); or (ii) were explicitly mentioned8 during the analyses carried out by the researchers - for example, Marins, Teixeira, and Savioli (2021) comment that the subdomains common content knowledge (CCK), horizon content knowledge (HCK), and knowledge of the content and curriculum (KCC) were not identified in the data analyses. Two subdomains related to the pedagogical content knowledge domain (Shulman, 1986; 1987) were the most highlighted in the research, especially knowledge of content and teaching. The horizon content knowledge subdomain was the least mentioned in the analysis of the articles. The only work that addressed all subdomains in its analyses was Amorim et al.'s (2020). However, the authors mention that some subdomains may be more prominent, namely, specialized content knowledge, knowledge of content and teaching, and knowledge of content and curriculum.

This finding that most (or perhaps all) studies do not cover all MKT subdomains is in line with the result presented by Patrono and Ferreira (2021). The authors conducted a bibliographic survey considering only dissertations and theses and concluded that none indicated "[...] the development of mathematical knowledge for teaching a mathematical topic in a way that encompassed all MKT categories, i.e., that provided discussions of all the knowledge necessary to carry out mathematics teaching" (Patrono & Ferreira, 2021, p. 18).

Perhaps because the nine investigations considered here were carried out in the context of formal teacher education spaces, it is clear that the main conclusions drawn in the research

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7 According to the authors, "Some activities worked with investigative processes linked to the history of mathematics, others related mathematics to art, there were those that worked with mathematics used by older people in their daily lives, as well as those that focused on the production of materials, with the aim of discussing mathematical content" (Silva et al., 2021, p. 772).

8 This choice was made because some articles present the MKT model in full in the theoretical basis section, but do not always explain its subdomains in the data analysis. Therefore, it was not considered sufficient for the article to mention the subdomains in the theoretical basis, but, on the contrary, that they were evidenced in the data analyzed.

refer to assessing whether the formative process provided opportunities for the development (or advances or mobilization or modification) of teachers' mathematical knowledge.

For example, Amorim et al. (2020, p. 224, emphasis added) conclude that "[...] prospective mathematics teachers *expanded their knowledge base* [...], especially in relation to specialized content knowledge, knowledge of content and teaching, and knowledge of content and curriculum." Wanderley and Souza (2020, p. 1, our emphasis) point out that "The main results showed that teachers *expanded and deepened their knowledge* of volume, previously limited to formulas [...]". Fonçatti and Morelatti (2023, p. 16, our emphasis) state that "[...] this formative context appears as a possible path to provide learning necessary for teaching and *develop pedagogical content knowledge* still in initial education [...]". Marins et al. (2021, p. 340, emphasis added) conclude that "We can infer that participation in this formative process promoted a *mobilization/development of SCK, KCS, and KCT* according to the proposal by Ball et al. (2008). Silva et al. (2021, p. 784, our emphasis) concluded that "[...] the involvement of prospective teachers in the extension project and their pedagogical work with the older public *contributed to their pedagogical content knowledge* [...]." Jardim et al. (2023) conjecture that prospective teachers

> [...] presented prior knowledge about procedures and concepts (CCK), but as they became involved with TAP and were questioned in the different phases of exploratory teaching (Canavarro et al., 2012), *mobilized content knowledge* related to school, the SRCK (Dreher et al., 2018) [...] In the end, their *knowledge is modified*, and these can be used for teaching purposes, which would be linked to the SCK. (Jardim et al., 2023, p. 18)

Armas's research (2020) presented conclusions slightly differently than the others since it used the MKT subdomains to indicate weaknesses in the teacher's mathematical knowledge. According to the author,

> The state of development of the epistemic facet of the didactic-mathematical knowledge of this group of pre-service teachers on functions highlights some formative deficiencies: while common content knowledge and horizon content knowledge present a good state of development, their specialized content knowledge presents serious limitations. (Armas, 2020, p. 127)

Finally, one last observation that can be made from the works analyzed concerns the gaps perceived by the authors in the MKT model. A perceived gap was mentioned in an article by Martins and Curi (2022), which discussed mathematical knowledge and the beliefs of the teachers who teach mathematics. To address beliefs, the authors seek to base themselves on other works –for example, Gómez-Chacón (2003) and Gómez-Chacón et al. (2006)– because the MKT model does not address issues regarding the teacher's belief about mathematics and its teaching and learning processes.

Another perceived gap came from Jardim et al.'s (2023, p. 3) work. The authors rely on Speer et al. (2015) to state that "[...] The six domains of MKT do not clarify the relationship between academic and school mathematics since the studies that originated these domains observed teachers working in contexts corresponding to the Brazilian initial and final years of elementary school." In other words, the MKT subdomains were proposed based on studies that did not include, for example, mathematics taught in high school and possible relationships with academic mathematics. For this reason, Jardim et al. (2023) complement their theoretical foundation with the teacher's mathematical knowledge, *school-related content knowledge* (SRCK) presented by Dreher et al. (2018). Such gaps will be deepened in the next section. The next section approaches the third source of *contrast corpus,* i.e., some models and types of knowledge that came after the MKT.

Some models and types of knowledge after the MKT

Most research examining teachers' content knowledge and its role in practice is conducted from a cognitive theoretical perspective, in which knowledge is considered one of several factors that influence decision-making in the classroom. Considering that the MKT unites knowledge, teaching practice, and student learning, divided into two domains, Speer et al. (2015) explored how the differences between contexts equivalent to Brazilian elementary and high school and higher education can lead to a different characterization of the MKT. These authors pointed out that teaching tasks[9] in elementary and high school are similar, suggesting that cognitive processes will function correspondingly. However, teaching approaches vary across school years, and teachers' practices may be more diverse in later school years than in elementary and high school. In high school, some types of mathematical knowledge are more relevant than others, and precise definitions are necessary, which is reflected in teachers' education aimed at these levels, such as the need to justify some procedures and use some demonstrations appropriate to the school grade.

For middle and high school, research has assumed that MKT is mobilized likewise, without questioning the differences between CCK and SCK at the different levels of education at stake. Most studies focused on high school teachers seek to identify elements of knowledge

[9]The task is understood as a segment of classroom activities dedicated to the development of a particular mathematical idea (Stein & Smith, 2009).

when unzipping the content of a mathematical idea or task. All of this is based on the knowledge possessed by those who do the unzip, usually with substantial knowledge – knowledge that emerges from mathematics. An example is the interpretation of a student's answer based on CCK-related content, a teacher's necessary skill.

At the university level, Speer et al. (2015) noted that students developed ideas and ways of thinking unfamiliar to the mathematician, who needed to do work to make sense of and determine the validity of students' contributions (characterized as SCK). This is exemplified by the mathematician's work in making sense of mathematical ideas and reasoning presented by someone else and determining whether that reasoning is correct (research work), which can be characterized as SCK.

The distinction between CCK and SCK for mathematicians does not seem explicit. Speer et al. (2015) suggest analyzing these distinctions when addressing students' reasoning for decision-making and the mathematical knowledge involved. To this end, there is a need for more studies on the types of knowledge involved in high school and undergraduate studies, similar to what the group of Deborah Ball performed. Such studies would help build theory, developing practical instruments to assess how the various opportunities teachers provide favor student learning. Generalizing the current theoretical framework may lead to the loss of a better understanding of the nature of teaching knowledge, which, in general, improves teacher learning to support their students' learning.

Speer et al. (2015) also state that this type of knowledge for teaching can be incorporated into existing courses that are based on the wisdom of practice and not just on theoretical or empirical results. The ways mathematics teachers use their knowledge in secondary education may have implications for helping teachers and how it can be assessed.

The distinction between the CCK and SCK, defined by Ball et al. (2008), may not be sufficient to address the knowledge mobilized by mathematics teachers in the final years of elementary school and high school. Thus, based on the discussion presented by Speer et al. (2015) about the differences between the knowledge mobilized by mathematics teachers at different levels of education, Dreher et al. (2018) point out that the MKT lacks focus on a dimension that goes beyond the distinction between common and specialized knowldge, concentrating on the non-trivial relationship between academic mathematics and school.

In this sense, *school-related content knowledge* (SRCK), proposed by Dreher et al. (2018), is knowledge of the mathematical content specific to the mathematics teacher profession, considering the non-trivial interrelations between academic mathematics and school mathematics, which is not explicit in the MKT model. The type of knowledge addressed by the

SRCK does not mix with pedagogical knowledge but shares knowledge with mathematicians when considering academic content knowledge, going beyond the knowledge of the content of school mathematics. Although the SRCK relies on the framework of academic mathematics, it does not have a systematic structure of its own. This knowledge encompasses three facets: the curriculum structure, the fundamental mathematical ideas related to school, and the relationships between academic and school mathematics, considering its two meanings.

Although the works of Speer et al. (2015) and Dreher et al. (2018) do not present a model of knowledge, their ideas are relevant for deepening the understanding of mathematics for teaching that is mobilized at different levels of education, especially when the focus is on the approximation between academic mathematics and school mathematics, knowledge necessary for the practice of the mathematics teacher (Moreira & David, 2008).

One model we mention is the *mathematics teachers' specialized knowledge* (MTSK). According to Flores-Medrano et al. (2014), MTSK emerges as a response to the difficulties detected in the MKT and is based on the potential of this and other models that characterize the mathematics teacher's knowledge.

One of these difficulties detected refers precisely to CCK and SCK (also passing through HCK). For Flores-Medrano et al. (2014), the MKT recognizes at least two ways teachers know the mathematics they will teach: CCK and SCK (HCK would be, according to the authors, a third way). However, the degree of association with the knowledge they teach students is unclear. Thus, the authors point out some problems with this way of defining subdomains through complementarity between them. For analytical purposes, this definition requires differentiation criteria whose degree of subjectivity may lead to considering non-equivalent aspects in different countries, school systems, or others (Flores-Medrano et al., 2014).

The MTSK considers the specialized nature of the teacher's knowledge integrally in all its subdomains and avoids alluding to external referents, such as knowledge from other professions (Flores-Medrano et al., 2014). Thus, as an alternative to describing what and how the mathematics teacher knows the topics he/she will teach, overcoming the difficulties detected in the MKT model on CCK and SCK, the authors of MTSK consider the subdomain *knowledge of mathematical topics* (KoT), which involves knowing mathematical content and its meanings in a well-founded way. It integrates the content that the student is expected to learn and allows the consideration of knowledge with a level of depth greater than that expected for students (Flores-Medrano et al., 2014).

Another aspect differentiating the MTSK model from the MKT model concerns the beliefs. While the MKT does not mention the role of teachers' beliefs, the MTSK model considers the impact of these beliefs on teaching practice. Thus, the MTSK considers beliefs as a component that allows the researcher to investigate the relationship between the conceptions that teachers have regarding mathematics and the processes of teaching and learning mathematics, and teachers' mathematical knowledge, the one he/she possesses and the one he/she uses in his/her practice (Climent et al. 2014).

We can observe in the works cited in this section that MKT, despite being a precursor on the knowledge necessary for mathematics teaching, was not the only model, just as it does not present enough subdomains to characterize what the teacher mobilizes when teaching. In this sense, the model was never presented as finalized in its proposals, and by examining how researchers have used them, we can advance in understanding the complexity of mathematical knowledge for teaching, starting from beliefs, connections between academic and school mathematics in favor of teaching, and the different relationships between the subdomains predicted by the MKT model presented in 2008. In the next section, we discuss the last source of the *contrast corpus*, i.e., recent studies developed by researcher Deborah Ball's group.

Recent studies by emerging researchers from Deborah's group

Hereafter, we follow the evolution of the MKT model through the work of emerging researchers from Deborah Ball's group. We selected articles whose authors were related to this research group and whose work address the MKT model in some way, allowing us to tell the story of the evolution of the MKT model by its original authors.

Even before Ball et al. (2008) published the complete MKT model, the University of Michigan group was already investing in empirical tests to validate the model, which consisted of creating multiple-choice items and applying these items in large-scale tests. Hill et al. (2004) described the development of these items, and the correlation of mathematical knowledge for teaching, as proposed in the MKT model, was validated through tests described in Hill et al. (2005). The methods described in both works are robust and contributed to the reliability and worldwide dissemination of the MKT model.

Recently, work on test development has made some progress in corroborating the model further and allowing an understanding of its limitations. Howell et al. (2016) show that the principles that support the creation of items for the initial years of elementary school can be extended to middle and high school in the sense that the answers align with the knowledge of the teachers that one wants to measure and that such knowledge is different from common mathematical knowledge. Charalambous et al. (2020) found that MKT correlates with students' mathematical performance but could not discern between the CCK, SCK, and KCT categories.

On the other hand, they acknowledge that this may be a limitation of the MKT model, suggesting that the domains are not, in fact, different. The authors argue that there is a weakness in the research method since multiple-choice tests may not be the best methodology to capture teachers' knowledge in practice, being more appropriate for capturing static aspects of the teacher's action.

This tendency to investigate the more static aspects of practice was also perceived in mathematics education as a whole, as per the survey presented by Thanheiser et al. (2014). They investigated 112 surveys conducted between 1978 and 2012 and observed that most studies on the content knowledge of prospective teachers focused on "static studies of knowledge" rather than on how that knowledge develops. Thus, Thanheiser et al. (2014) suggested that further research should focus on characterizing prospective teachers' content knowledge and investigating how this knowledge develops.

In relation to MKT, one of the challenges mentioned is the difficulty of progress due to the lack of a visibly understood and practical methodology for studying and developing mathematical knowledge for teaching (Hoover et al. 2016). The authors also reflect on two methodological approaches used to study the nature and composition of mathematical knowledge for teaching: interview studies and observation of practice.

Hoover et al. (2016) state that since the 1980s and 1990s, studies began to use interviews with teachers to investigate their knowledge. However, the interviews had two limitations: first, they tended to focus on identifying gaps in teachers' mathematical knowledge rather than elucidating the mathematical knowledge requirements for teaching, and second, although the interviews addressed the importance of teacher expertise, they did not provide additional high-quality information about that expertise (Hoover et al. 2016). Therefore, for Hoover et al. (2016), the weaknesses of these interview-based studies lay in the fact that they focused the conversation on teachers' lack of knowledge while providing little information on how to correct those deficiencies.

Regarding observational studies of practice, the authors mention that due to the perceived limitations in the approach via interviews with teachers, the research group at the University of Michigan developed a practice-based approach to the study of instructional video records. However, this is a time-consuming and expensive approach, requiring conceptualizing the teaching work and the mathematical demands of that work simultaneously (Hoover et al., 2016). Based on these limitations of the mentioned methods (interviews and observations), Hoover et al. (2016) suggest directions for potential innovations. They discuss some of the ways they began to explore methodological approaches to studying mathematical knowledge for teaching. One approach the authors consider promising is to use places where deliberations about teaching are happening as places where one can productively research the work of teaching and its mathematical demands. For example, Hoover et al. (2016) cite Kim (2016), who, to investigate the work involved in providing feedback in writing to students, asks interviewees to justify the feedback. Rather than videotaping classroom instruction and analyzing the mathematical demands of teaching, Kim (2016) analyzes these demands as they unfold in a narrow slice of the mathematical work of teaching. It means, therefore, using places of professional deliberation on teaching as places of research to study teaching.

Regarding advances toward developing ways of measuring mathematical knowledge for teaching, Selling et al. (2016) present the *mathematical work of teaching* (MWT), a framework built based on mathematical teaching tasks (Ball et al., 2008), whose main objective is to inform and assist the development of instruments to evaluate and measure MKT. The framework focuses on mathematical work for teaching, going beyond CCK, but without delving into the teacher's pedagogical choices (Selling et al., 2016). That is, the framework is intended more for the CK domain than for the PCK domain. Selling et al. (2016) recognize that the distinction between SCK and CCK may not be well defined. Therefore, when constructing the framework, they took into account the difficulties that MKT reviewers could encounter not only in understanding the differences between these two subdomains but also in identifying and instigating teachers' SCK. In this way, discerning the domains of the MKT model is also difficult within the framework presented by Selling et al. (2016).

Thus, based on the work of emerging researchers from Deborah Ball's research group, it is clear that empirical tests were critical to validate the model and continue to be so, but are ineffective in discerning between the model's subdomains and capturing dynamic moments of practice. Some other methodologies have been suggested to overcome these limitations, such as interviews, video recording analysis, and professional deliberation venues, all with advantages and disadvantages. Finally, from this group, there were attempts to refine the model, with advantages and disadvantages. In the next section, we will present a critical analysis, comparing the four sources that comprise the *contrast corpus*.

Critical analysis: telling a story about the MKT model based on research in Brazil

Based on a theoretical and speculative perspective, this research aimed to critically analyze the MKT model, focusing on how research in Brazil has been using it. MKT is a moving model, which was and continues to be theoretically constituted based on research on teaching practice. It arose from the need to specify, for the context of mathematics, the domains of fundamental knowledge for teaching proposed by Shulman (1986; 1987), encouraging the proposition of other models (such as the already mentioned SRCK and MTSK, but there are others). It remains under constant review or evaluation, in terms of the division and characterization of its subdomains and in methodological aspects, to make its nature effective as a model that involves the mathematical knowledge necessary for the teacher to exercise his/her role of teaching mathematics.

The research that appeared in the survey and assumes MKT as a theoretical reference does not seem to be part of evaluating the model and verifying its potential or weaknesses in the specific context in which the investigation is carried out. There is a predominance of research that aims to understand whether the proposed formative processes provided opportunities for the teacher's mathematical knowledge's development (or advances, mobilization, or modification). However, little is said about the nature of this knowledge or how it developed. Often, conclusions made in research are based on the identification of MKT subdomains, especially those related to PCK. In other words, much research seems to focus on what Thanheiser et al. (2014) have called "static studies of knowledge" rather than on how that knowledge develops.

This does not mean that the research identified in the survey does not reveal weaknesses in the MKT model. Some research has used other theoretical frameworks to overcome gaps left by MKT, such as, for example, the issue of beliefs (about mathematics or its teaching and learning processes), the absence of a discussion about the relationship between academic and school mathematics, and the perception that some subdomains may not be well established, especially regarding the distinction between CCK and SCK. As mentioned above, such findings are being bypassed in articles through other theoretical models of the teacher's mathematical knowledge.

However, it is worth noting that the concerns presented in the most recent work by researcher Deborah Ball and her collaborators –for example, Hoover et al. (2016), Selling et al. (2016), and Charalambous et al. (2020)– were not noticed in the works carried out in Brazil. Thus, on the one hand, the many publications on MKT in Brazil (as mentioned in the introduction of this article) suggest that the topic may somehow be exhausted in the country; on the other hand, it is clear that there is progress to be made in order to understand better the nature and development of teachers' MKT in the Brazilian context.

Since the MKT is a model in motion connecting with practice and, therefore, developing from research in different contexts, research in the Brazilian context must go beyond the mere use of MKT subdomains as assumed *a priori* categories. These categories must be investigated, questioned, expanded, and, eventually, reformulated, considering the specific context. An example of this characteristic of the MKT being a dynamic model can be seen in Ball et al. (2008) when they present the well-known figure that illustrates the subdomains (here translated in Figure 1). The authors call the figure a diagram to represent the *current hypotheses* regarding the teacher's mathematical knowledge, and it is complemented by informing that the KCC and HCK subdomains are provisionally allocated in the PCK and CK domains, respectively. This provisional nature opens space for other similar models to be proposed and for MKT to be re-discussed.

In this sense, questions and future research possibilities arise to propose going beyond taking MKT subdomains as *a priori* categories to refine the model for the Brazilian context. Considering the initial education of teachers in Brazil, how does the development of MKT occur? Considering the reality of schools in the Brazilian context, how does MKT develop? How does MKT develop in teaching practice, considering the different levels of education? How can we promote the development of SCK in teachers who work in the initial years of elementary school and who generally have a degree in pedagogy? To what extent do sociocultural aspects interfere in the development of the MKT and, consequently, in the characterization of subdomains? How do we propose instruments that help evaluate the MKT considering the Brazilian context?

These questions emerge from the interpretations and arguments based on the corpus of this work, indicating future research possibilities, seeking to include the Brazilian context in the set of investigations on the MKT model, and preventing Brazilian research from being just a reproduction of theoretical references developed in such distinct sociocultural contexts.

For example, considering that the initial education of mathematics teachers in the United States of America (the country where the MKT model originated) differs from the formative processes in Brazil, investigating how school mathematics is approached in teacher education in both contexts –seeking to understand how this impacts the teacher's mathematical knowledge before starting their career or during the first years of their career– can deepen discussions about how the teacher develops the subdomains related to specific content knowledge.

Thus, the questions listed above, suggesting investigations in the Brazilian context, can contribute to evaluating the model and verifying its strengths or weaknesses, bringing new elements to discussions on each subdomain of the MKT.

Dedication

We dedicate this work to Marlova Estela Caldatto (UTFPR), who passed away on November 28, 2022, before the research was completed. Her contributions made a difference. We thank her for all her dedication to mathematics education!

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