ISSN 1983-3156
Qualis A1
http://dx.doi.org/10.23925/1983-3156.2024v26i3p280-303

Epistemological model of reference in anthropological theory of the didactic: hypothesis and application in didactic problems of differential and integral calculus

Modelo epistemológico de referencia en teoría antropológica de lo didáctico: Hipótesis y aplicación en problemas didácticos de cálculo diferencial e integral

Modèle épistémologique de référence en théorie anthropologique du didactique : hypothèse et application aux problèmes didactiques de calcul différentiel et intégral

Modelo epistemológico de referência na teoria antropológica do didático: hipótese e aplicação em problemas didáticos de cálculo diferencial e integral

Renato da Silva Ignácio[1]
Universidade Federal de Campina Grande (UFCG)
Doutor em Educação Matemática
https://orcid.org/0000-0003-0448-3241

Valdir Bezerra dos Santos Júnior[2]
Universidade Federal de Pernambuco (UFPE)
Doutor em Educação Matemática
https://orcid.org/0000-0002-3869-201X

Marlene Alves Dias[3]
Instituto Federal de Educação, Ciência e Tecnologia do Estado do Ceará
Doutora em Matemática-Didática da Matemática
https://orcid.org/0000-0001-9168-9066

Abstract

This article is an excerpt from doctoral research that investigated the limits and possibilities of the didactic methodology study and research path (SRP), of the anthropological theory of the didactic (ATD), as a teaching alternative for basic education in Brazil. We started the research by considering the following question: How can we find the shortest possible route connecting an origin (O) and a destination (D)? This problem led us to consider the development of what in ATD is called the reference epistemological model (REM). The REM model must be explained whenever you want to formulate an authentic didactic problem or study-generating question. This work initially presents a comparison of the modeling between ATD and other theoretical currents and then an example of an epistemological reference model based on the problem of choosing the shortest access path from one point to another under certain conditions. We consider the objective of this article to relate, through the notion of praxeology, the

[1] renatosignacio@gmail.com
[2] valdir.bezerra@gmail.com
[3] maralvesdias@gmail.com

possibility of modeling in the light of ATD. The comparative study shows us that modeling within the scope of ATD is not an object to be taught nor a means to learn and teach certain mathematical concepts. The main characteristic of modeling in ATD consists of the elaboration and experimental contrast of epistemological mathematics models to approach didactic problems as a provisional hypothesis, which can be modified while developing an SRP.

Keywords: Anthropological theory of the didactic, Modeling, Praxeology, Path of study and research, Epistemological model of reference

Resumen

Este artículo es un extracto de una investigación doctoral que investigó los límites y posibilidades de la metodología didáctica recorrido de estudio e investigación (REI), de la teoría antropológica de lo didáctico (TAD), como alternativa de enseñanza para la educación básica en Brasil. Comenzamos la investigación considerando la pregunta: ¿Cómo encontrar la ruta más corta posible que conecte un origen (O) y un destino (D)? Este problema nos llevó a considerar el desarrollo de lo que en TAD se denomina modelo epistemológico de referencia (MER). El MER es el modelo que hay que explicar cada vez que se quiere formular un auténtico problema didáctico o una pregunta generadora de estudio. Este trabajo presenta inicialmente una comparación de la modelización entre la TAD y otras corrientes teóricas, y luego un ejemplo de modelo epistemológico de referencia basado en el problema de elegir el camino de acceso más corto de un punto a otro bajo ciertas condiciones. Consideramos el objetivo de este artículo relacionar, a través de la noción de praxeología, la posibilidad de modelar a la luz de la TAD. El estudio comparativo nos muestra que la modelización en el ámbito de la TAD no es un objeto a enseñar ni un medio para aprender y enseñar determinados conceptos matemáticos. La principal característica de la modelación en la TAD consiste en la elaboración y contrastación experimental de modelos matemáticos epistemológicos con el propósito de abordar problemas didácticos como una hipótesis provisional, que puede ser modificada durante el proceso de desarrollo de un REI.

Palabras clave: Teoría antropológica de lo didáctico, Modelización, Praxeología, Recorrido de estudio e investigación, Modelo epistemológico de referencia.

Résumé

Cet article est un extrait d'une recherche doctorale qui a étudié les limites et les possibilités de la méthodologie didactique Parcours d'Etude et de Recherche (PER), de la Théorie Anthropologique de la Didactique (TAD), comme alternative pédagogique pour l'éducation de base au Brésil. Nous avons commencé la recherche en considérant la question : Comment trouver l'itinéraire le plus court possible reliant une origine (O) et une destination (D) ? Cette problématique nous a amené à envisager le développement de ce que l'on appelle en TAD le modèle épistémologique de référence (MER). Le MER est le modèle qu'il faut expliquer chaque fois que l'on souhaite formuler un véritable problème pédagogique ou une question génératrice d'étude. Ce travail présente dans un premier temps une comparaison de la modélisation entre TAD et d'autres courants théoriques, puis un exemple de modèle épistémologique de référence basé sur la problématique du choix du chemin d'accès le plus court d'un point à un autre sous certaines conditions. Nous considérons que l'objectif de cet article est de relier, à travers la notion de praxéologie, la possibilité de modélisation à la lumière du TAD. L'étude comparative nous montre que la modélisation dans le cadre de la Théorie Anthropologique de la Didactique n'est pas un objet à enseigner ni un moyen d'apprendre et d'enseigner certains concepts mathématiques. La principale caractéristique de la modélisation en TAD consiste en l'élaboration et le contraste expérimental de modèles épistémologiques des mathématiques dans le but d'aborder les problèmes didactiques comme une hypothèse provisoire pouvant être modifiée au cours du processus d'élaboration d'un PER.

Mots-clés : Théorie anthropologique du didactique, Modélisation, Praxéologie, Parcours d'étude et de recherche, Modèle épistémologique de référence.

Resumo

Este artigo é um recorte de uma pesquisa de doutorado que investigou os limites e possibilidades da metodologia didática percurso de estudo e pesquisa (PEP), da teoria antropológica do didático (TAD), como alternativa de ensino para a educação básica do Brasil. Iniciamos a pesquisa considerando a pergunta: Como encontrar o menor percurso possível interligando uma origem (O) e um destino (D)? Este problema nos conduziu a considerar a elaboração do que na TAD se denomina modelo epistemológico de referência (MER). O MER é o modelo que precisa ser explicitado toda vez que se deseja formular um autêntico problema didático ou questão geradora de estudo. Este trabalho apresenta, inicialmente um comparativo da modelagem entre a TAD e outras correntes teóricas, e em seguida um exemplo de modelo epistemológico de referência a partir do problema da escolha do caminho mais curto de acesso de um ponto a outro sobre determinadas condições. Consideramos como objetivo deste artigo relacionar, por meio da noção de praxeologia, a possibilidade de uma modelagem sob a luz da TAD. O estudo comparativo nos mostra que a modelagem no escopo da TAD não é um objeto a ser ensinado e nem um meio para aprender e ensinar certos conceitos matemáticos. A principal característica da modelagem na TAD consiste na elaboração e contraste experimental de modelos epistemológicos de matemática com a finalidade de abordar problemas didáticos como uma hipótese provisória, que pode ser modificada durante o processo de desenvolvimento de um PEP.

Palavras-chave: Teoria antropológica do didático, Modelagem, Praxeologia, Percurso de estudo e pesquisa, Modelo epistemológico de referência.

Reference epistemological model in ATD: hypothesis and application in didactic problems of differential and integral calculus

This article is an excerpt from a doctoral research on the limits and possibilities of the teaching methodology of the anthropological theory of the didactic (ATD) called study and research path (SRP) as a teaching alternative for basic education in Brazil. We began the research by considering: How can we find the shortest route connecting an origin (O) and a destination (D)?

This work will present alternative answers to this question. At this point, we will preliminary analyze the question and state that the problem of choosing the shortest path from one point to another is a daily life example of a situation.

The rationalization of problems of this nature arises when humans decide to replace their naive view of reality with a critical stance. Rational thinking requires appropriate language to express a fact or situation through scientific analysis. In this way, the evolution of human thought manifests through its ability to generate new representations as conceptual references that generate solutions to societal problems.

Therefore, when trying to reflect, explain, understand, or modify a portion of reality, the usual process of scientific analysis is formalized through an artificial process. This formalization selects, in the system under study, arguments or parameters considered essential, in a process we call a model.

The construction of models has accompanied our history since humans have sought to understand phenomena around them and describe them clearly since the most primitive times.

Artifacts and machines are examples of models generated by the thoughts of their inventors. Regarding the definition of praxeology, we refer to Chevallard (2011, p. 1): "[...] Every human activity consists of carrying out a task t of a certain type T, through a technique τ, justified by a technology θ that allows at the same time to think about it, even to produce it, and that is justifiable by a theory Θ." Thus, these examples are praxeologies built to meet people's needs since ancient times and constitute models created by technological development. Ancient peoples created the *shadoof*, or *picota*, as it is known in Brazil, to manage water for field irrigation. This simple machine used to extract water from wells demonstrates important mathematical and physical principles such as the lever.

The need to understand real-world processes[4] from the interaction between multiple components and predict what might happen has been a powerful motivation for developing knowledge in contemporary society.

We can also cite neuroscience studies, whose central interest is to describe the cognitive system of living beings, particularly that of humans. According to neuroscience, the brain acts based on the organization of neuronal structures; these configurations explain processes such as perception. Inputs to the sensory system activate existing configurations, comparing features and finding similarities while at the same time acting to modify existing structures, consolidating their configuration, or generating changes in the structure.

The result is a mental model representing each individual's world at specific moments. These mental models are systems that make inferences about possible environmental changes and their consequences for the individual. The mental model may have a replica of itself in physical symbol systems, such as a diagram or an algebraic formula (Holland et al., 1986). In this line of thought, the study on modeling competence links the creation of two models: a mental one and an external one, also known as the conceptual model.

The authors highlight that projecting or transforming the mental model into some form of language to make it visible is inherent to human beings. They further argue that the sciences are constructed under systems of conceptual models and fulfill a communicative function with which collaborative actions are intertwined; these can be oriented to act on the environment and are the basis for regulating the mental models of communication actors.

Brazilian neuroscientist Nicolelis (2011) highlights the human brain's ability to model, design, and simulate reality scenarios. In an interview, he stated that the brain is a shaper, a reality sculptor.

This ability of the human brain to model reality can be observed in contemporary science, which has also resorted more frequently to the construction of mathematical models to solve complex problems. Computer science and artificial intelligence are two examples of fields of research that have been dedicated to solving problems of general interest and that, through the construction of models, have presented solutions to problematic situations that are sensitive to society.

The examples cited above show that the increasing use of mathematical models causes a pendulum shift in mathematics, transferring the emphasis from a type of thinking resulting from complete information or deductive (monotonic) thinking to inductive (non-monotonic)

---

[4] Real world, in a broad sense, from everyday reality to other areas of knowledge.

thinking, a type of thinking characterized by having incomplete information about situations and more aligned with the dynamics of daily life.

The ability to apply mathematics in other areas requires taking a problem defined in some practical situation, transforming it into a mathematical model, and seeking a *solution* that can be reinterpreted in terms of the original situation.

Borroneo (2006) and Maaß (2006) describe a modeling cycle as the process that transitions from the real world to the mathematical world. In the modeling cycle, real problems are mathematized by transferring objects, data, and information from reality to the world of mathematics. In this way, a mathematical model is obtained, and a mathematical solution is sought with it in hand. Then, it can be interpreted and validated in the real world (where the problem arose), giving rise to a real solution.

In Brazil, several authors conceive mathematical modeling differently and use the terms real situation, problematization, and investigation to define it. We can initially cite D'Ambrosio (2002), who considers that the origins of the central ideas of mathematics result from a process that seeks to understand and explain facts and phenomena observed in reality. The development of these ideas and their intellectual organization occur from the elaboration of representations of reality, and for this author, mathematical modeling is mathematics par excellence.

Below, we present other definitions of mathematical modeling: a) Modeling transforms real-world problems into mathematical problems to solve them, interpreting their solutions in the language of the real world (Bassanezi, 2002); b) Mathematical modeling is the process of obtaining a model, and a mathematical model is a set of symbols and mathematical relationships that seeks to translate in some way a phenomenon in question or problem of a real situation (Biembengut & Hein, 2003); c) Mathematical modeling is an alternative strategy for teaching mathematics in an environment and represents a perspective that includes socio-school experiences, construction, and consolidation of knowledge and significant learning (Scheffer, 1999).

In an approximate classification, we could say that there are three possibilities for interpreting mathematical modeling: as a learning object, as a teaching methodology, and as a research method.

Table 1, presented by Villa-Ochoa (2007), summarizes some aspects that differentiate the mathematical modeling process as a scientific activity and a tool for constructing mathematical concepts in the classroom.

Table 1.

Aspects of the modeling process (Villa-Ochoa, 2007, p. 52)

Criteria	As a scientific activity	As a teaching tool
Purpose of the model	The model is constructed based on the analysis of some situations, through which it seeks to explain phenomena and solve problems.	The model is designed to construct a mathematical concept endowed with meaning and to awaken motivation and interest in mathematics due to its applicable nature in a theory or science.
Mathematical concepts	They emerge through a process of abstraction and simplification of the phenomenon.	They must be considered a priori, based on the teachers' context preparation and selection, following the purposes of the class.
Contexts	They address problems that are not commonly addressed or are addressed in a different way within science.	They must comply with problems previously addressed by the teacher to evaluate their relevance to educational purposes.
Other factors	They generally occur in an environment specific to the science in which they are applied and are usually external to educational factors.	They are regularly presented in the classroom with motivation specific to everyday contexts and other sciences.

However, in Brazil, the experience with modeling remained confined for many years to the application of previously introduced mathematical knowledge, simulating real situations as a motivational strategy for learning.

Mathematical modeling, therefore, aims to involve students in authentic scientific practices, as opposed to routines in which they are merely consumers of scientific knowledge products. Modeling does not just help us understand the central ideas of the different scientific disciplines, but also the acquisition of epistemological knowledge.

After this brief reflection on the importance of using mathematical models, particularly in the development of study sequences centered on mathematical modeling, we consider elements that led us to relate, through the notion of praxeology, the possibility of modeling under the light of the anthropological theory of the didactic (ATD), since, during the development of a study and research process, it is up to the participants in the activity (students) to ask new questions and seek answers through research, guided by the teacher.

Modeling in the light of the anthropological theory of the didactic

From a specific praxeology, modeling sets off in an isolated system that serves only as a pretext for the student to build a model that represents it. Once the model is built, the system loses its importance and is abandoned, as the objective seems to consider that the model is only a part of the student's mathematical heritage.

To Chevallard, Bosch, and Gascón (1997), those models, limited to isolated concepts, techniques, and problems, ignore the presence of issues surrounding the systems that motivated their construction.

For the anthropological theory of the didactic, mathematical modeling (MM) is a kind of heart of mathematical activity. Authors Chevallard, Bosch, and Gascón (1997) place MM in the following terms:

> An essential aspect of mathematical activity consists of constructing a (mathematical) model of the reality we want to study, working with this model, and interpreting the results obtained in this work to answer the questions initially raised. Therefore, much of mathematical activity can be identified as mathematical modeling activity. (Chevallard, Bosch, & Gascón, 1997, p. 51, our translation)

The praxeological structure expands the notions of system and model, as the components of a praxeology are interrelated. This characteristic does not allow the modeling of an element (type of task, technique, technology, and theory) to be considered independently of the others.

According to the ATD, we can describe any mathematical activity through the interrelation between systems[5] and models because, in this theory, any part of reality that can be isolated from it is a system that can be modeled mathematically, which means that we can identify, around any system, problematic issues at the origin of the model construction.

Therefore, the ATD describes modeling processes as processes of reconstruction and articulation of praxeologies of increasing complexity: particular→local→regional→global, generated from questioning the meaning or reason for being of the phenomenon studied.

According to Chevallard (1989, 1992), modeling in light of the ATD meets some requirements inherent to it: a) A model is an artificial construction that establishes an adequate relationship with reality, refuting the representational illusion, i.e., the idea of a model as a copy

[5] The anthropological theory of the didactic considers that any part of reality that can be isolated is a system that can be modeled mathematically.

of the real world. Its primary function is not to "resemble" the system it models but to provide knowledge as most economically and effectively as possible; b) A model must be proficient to allow the construction of knowledge that would be more difficult to obtain if we used another model; c) Modeling not to just build praxeologies but to answer problematic questions; d) Mathematics works as a tool in the construction of models.

It is important to emphasize that the ATD does not consider modeling as an object to be taught and not a means to learn and teach specific mathematical concepts. Thus, the reference epistemological model (REM) notion appears in the ATD as a model that must be made explicit whenever you want to formulate an authentic teaching problem.

Fonseca, Gascón, and Lucas (2014) state that the formulation of a didactic problem in the didactics of mathematics contains, more or less explicitly, an interpretation of the activity that will be used or even a model of such activity, even if it is not very precise, but which will accompany this notion within the scope of school mathematics in a given institution.

Gascón (2011) points out that such an explanation corresponds to the epistemological dimension of the problem, which, in turn, corresponds to a basic dimension of the didactic problem and which is materialized through a model called by the researcher the reference epistemological model (REM).

The structure of the REM, according to Fonseca, Gascón, and Lucas (2014), is made up of a network of mathematical praxeologies whose dynamics allow for progressive expansions and complements and underpin a study and research path (SRP). Therefore, a REM must be considered a provisional hypothesis to be verified experimentally, and can be constantly modified.

According to these authors, the ATD interprets that the mathematical activity is an institutionalized human practice; i.e., a REM and the generating question that will attempt to answer it are developed around an institution. However, institutions are not watertight compartments, and problematic issues develop as they are studied, so it is possible to design a REM that can support study processes situated partially in two or more institutions and at two or more educational levels.

According to Bosch and Gascón (2010), the ATD also shows us that it is necessary to develop an epistemological model that can serve as a reference, both for the analysis of spontaneous epistemologies present in the observed institutions and for the elaboration of new proposals for didactic praxeological organizations.

Let us consider Jovignot-Candy's (2018) research, which aimed to develop a REM that explains the different paths taken to study the didactic transposition of the concept of ideal**,** which can be valuable for building a REM for other concepts. In a presentation about her thesis: “Le Modèle épistémologique de référence: un outil pour l'étude de la transposition didactique du concept d'idéal” (The epistemological model of reference: a tool for the didactic transposition of the concept of ideal), the researcher shows the different paths taken in the study to constitute the proposed REM.

In her research methodology, Jovignot-Candy (2018, s/n) conducts different studies: “The study of the historical epistemology. Diachronic historical study through textbooks (throughout time). Study of handouts, guided work material, and revised exercises. Contemporary epistemological study: questionnaire for researchers. Synchronous textbook study.”

We also emphasize that modeling allows the elaboration and experimental contrast of epistemological models of mathematics to address didactic problems. This feature is in tune with Guy Brousseau, who initially considered the didactics of mathematics as an experimental epistemology (Brousseau, 1986).

After this brief highlight on the importance of REM for the construction, development, and analysis of an SRP, we will now evaluate the conditions and type of relationship that educating through study and research paths can find in Brazilian legislation, i.e., the guidelines and bases of Brazilian education and the National Common Curriculum Base (BNCC) (Brazil, 2018).

The study and research path as a teaching and learning tool in Brazilian education

According to Chevallard (2009), an SRP corresponds to a codisciplinary investigation in which a teacher or a group of teachers and students accept the challenge of studying and researching a question Q_0.

Florensa et al. (2020) observe that many teaching problems are related to the absence of epistemological tools to design, manage, and evaluate study processes. This led them to propose question-and-answer maps as a tool, which partially represent a reference epistemological model, as we will see below.

Depending on the institution where Q_0 is addressed, there will be a study path and a peculiar research. We present as an example the idealized SRP, which we seek to limit to the movement of people and objects.

Initially, we will revisit the analysis of the conditions and restrictions of posing question Q_0 presented in Ignacio et al. (2020, p.804), which seeks to respond if the proposed Q_0 can generate new questions. Q_0: How can we find the shortest route connecting an origin (O) and a destiny (D)?

We identify through an analysis *a priori* from the initial question that the problematic element of Q_0 consists of explaining the maximum variations of trajectories to be covered between O and D. We can take the uncertain aspects as questions that need to be answered in order to present a "good" answer to Q_0. We must answer questions like: 1. Is the path on the Euclidean plane? 2. Are the origin and destination of the trajectory previously defined, or is only the starting point informed? 3. The route between origin (O) and destiny (D) is direct, without stops or detours or not? 4. The number of possible paths and their respective distances that connect (O) to (D) are previously defined or not? 5. The shape of (O) and (D) should be considered for finding the shortest route or not? 6. Is the shortest route the one that occurs in the shortest absolute distance, the shortest time, or the one with the lowest travel cost?

In our *a priori* analysis, we establish the following restrictions: a. We choose the two-dimensional Euclidean plane or symbolically the R^2, even though we know that the path sought may occur in a space of other dimensions; b. We chose to use elements of geometry such as point, line, and circumference to represent the origin, destination, and their interconnections.

In our trajectory simulations, we will consider direct paths from (O) to (D) and paths that must make stops or interconnections (I) before reaching the destination. Therefore, we must remember that the geometric elements considered here are mathematical models that represent possible origins, interconnections, and destinations. In the hypothesis of the shortest path involving a circumference c and a point P of the plan, the elements will always be under the following condition: $P \notin c$. It means that P is external to c.

Figure 1 corresponds to a map of questions and answers, as proposed by Florensa et al. (2020), which summarizes the path we try to seek as possible answers to Q_0.

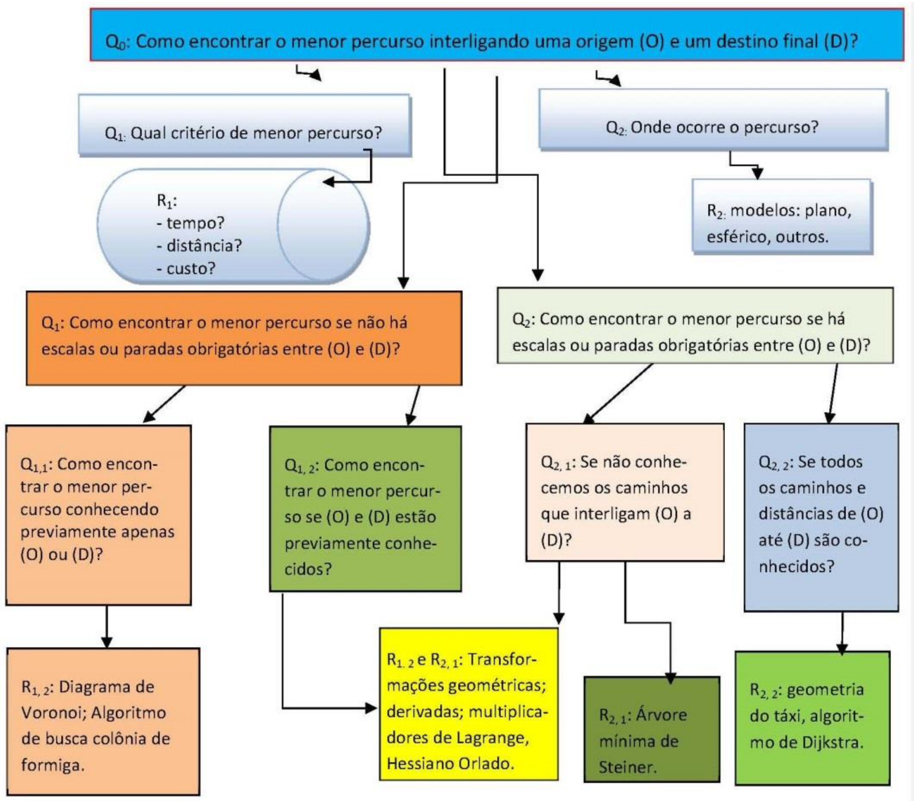

Figure 1.

Map of possible paths for the study of Q_0 (Ignacio, Bosch, & Dias, p. 808, 2020).

The hypotheses of the proposed path are problematic situations that can be stated generically: finding the shortest path for objects on the plane, each of which can be points, lines, or circles.

We discuss some path variations between geometric objects. Solutions like the ones we will show below can emerge from any domain of mathematics. The examples of hypothetical path situations we analyzed gave rise to a network of mathematical praxeologies in the study of differential and integral calculus. Below, we present the description of this praxeological network generated from the variations in paths between the geometric objects considered.

The shortest (direct) route between an origin O and destination D

Let us initially respond to the hypothetical situation by projecting scenarios with the shortest possible route. For each path hypothesis, one or more models are presented as a possible response to the projected hypothetical situations.

These situations are stated in the form of problematic tasks, and we use the tools that mathematics provides to solve them. Some hypotheses of a path may occur if we use these models (point, line, and circumference) to represent the origin O and destiny D. Three path hypotheses (H) will be studied:

a. $H_{1.1}$: the shortest route between two points;

b. $H_{1.2}$: the shortest route between a point and a straight line;

c. $H_{1.3}$: the shortest route between two straight lines.

$H_{1.1}$: the shortest route between two points;

What is the shortest route between two points?

We are considering as an optimization criterion –i.e., as the shortest route– the route that covers the shortest distance. Through Euclidean geometry, the solution to the above question is the line segment that connects the origin O to the destination D.

The primitive elements in Euclidean geometry (point, line, and plane) do not need definition. But this does not prevent us from accepting the definition of a straight line used by Markushevich (1977) as a particular case of a curve. According to Markushevich, a curve or curved line is the trail of a moving point. In this way, the line will be the solution to our question Q0, knowing that: "[...] a moving point effectively describes a line if it passes from one position to another by the shortest path (Markushevich, 1977, p. 3)".

As it is an Euclidean plane, it is the most straightforward circumstance because Euclidean geometry defines the line segment connecting the two points as the shortest distance between them.

The analytical model starts from the premise that each point in the Euclidean plane can be designated as an ordered pair of real numbers. The most common representation model uses two lines x e y perpendicular to each other that are used as axes, creating a Cartesian coordinate system to associate each real number with the intersection point of these axes, which we call an ordered pair $(0, 0)$. The other points are designated as ordered pairs, as shown in the figure below. We denote the coordinates of a point $P = (x_1, y_1)$.

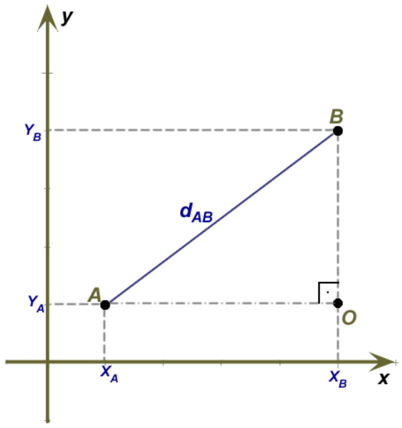

Figure 2.

The shortest distance between two points on the Cartesian plane (Ignacio, 2018, p. 170).

We use the Euclidean metric[6] to calculate distance; therefore, the shortest distance between points A and B is the measurement of the segment with the two endpoints. The point representation model is defined by pairs $A = (x_a, y_a)$ e $B = (x_b, y_b)$ and the shortest distance between these two by $d_e\ (A, B)$. If the segment connecting a point A to a point B is parallel to an O_x axis a $d_e(A, B) = \|x_b - x_a\|$, and if the segment is parallel to the O_y axis, then $d_e(A, B) = \|y_b - y_a\|$. If it is not parallel to either of the two axes, we can represent this possibility geometrically in a generic way using Figure 2.

Given that we consider the orthogonality between the coordinate axes of the Cartesian plane, we can indicate point O and points A and B as vertices of a right triangle under the conditions shown in Figure 2.

The model for calculating the distance between two points applies the Pythagorean theorem, as the segment AB is the hypotenuse of triangle AOB, and the measurement of AB corresponds to the distance between these two points. Since it is a right triangle, we can apply:

$$d_E(A, B)^2 = d_e(A, O)^2 = (x_B - x_A)^2 + (y_B - y_A)^2$$

Therefore, the expression is as follows:

[6]A metric M is a way of measuring distance in a different set of $\emptyset$ which, in the Cartesian plane, associates each ordered pair of elements of M as a function of a real number $d(x, y)$, so as to meet the following properties for any $x, y, z \in M$.

1. $d(x, y) = 0$
2. $d(x, y) > 0\ se\ \neq y$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leq d(x, y) + d(y, z)$

$$d_{E(A,B)} = \sqrt{(x_B - x_A)^2 + (y_B^2 - y_A)^2}$$

H1.2: The shortest distance between a point and a straight line.

What is the shortest distance between a straight line and a point A outside the straight line?

It is r of general equation $+by + c = 0$, with $a\ e\ b$ not simultaneously null, $a, b, c \in \mathbb{R}$ and be $A(x_0, y_0)$.

The distance between a point $P(x, y)$ belongs to line , and point A is given by:

$$d = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

Note that the distance function depends on two variables $x\ and\ y$. This factor leads us to take $y = mx + n$, with $m = -\frac{a}{b}$ and $n = -\frac{c}{b}$ $, com\ b \neq 0$, given that $P \in r$.

In this way, replacing y in d, we will obtain:

$$d = \sqrt{(x - x_0)^2 + (mx + n - y_0)^2}$$

Now, we have a function of just one variable x. Let us minimize $f(x) = d^2$, which is much easier to work with, and we find the same result when we use d.

$$f(x) = (x - x_0)^2 + (mx + n - y_0)^2$$

A maximum or minimum point of a function occurs at a critical number, which, in turn, is a number in the domain of the function where the derivative vanishes or does not exist.

$$f'(x) = 2(x - x_0) + 2(mx + n - y_0)m$$
$$= 2x - 2x_0 + 2m^2x + 2mn - 2my_0$$

As $f'(x)$ exists for every real number, just do $f'(x) = 0 \Rightarrow x(2 + 2m^2) = 2x_0 + 2my_0 - 2mn$

$$\Rightarrow x = \frac{2x_0 + 2my_0 - 2mn}{2 + 2m^2}$$
$$\Rightarrow x = \frac{x_0 + my_0 - mn}{1 + m^2}$$

This is the only critical number, the minimum of the function f in it.

We use the second derivative test for local extremes to justify this statement. The test ensures that being c a critical number:

$$(i) f'(c) = 0\ e\ f"(c) < 0 \Rightarrow f(c)\ is\ the\ local\ maximum$$
$$(ii) f'(0) = 0\ e\ f"(0) > 0 \Rightarrow f(c)\ is\ the\ local\ minimum$$

We have $f''(x) = 2 + 2m^2 > 0 \; \forall \; x \in \mathbb{R}$, in particular $f''(x) > 0$, for the critical number found. With $y = mx + n, we\ will\ have\ y = m\left(\frac{x_0+my_0-mn}{1+m^2}\right) + n$

$$\Rightarrow y = \frac{mx_0 + m^2y_0 - m^2n + n + \ m^2n}{1 + m^2}$$

$$\Rightarrow y = \frac{m^2y_0 + mx_0 + \ n}{1 + m^2}$$

Thus, the point on line r that is closest to point A given is:

$$P\left(\frac{x_0 + my_0 - mn}{1 + m^2}, \frac{m^2y_0 + mx_0 + n}{1 + m^2}\right)$$

With $m = -\frac{a}{b}\ e\ n = -\frac{c}{b}, b \neq 0,$, we have:

$$P\left(\frac{x_0 + \left(-\frac{a}{b}y_0\right) - \left(-\frac{a}{b}\right)\left(-\frac{c}{b}\right)}{1 + \left(-\frac{a}{b^2}^2\right)}, \frac{\left(-\frac{a}{b}\right)^2\ y_0 + \left(-\frac{a}{b}x_0 + \left(-\frac{c}{b}\right)\right)}{1 + \left(-\frac{a^2}{b^2}\right)}\right), \Rightarrow$$

$$P\left(\frac{b^2x_0 - aby_0 - ac}{a^2 + b^2}, \frac{a^2y_0 - abx_0 - bc}{a^2 + b^2}\right)$$

Therefore, the shortest distance between the straight line r and the point A is:

$$d = \sqrt{f(x)} = \sqrt{\left(\frac{b^2x_0 - ab - ac}{a^2 + b^2} - x_0\right)^2 + \left(\frac{a^2y_0 - abx_0 - bc}{a^2 + b^2} - y_0\right)^2}$$

$$d = \sqrt{\left(\frac{a^2x_0 + aby_0 + ac}{a^2 + b^2}\right)^2 + \left(\frac{b^2y_0 + abx_0 + bc}{a^2 + b^2} - y_0\right)^2} \Rightarrow$$

$$d = \sqrt{\frac{(ax_0 + by_0 + c)^2}{a^2 + b^2}} \Rightarrow d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

The definition of a function used here is paramount, i.e.:

Given the sets X e Y, a function f: is a rule (or set of instructions) that says how to associate each element of X with an element of Y. Set X is called the domain, and Y is the codomain of the function$f$. For each $x \in$ X, element $f(x) \in$ Y, the image is called x by function f, or the value assumed by the function f on point x.

Using the Lagrange multiplier method: Let r be a straight line equation $ax + by + c = 0$, with $a, b, c \in \mathbb{R}$, being $a\ e\ b$ not simultaneously null and $A(x_0, y_0)$ a given point not belonging to r. Consider $P(x, y)$ to be any point of r; let us find its coordinates so that this is the point of r closest to A.

The distance between $A\ e\ P$ is given by:

$$d(A,P) = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

The solution becomes simpler if we minimize the square of this distance:

$$d^2 = f(x,y) = (x-x_0)^2 + (y-y_0)^2$$

With the restriction, point (x,y) belongs to the line, that is:

$$g(x,y) = ax + by + c = 0$$

According to Lagrange's theorem, the extremals arise from the system below, given by a vector equation and a scalar equation:

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 0 \end{cases}$$

Where:

$$\nabla f(x,y) = (2x - 2x_0, 2y - 2y_0)$$
$$\nabla g(x,y) = (a,b)$$

Thus:

$$\begin{cases} 2x - 2x_0 = \lambda a \\ 2y - 2y_0 = \lambda b \\ ax + by + c = 0 \end{cases}$$

From the system, we can conclude that $x = \frac{2x_0 + \lambda a}{2}$ e $y = \frac{2y_0 + \lambda b}{2}$

Replacing these values into the third equation, it follows that:

$$a\left(\frac{2x_0 + \lambda a}{2}\right) + b\left(\frac{2y_0 + \lambda b}{2}\right) + c = 0$$
$$2ax_0 + \lambda a^2 + 2by_0 + \lambda b^2 + 2c = 0$$
$$\lambda = \frac{-2c - 2ax_0 - 2by_0}{a^2 + b^2}$$

Now, replacing λ in x e y:

$$x = x_0 + \frac{a}{2}\lambda = x_0 + \frac{a}{2}\left(\frac{-2c - 2ax_0 - 2by_0}{a^2 + b^2}\right)$$
$$x = \frac{b^2x_0 - aby_0 - ac}{a^2 + b^2}$$

$$y = y_0 + \frac{b}{2}\lambda = y_0 + \frac{b}{2}\left(\frac{-2c - 2ax_0 - 2by_0}{a^2 + b^2}\right)$$
$$\Rightarrow y = y_0 - \frac{bc + abx_0 + b^2y_0}{a^2 + b^2} = \frac{a^2y_0 + b^2y_0 - bc - abx_0 - b^2y_0}{a^2 + b^2}$$
$$y = \frac{a^2y_0 - abx_0 - bc}{a^2 + b^2}$$

Therefore, the minimum point is $P = \left(\frac{b^2x_0 - aby_0 - ac}{a^2 + b^2}, \frac{a^2y_0 - abx_0 - bc}{a^2 + b^2}\right)$

And the shortest distance between A and the straight line r is $d = \sqrt{f(P)} = d(P, A)$

$$d(P,A) = \sqrt{\left(x_0 - \frac{b^2x_0 - aby_0 - ac}{a^2+b^2}\right)^2 + \left(y_0 - \frac{a^2y_0 - abx_0 - bc}{a^2+b^2}\right)^2}$$

$$d(P,A) = \sqrt{\left(\frac{a^2x_0 + aby_0 + ac}{a^2+b^2}\right)^2 + \left(\frac{b^2y_0 + abx_0 + bc}{a^2+b^2}\right)^2}$$

$$d = \sqrt{\frac{(ax_0 + by_0 + c)^2}{a^2+b^2}} \Rightarrow d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2+b^2}}.$$

$H_{1.3}$: The shortest distance between two straight lines.

What is the shortest distance between two straight lines?

We define the distance between two straight lines $r\ e\ r'$ as being the shortest distance between a point of r and a point of r'.

That is, $d(r;\ r') = min\,\{d(P;\ P')|P\epsilon\ r\ e\ P'\epsilon\ r'\}$. Then, $d(r;\ r') = 0\ se\ r\ e\ r'$they are coincident or concurrent.

Be r and r' parallel lines. We know that, given $P \in r$, there is a single point $P^* \in r'$, foot of the perpendicular a r'drawn by P, such that $d(P;\ P') \geq d(P; P^*)$, for all $P' \in r'$.

As $r \parallel r'$, we have $d(Q;\ Q^*) = d(P;\ P^*)$, whatever $P;\ Q \in r$, given that $QPP^*Q^*$is a rectangle. Then $d(Q;\ Q') \geq d(Q; Q^*) = d(P; P^*) = d(P;\ r')$, whatever $Q \in r\ e\ Q' \in r'$. Therefore, $d(r;\ r') = d(P;\ r')$; whatever $P \in r$.

As a consequence of the theorem, we have the following corollary:

Corollary: *Be* $r: ax + by = c\ e\ r': ax + by = c'$ *parallel* $(c \neq c')$ *or coincidental lines* $(c = c')$. Then,$d(r\ ;\ r') = \frac{|c-c'|}{\sqrt{a^2+b^2}}$.

Proof: Let $P = (x_0;\ y_0)$ be a point on the line r. Then, $(r;\ r') = d(P;\ r') = \frac{|ax_0+by_0-c'|}{\sqrt{a^2+b^2}}$.

As $ax_0 + by_0 = c$, we obtain, $d(r;\ r') = \frac{|c-c'|}{\sqrt{a^2+b^2}}$.

The three path hypotheses are merely illustrative. They are intended to clarify that mathematical models emerge as a response to questions that may arise throughout the study. More hypotheses of a shorter route were not explored in this work.

Some considerations

Research conducted on mathematical modeling in mathematics education points to promising aspects for using modeling in the classroom as an alternative to allow the establishment of relationships between everyday life and other areas of knowledge.

This article sought to demonstrate that mathematical modeling is roughly presented in three aspects of interpretation: as a learning object, teaching methodology, and research method.

A common characteristic among these strands is the acknowledgment that mathematical modeling, therefore, aims to involve students in authentic scientific practices instead of routines in which they merely consume scientific knowledge products. Modeling does help us understand both the central ideas of the different scientific disciplines and epistemological knowledge acquisition.

Researchers of the anthropological theory of the didactic observe that in the above aspects, modeling is considered through a specific praxeology and sets off from an isolated system that serves only as a pretext for the student to construct a model that represents him. Therefore, once this model is built, the system loses its importance and is subsequently abandoned, as the objective seems to be restricted to considering that the model is only part of the student's mathematical heritage.

However, the ATD praxeological structure expands the notions of system and model, as the components of praxeology are interrelated, and this characteristic does not allow the modeling of an element (type of task, technique, technology, and theory) to be considered independently of the others.

A model for anthropological theory is an artificial construction that establishes an adequate relationship with reality, refuting the representational illusion, i.e., the idea of a model as a copy of the real world. Its main function is not to "resemble" the system it models but to provide knowledge in the most economical and effective way possible. It must be proficient to allow the construction of knowledge that would be more difficult to obtain if we used another model.

We say modeling not as a mere way of constructing praxeologies but of answering problematic questions in the process of reconstruction and articulation of praxeologies of increasing complexity (particular→local→regional→global) generated from questioning the meaning or reason for being of the phenomenon studied.

In the example of constructing a reference epistemological model described in this article, we start from a problematic question: Q_0: How can we find the shortest route connecting an origin (O) and a destiny (D)?

Depending on the institution where Q_0 is addressed, there will be a study path and a peculiar research. In the *a priori* analysis of the initial question, we see that the problematic element of Q_0 consists of explaining the maximum variations of trajectories to be covered between *O* and *D*. To present a "good" answer to Q_0, we must consider uncertain aspects such as: 1. Is the path on the Euclidean plane? 2. Are the origin and destination of the trajectory previously defined, or is only the starting point informed? 3. Is the route between origin (O) and destiny (D) direct, without stops or detours? 4. Are the number of possible paths and their respective distances that connect (O) to (D) previously defined? 5. The shape of (O) and (D) should be considered to find the shortest route or not? 6. Is the shortest route the one that occurs in the shortest absolute distance, the shortest time, or the one with the lowest travel cost?

A priori, we did not establish the institution, stage of education, or year of schooling in which the question would be studied. The possible paths envisioned by the authors are shown on the map in Figure 1. When exploring the route hypotheses, we present only three possible situations.

For this article, we established the following restrictions: a. We chose the two-dimensional Euclidean plane or symbolically the R^2, even though we know that the path sought may occur in a space of other dimensions; b. We chose to use elements of geometry such as point, line, and circumference to represent the origin, destination, and their interconnections.

Finally, it is worth highlighting the fortuitous encounter of this question with concepts studied in differential and integral calculus, in particular, the method of Lagrange multipliers in determining the global optimum point for a problem subject to an equality restriction, which enabled a theoretical-mathematical foundation on real functions of several variables and conditioned extremes that can be maximized or minimized, as well as basic concepts on optimization and methods for determining local or global optimum points with Lagrange multipliers.

Acknowledgement

We would like to thank CAPES for funding a sandwich doctoral scholarship, which helped in the development of the research, carried out at the Ramon Lull University, Barcelona-Spain, under the supervision of Prof. Marianna Bosch.

References

Barquero, B., Bosch, M. & Gascon, J. (2011) Los recorridos de estúdio e investigación y la modelización matemática en la enseñanza universitaria de las ciencias experimentales. *Enseñanza de las Ciencias*, 29(3), (pp. 339-352).

Bassanezzi, R. C (2002). *Ensino – aprendizagem com modelagem matemática: uma nova estratégia*. São Paulo: Editora Contexto.

Biembengut, M. S; Hein, N. (2003) *Modelagem Matemática no Ensino*. 3ª ed. São Paulo: Contexto.

Blum, W. et al. (Eds.). (1989). Applications and Modelling in Learning and Teaching Mathematics. Chichester: Ellis Horwood. deLange, J. et al. (Eds.). (1993). *Innovation in Maths Education by Modelling and Applications*. Chichester: Ellis Horwood.

Blum, W. & Niss, M. (1991). Applied Mathematical Problem Solving, Modeling, Aplications, and Links to Other Subjects-State, Trends and Issues in Mathematics Instructions, *Educational Studies in Mathematics*, nº 23, (pp. 37-68).

Bolea, P. (2003). *Los procesos de algebrización de las Organizaciones Matemáticas Escolares*. Tesis Doctoral, Universidad de Zaragoza; Zaragoza, España.

Borromeo Ferri, R. (2006). Theoretical and empirical differentiations of phases in the modelling process. *ZDM*, *38*(2), 86-95.

Bosch, M.; & Gascón, J. (2010). Fundamentación antropológica de las organizaciones didácticas: de los “talleres de prácticas matemáticas” a los “recorridos de estudio e investigación”. En A. Bronner, M. Larguier, M. Artaud, M. Bosch, Y.Chevallard, G. Cirade y C. Ladage (Eds.): *Diffuser les mathématiques (et les autres savoirs) comme outils de connaissance et d’action* (pp. 1-10). Montpellier: Université de Montpellier.

Brasil (2018). *Base Nacional Comum Curricular*. Brasília: MEC.2018.

Brasil (2004). Ministério da Educação. Secretaria da Educação Média e Tecnológica. *Parâmetros Curriculares Nacionais + (PCN+) - Ciências da Natureza e suas Tecnologias*. Brasília: MEC, 2004.

Brousseau, G. (1986), Fondements et méthodes de la didactique des mathématiques. *Recherches en Didactique des Mathématiques*, 7/2, 33-115.

Chevallard, Y. (1989). Le passage de l’arithmétique à l’algébrique dans l’enseignement des mathématiques au collège - Deuxième partie: Perspectives curriculaires: la notion de modelisation. *Petit X*, 19, (pp. 45-75).

Chevallard, Y. (1992). Concepts fondamentaux de la didactique: Perspectives apportées par une approche anthropologique. *Recherches en Didactique des Mathématiques*. 12/1. (pp. 73-112).

Chevallard, Y. (2007) *Le développement actuel de la TAD: pistes et jalons*. http://yves.chevallard.free.fr/spip/spip/article.php3?id_article=150

Chevallard, Yves (2011). Organiser l’étude 1. Structures & Fonctions. In: *Actes de la XI école d’été*. (pp.1-19) http://yves.chevallard.free.fr/spip/spip/IMG/pdf/Organiser_l_etude_1.pdf

Chevallard, Y., Bosch, M & Gascón, J. (1997). *Estudiar Matemática:* El eslabón perdido entre la enseñanza y el aprendizaje. ICE/Horsori, Barcelona.

D’Ambrósio, U. (2002). Prefácio. In. BASSANEZI, R. C. *Ensino-Aprendizagem com Modelagem Matemática:* uma nova estratégia. São Paulo: Contexto. (pp. 11-12).

Florensa, I., Bosch, M., & Gascon, J. (2020). Reference epistemological model: what form and function in school institutions?. *Educação Matemática Pesquisa. Revista Do Programa*

De Estudos Pós-Graduados Em Educação Matemática, 22(4), 240–249. https://doi.org/10.23925/1983-3156.2020v22i4p240-249

Fonseca Bon, Cecilio, Gascón Pérez, Josep & Oliveira Lucas, Catarina. (2014). Desarrollo de un modelo epistemológico de referencia en torno a la modelización funcional. *Revista latinoamericana de investigación en matemática educativa*, 17(3), 289-318. https://doi.org/10.12802/relime.13.1732

García, F. J.; Higueras, L. R. (2011). Modifying Teachers' Practices: the case of a European Training Course on Modelling and Applications. In: KAISER, G. et al. (Org.). *Trends in teaching and learning of Mathematical Modelling: international perspectives on the teaching and learning of Mathematical Modelling*. London: Ed. Springer, p. 569-578.

Gascón, Josep (2011). Las tres dimensiones fundamentales de un problema didáctico. El caso del álgebra elemental. *Revista latinoamericana de investigación en matemática educativa*, 14(2), 203-231.

Holland, John H., Keith J. Holyoak, Richard E. Nisbett and Paul R. Thagard, (1986) Induction: Processes of Inference, *Learning, and Discovery*, Cambridge: M.I.T. Press.

Ignácio, R. (2018). *Percurso de estudo e pesquisa na educação básica: limites e possibilidades*. (tese de doutorado). Universidade Anhanguera de São Paulo.

Ignácio, R., Bosch, M., & Dias, M. A. (2020). Parcours d'étude et de recherche : une étude sur les chemins minimaux. *Educação Matemática Pesquisa Revista Do Programa De Estudos Pós-Graduados Em Educação Matemática*, 22(4), 801–808. https://doi.org/10.23925/1983-3156.2020v22i4p801-808

Jovignot-Candy, J. (2018). *Le Modèle épistémologique de référence un outil pour l'étude de la transposition didactique du concept d'idéal*. https://www.academia.edu/37207013/Le_Mod%C3%A8le_%C3%A9pist%C3%A9mologique_de_r%C3%A9f%C3%A9rence_un_outil_pour_l%C3%A9tude_de_la_transposition_didactique_du_concept_did%C3%A9al

Lucas, C. O. (2015). *Una Posible "Razão de Ser" del Cálculo Diferencial Elemental en el Ámbito de la Modelización Funcional*. Tesis de doctorado. Universidade de Vigo.

Maaß, K. (2006). What are modelling competencies? *Zentralblatt für Didaktik der Mathematik*, 38(2), 113-142.

Nicolelis, Miguel (2011). *Muito além do nosso eu: a nova neurociência que une cérebro e máquinas — e como ela pode mudar nossas vidas*. São Paulo: Companhia das Letras.

Ruiz-Munzón, N.; Bosch, M. & Gascón, J. (2011). Un modelo epistemológico de referencia del algebra como instrumento de modelización. En M. Bosch et al (Eds.), *Un panorama de la TAD* (Vol. 10, p. 743-765). Barcelona, España: Centre de Recerca Matemàtica.

Scheffer, Nilce Fátima (1999). Modelagem matemática: uma abordagem para o ensino aprendizagem da matemática. *Educação Matemática em Revista*, n. 1, p. 11-16.

Sierra, T. A (2006). *Lo Matemático em el Diseño y Analisis de Organizaciones Didácticas: los Sistemas de Numeráción y la Medida de Magnitudes*. Tesis del Doctorado. Universidad Complutense de Madrid.

Villa-Ochoa, J. A. (2007). La modelación como proceso en el aula de matemáticas. Un marco de referencia y un ejemplo. *Tecno Lógicas*, (19) pp.51-81.

Reviewer and translator: Maria Isabel de Castro Lima