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The use of figures related to the complex integral and Cauchy's integral theorem in university textbooks of complex analysis used in Spain

El uso de figuras relacionadas con la integral compleja y al teorema integral de Cauchy en libros de texto universitarios de variable compleja utilizados en España

L'utilisation de figures liées à l'intégrale complexe et au théorème intégral de Cauchy dans les manuels universitaires d'analyse complexe utilisés en Espagne

O uso de figuras relacionadas à integral complexa e ao teorema integral de Cauchy em livros didáticos universitários de variável complexa usados na Espanha

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Abstract

This study addresses a question about the similarities and differences between original mathematical works in complex analysis and contemporary textbooks, regarding the use of figures (conceived as two-dimensional drawings) to address concepts in this branch of mathematics. In order to answer this question, we analyzed the four main textbooks that are referenced in the teachers' guides of all Spanish public universities that offer a degree in mathematics. Specifically, we present how these four textbooks structure the concept of complex integral and the proof of Cauchy's integral theorem. To carry out our analysis, we retrieved a reference epistemological model that describes how historical subjects used figures to develop complex analysis from the first quarter of the 19th century to the first half of the 20th

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century. The study shows that the four textbooks structure these concepts in such a way that they are related to the most contemporary forms in which they were attended in their historical development. Although, we are not against structuring the content of the textbooks in this way, we will argue how the reference epistemological model can serve as an epistemological alternative for the elaboration of didactic material that takes into account a historical development of complex analysis.

Keywords: Reference epistemological model, Textbook content analysis, Figures, Didactic transposition, Complex analysis.

Resumen

Este estudio aborda una cuestión sobre las semejanzas y diferencias entre obras matemáticas originales de análisis complejo y libros de texto contemporáneos, en relación con el uso de figuras (concebidas como dibujos bidimensionales) para abordar conceptos en esta rama de las matemáticas. Para responder a esta pregunta, analizamos los cuatro principales libros de texto referenciados en las guías docentes de todas las universidades públicas españolas que imparten la licenciatura en matemáticas. En concreto, presentamos cómo estos cuatro libros de texto estructuran al concepto de integral compleja y demuestran el teorema integral de Cauchy. Para llevar a cabo nuestro análisis, recuperamos un modelo epistemológico de referencia que describe cómo sujetos históricos utilizaron figuras para desarrollar la variable compleja desde el primer cuarto del siglo XIX hasta la primera mitad del siglo XX. En el estudio se muestra que los cuatro libros de texto estructuran estos conceptos de tal manera que se relacionen con las formas más contemporáneas en las que fueron atendidos en su desarrollo histórico. Aunque no estamos en contra de estructurar el contenido de libros de texto de esta manera, argumentaremos cómo el modelo epistemológico de referencia puede ser servir como una alternativa epistemológica para la elaboración de material didáctico que tome en consideración un desarrollo histórico de la variable compleja.

Palabras clave: Modelo epistemológico de referencia, Análisis del contenido de libros de texto, Figuras, Transposición didáctica, Variable compleja.

Résumé

Cette étude porte sur une question relative aux similitudes et aux différences entre les travaux mathématiques originaux en analyse complexe et les manuels contemporains, concernant l'utilisation de figures (conçues comme des dessins en deux dimensions) afin aborder les concepts de cette branche des mathématiques. Pour répondre à cette question, nous avons analysé les quatre principaux manuels qui sont référencés dans les programmes de l'enseignant de toutes les universités publiques espagnoles qui proposent un diplôme en mathématiques. Nous présentons comment ces quatre livres structurent le concept d'intégrale complexe et la preuve du théorème intégral de Cauchy. Pour mener à bien notre analyse, nous avons opté pour un modèle de référence épistémologique qui décrit la manière avec les sujets historiques ont utilisé les figures pour développer l'analyse complexe entre le premier quart du 19e siècle et la première moitié du 20e siècle. L'étude montre que les quatre manuels universitaires structurent ces concepts de manière à les mettre en relation avec les formes les plus contemporaines, auxquelles ils ont été associés dans leur développement historique. Bien que nous ne soyons pas opposés à ce que le contenu des manuels soit structuré de cette manière, nous expliquerons comment le modèle de référence épistémologique peut servir d'alternative épistémologique pour l'élaboration d'un matériel didactique qui tienne compte du développement historique de l'analyse complexe.

Mots-clés : Modèle de référence épistémologique, Analyse du contenu des manuels, Figures, Transposition didactique, Analyse complexe.

Resumo

Este estudo aborda uma questão sobre as semelhanças e diferenças entre trabalhos matemáticos originais em análise complexa e livros didáticos contemporâneos, com relação ao uso de figures (concebidas como imagens bidimensionais) para abordar conceitos nesse ramo da matemática. Para responder essa pregunta, analisamos os quatro principais livros didáticos que são referenciados nos guias de professores de todas as universidades públicas espanholas que oferecem graduação em matemática. Especificamente, apresentamos como esses quatro livros didáticos estruturam o conceito de integral complexa e a prova do teorema da integral de Cauchy. Para realizar nossa análise, recuperamos um modelo de referência epistemológica que descreve como os sujeitos históricos usaram figuras para desenvolver a análise complexa desde o primeiro quarto do século XIX até a primeira metade do século XX. O estudo mostra que os quatro livros didáticos estruturam esses conceitos de tal forma que eles estão relacionados às formas mais contemporâneas em que foram utilizados em seu desenvolvimento histórico. Embora não sejamos contra a estruturação do conteúdo dos livros didáticos dessa forma, argumentaremos como o modelo de referência epistemológica pode servir como uma alternativa epistemológica para a elaboração de material didático que leve em conta o desenvolvimento histórico da variável complexa.

Palavras Clave: Modelo epistemológico de referência, Análise de conteúdo de livros didáticos, Figuras, Transposição didática, Variável complexa.

The use of figures related to the complex integral and Cauchy's integral theorem in university textbooks of complex analysis used in Spain

According to Manning (1975), complex function theory, also called complex analysis, emerged in the 19th century when Cauchy, Riemann and Weierstrass provided mathematical foundations for techniques used in the 18th century for complex magnitudes and functions. Decades later, since the second half of the 20th century, complex analysis is structured in university textbooks in much the same way as the content of university calculus textbooks. For example, Figure 1 shows that Spivak (1996) and Ahlfors (1979) arrange the content of their books by means of the same concepts (number, function, differentiation of functions, integration of functions, and infinite series of functions) defined over different sets ($\mathbb{R}$ and $\mathbb{C}$).

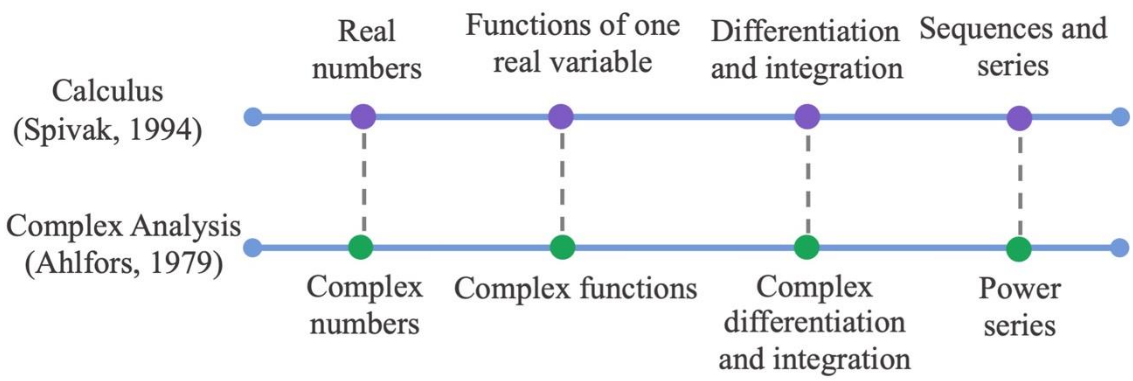

Figure 1.

Sequential structure of the concepts covered in complex analysis and real calculus textbooks (Own elaboration)

Despite these similarities between the way the content of complex analysis and university calculus is structured in textbooks, it is important to emphasize that these two areas of mathematics present differences, since there are results that are specific to complex analysis that cannot be adapted to the case of calculus in one real variable. Two examples are given below to illustrate this difference.

i. Liouville's theorem states that if a function $f: \mathbb{C} \longrightarrow \mathbb{A} \subseteq \mathbb{C}$ is bounded and complex differentiable in its domain, then f is a constant function. A counterexample in one real variable calculus is the function $sin(x): \mathbb{R} \longrightarrow [-1, 1] \subseteq \mathbb{R}$, which is bounded and infinitely differentiable in its domain, and yet $\sin(x)$ is not a constant function.

ii. In complex analysis, the existence of the first derivative of a complex function f implies the existence of all higher derivatives of the function f. Meanwhile, the function

$$g(x) = \begin{cases} \frac{1}{2}x^2 \; si \; x \geq 0 \\ -\frac{1}{2}x^2 \; si \; x < 0 \end{cases}$$

defined from ℝ to ℝ is differentable only once in $x = 0$.

These types of differences are often made clear in complex analysis textbooks. For example, according to Zill and Shanahan (2013), there is little similarity between derivatives of real value functions and complex value functions. In the authors' words:

> Although many of the concepts […] will seem familiar […] there are important differences between this material and the calculus of real functions $f(x)$. As the subsequent chapters of this text unfold, you will see that except for familiarity of names and definitions, there is little similarity between the interpretations of quantities such as $f'(x)$ and $f'(z)$ (p. 142).

Studies in mathematics education, such as that of Soto-Johnson and Hancock (2019), have made it possible to counter conceptions like those of Zill and Shanahan. In particular, Soto-Johnson and Hancock give a geometric meaning to complex differentiation by conceptualizing real valued functions as mappings with certain characteristics. This allows them to conclude that "the geometric interpretation of the derivative $f'(x)$ can be viewed in a new light, as a special case of the complex amplitwist" (p. 434).

Studies such as that of Soto-Johnson and Hancock (2019) are part of a type of studies (Troup et al., 2023; Soto and Oehrtman, 2022; Oehrtman et al., 2019; Dittman, 2016) in mathematics education that address a geometric perspective of certain concepts from complex analysis. There are also research studies in the field that have analyzed how different representations of complex numbers are being used in classroom scenarios (Danenhower, 2000; Panaoura et al., 2006; Nemirovsky et al, 2012). And some studies have even resorted to the use of digital technologies with the aim of making different concepts of complex analysis accessible in classroom scenarios (D'azevedo and Dos Santos, 2021; Ponce, 2019).

However, in this research we recover some results of a type of studies that have analyzed the mathematical activity of historical subjects (Piña-Aguirre and Farfán, 2023; Piña-Aguirre and Farfán, 2022; Hanke, 2022; Cantoral and Farfán, 2004). With the aim of answering the question: how the different ways in which historical subjects did mathematics have been modified in order to configure a discourse in complex analysis textbooks?

To provide an answer to the previous question, in this study we recover three categories from Piña-Aguirre and Farfán (2023) that describe three different ways in which historical subjects did mathematics in what we now call complex analysis. By using these three categories as a frame of reference, we will show that there are some differences and some similarities in how complex analysis textbooks treat certain concepts in complex analysis, when compared to how these concepts were treated in their historical development.

We believe that by identifying these modifications, we can provide evidence of how mathematical knowledge undergoes modifications when it is restructured with the goal of transforming it into teachable knowledge. This opens the door to studying how the different ways of doing mathematics, that were used by historical subjects, are nuanced in contemporary teaching and learning scenarios.

Theoretical framework

Research such as that of Bosch and Gascón (2006) suggests that the mathematical knowledge presented in contemporary teaching and learning scenarios is the result of a didactic transposition process that transforms scientific knowledge into teachable knowledge. This indicates that Bosch and Gascón recognize that the ways of doing mathematics in their respective scenarios of origin are disrupted when mathematics is presented in textbooks or when it is structured in study plans and programs. According to these authors, the disruption of the ways of conceiving and doing mathematics with the aim of making mathematical knowledge teachable has repercussions. For example, different didactic phenomena (unintentional regularities) may arise in the processes of generating and disseminating mathematics in teaching and learning scenarios. And even these unintentional regularities are, in principle, different from the phenomena that arose when mathematical knowledge was being developed in their respective scenarios of origin.

According to Gascón (2014), researchers in mathematics education can take the processes of didactic transposition as an object of study by questioning the codes that decamine the different ways of doing and conceiving mathematics that are used in an acritical way in teaching and learning scenarios. Specifically, Gascón proposes that in order to emancipate from these codes, which he calls *dominant epistemological models*, it is necessary to construct *reference epistemological models* (REM) that provide evidence of other ways of doing and conceiving mathematics that are not usually recognized by dominant epistemological models.

As mentioned in the introduction of this paper, there is research that addresses a geometric perspective of certain concepts embedded in complex analysis. In these studies, researchers have analyzed the mathematical activity of professional mathematicians (Hanke, 2022; Oehrtman, 2019), undergraduate mathematics students (Troup et al., 2023; Soto and Oehrtman, 2022), and pre-service secondary mathematics teachers (Dittman, 2016) as they solve a series of tasks related to provide geometric interpretations of different complex analysis concepts. The results of these studies can conform a REM based on contemporary ways of dealing with different concepts framed in complex analysis using figures. However, we believe that, in order to provide an answer to our research question, it would be useful to have at our disposal a reference epistemological model grounded in historical evidence.

Table 1.

Different ways of doing mathematics in the historical development of complex analysis. Own elaboration from (Piña-Aguirre and Farfán, 2023).

Figures as means of representation	Figures as means of construction	Figures with epistemic value
This first category is the result of a conjecture that depicts that one way in which historical subjects did mathematics is characterized by the exclusive use of algebraic symbolism as the only means of mathematical justification. Therefore, in this category, the use of figures can be associated at most as a means of representing algebraic expressions.	This second category is characterized by the recognition that every figure used by historical subjects for the production of mathematical knowledge is accompanied by its counterpart via algebraic expressions.	Finally, this third category encapsulates the mathematical activity of historical subjects who used figures, without the need for them to be accompanied by their counterpart in algebraic symbolism, as a means of mathematical justification. That is, figures constitute a means of argumentation in the production of mathematical knowledge.

In this order of ideas, Piña-Aguirre and Farfán (2023) constructed a reference epistemological model of some aspects related to mathematical knowledge production in complex analysis in the context of complex integration. This REM consists of three categories (Table 1) configured after the analysis of foundational original works related to the well-known Cauchy's integral theorem, which states that the integral of a complex-valued function f that is differentiable in and on a closed contour C is equal to zero.

Their analysis is framed by a *history approach* in the sense of Grattan-Guinness (2004), because the authors do not seek a historical reconstruction of the events that led to a contemporary treatment of Cauchy's integral theorem. Instead, they propose a reconstruction that recognizes as valid the different ways in which historical subjects did mathematics in complex analysis. Specifically, by analyzing the historical development of the theorem in five original works spanning over seventy-five years, Piña-Aguirre and Farfán (2023) conjectured that during this period complex analysis evolved through the gradual incorporation of figures (conceived as two-dimensional drawings presented *per se* by historical subjects, or, alternatively, described via narrative expressions) into the purely symbolic apparatus (framed by the use of algebraic expressions such as the use of equations and functional relations) as a means of mathematical justification.

Table 1 summarizes the chronological development of this gradual incorporation of figures through three different ways of doing mathematics employed by historical subjects. The main difference between the three categories of the REM lies in how different historical subjects used figures (with different roles) while producing mathematical knowledge in the context of complex analysis. For example, according to Piña-Aguirre and Farfán (2023), in Cauchy's memoir *Sur les intégrales définies, prises entre des limites imaginaires* (Cauchy, 1825), the author used figures as a means of representation to give a meaning to the concept of complex integration, as well as to prove Cauchy's integral theorem. However, the authors comment that these mathematical objects were attended by Goursat in his work *Démonstration du Théorème de Cauchy* (Goursat, 1884) by incorporating the use of figures with epistemic value.

Studies such as those by Hanke (2022) and Garcia and Ross (2017) provide an overview of different ways in which it is possible to attend to the concept of complex integration and to prove Cauchy's integral theorem, respectively. Specifically, Hanke recovers how the concept of complex integration is presented in different resources, such as contemporary complex analysis textbooks, historical sources, and didactic innovations constructed by professional mathematicians. Table 2 shows seven categories, adapted from the work of Hanke (2022), that share the property that the complex integral $\int_\gamma f\, dz$ is conceptualized via an integration path $\gamma\colon [a,b] \longrightarrow \Omega$, with real part γ_1 and imaginary part γ_2, defined piecewise and continuously differentiable on a domain $\Omega \subseteq \mathbb{C}$, while $f\colon tr\,(\gamma) \longrightarrow \mathbb{C}$ is a function with real part u and imaginary part v whose domain is the trace of the curve γ.

Table 2.

Characterization of the concept of complex integral. Own elaboration from (Hanke, 2022)

The complex integral as an infinite sum $$\int_\gamma f(z)dz \coloneqq \lim_{\substack{P^{(n)},\xi^{(n)} \\ n\longrightarrow\infty}} \sum_{k=1}^{\nu_n} f\left(\gamma\left(\xi_k^{(n)}\right)\right)(\Delta\gamma)_k^{(n)}$$ For every sequence of partitions $P^{(n)}$ of $[a,b]$ of length $\nu_n \in \mathbb{N}$, $\xi_k^{(n)} \in [t_{k-1},\ t_k] \subseteq [a,b]$ and $(\Delta\gamma)_k^{(n)} = \gamma(t_k) - \gamma(t_{k-1})$ for $k = 1,2,\dots,\ \nu_n$, $n \in\ \mathbb{N}$, in such a way that the norm of $\ell(P^{(n)})$ converges to zero as $n \longrightarrow \infty$.
The complex integral via a mean value The complex integral can be defined as follows:

$$\int_\gamma f(z)dz \coloneqq L(\gamma) \underset{z \in tr(\gamma)}{av} (f(z)T(z))$$

Where $T\big(\gamma(t)\big) = |\gamma'(t)|^{-1}\gamma'(t)$ is the unit vector associated with the points of the γ curve. $\underset{z \in tr(\gamma)}{av}(f(z)T(z))$ is the mean value of the function $f \cdot T$ over the oriented path γ. And $L(\gamma)$ is the length of the γ curve.

The complex integral from a vector analysis perspective

If $f = u + iv$ is defined in the trace of a rectifiable and continuous differentiable γ path, then:

$$\int_\gamma f(z)dz \coloneqq \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy$$

The complex integral from Green's perspective

If u y v sare continuous differentiable functions in Ω, $f = u + iv$, and γ is a simple rectifiable path in $\Omega \subseteq \mathbb{C}$ whose interior is contained in Ω, then:

$$\int_\gamma f(z)dz := 2i \iint_{int(\gamma)} \bar{\partial} f\, d\mathcal{A}$$

where $\bar{\partial}$ is a differentiable operator and $d\mathcal{A}$ is an area differential of the form $dxdy$.

The complex integral as an antiderivative

If f is analytic in a simply connected domain, then there exists a primitive function F of f such that,

$$\int_\gamma f(z)dz \coloneqq F\big(\gamma(b)\big) - F\big(\gamma(a)\big)$$

The complex integral via residues

If f is analytic in $\Omega - A$, A is finite and $A \subseteq \Omega$, then:

$$\int_\gamma f(z)dz \coloneqq 2\pi i \sum_{\omega \in A} \underset{\omega}{Res(f)} \underset{\gamma}{Ind(\omega)}$$

Where $\underset{\omega}{Res(f)}$ are the residues of the function f and $\underset{\gamma}{Ind(\omega)}$ represents the index of the path γ around the point ω. That is, the integral depends on the sum of the residues of f.

The complex integral as a substitution

If $z = \gamma(t)$ and $dz = \gamma'(t)dt$, then:

$$\int_\gamma f(z)dz \coloneqq \int_a^b f\big(\gamma(t)\big)\gamma'(t)\,dt$$

On the other hand, Table 3 summarizes what Garcia and Ross (2017) consider to be four different ways in which complex analysis textbooks prove Cauchy's integral theorem, accompanied by didactic comments based on their expertise as working mathematicians and professors of complex analysis courses.

Table 3.

Proofs of Cauchy's integral theorem and some didactic comments. Own elaboration from (García and Ross, 2017)

Proofs of Cauchy's integral theorem	**Didactic comments from Garcia and Ross**	
	Advantages	Disadvantages
Through Green's theorem: if $f = u + iv$ where u, v are two harmonic functions, then: $\int_\gamma f(z)dz = \int_\gamma (u+iv)(dx+idy)$ $= \int_\gamma (udx - vdy)$ $+ i\int_\gamma (udy + vdx)$ $= \iint_\Omega \left[-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] dxdy$ $+ i\iint_\Omega \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] dxdy = 0$	As shown in the column on the left, the proof is simple and short	Green's theorem is not usually remembered by students. Moreover, this proof of the theorem requires f' to be continuous, and the usual proof of this fact requires Cauchy's integral formula, which is proved by Cauchy's integral theorem. This means that there is a logical inconsistency in the proof.
Following Leibniz Rule: Leibniz rule stablishes that if $F(x,t)$ and $\frac{\partial}{\partial x}F(x,t)$ are continuous functions in the closed rectangle $\{(x,t)\colon x_0 \leq x \leq x_1, a \leq t \leq b\}$, then: $\frac{d}{dx}\int_a^b F(x,t)dt = \int_a^b \frac{\partial}{\partial x}F(x,t)\,dt$	Leibniz rule allows to prove Cauchy's integral formula, which then in turn allows to prove that f' is continuous. Ergo, by applying Green's theorem, Cauchy's integral theorem can be proved without logical inconsistencies.	The logical chains of implications conveyed in the advantages section of this proof rely on purely algebraic expressions, bypassing a geometric perspective.
Through Goursat's lemma: Goursat's lemma states that $\int_\gamma f(z)dz = 0$ For every triangular/rectangular γ path contained in $\Omega \subseteq \mathbb{C}$, as long as f is analytic in Ω.	The continuity of f' is not necessary.	It requires a great deal of mathematical analysis: recursive constructions, linear approximations of the form $f(z) = f(z_o) + f'(z_o)(z - z_o) + \varepsilon(z - z_o)$, and the concepts of continuity and compacity.

Via deformations and homotopy: Non rigorous perspective: by taking as a starting point the Cauchy-Riemann equations, and based on physical principles, it follows that: $$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$ Then, the proof of the theorem rests by reducing γ_2 to a point. Rigorous perspective: By subdividing the homotopy domain $H: [0,1]^2 \longrightarrow \Omega$ that connects γ_1 with γ_2 through small rectangles whose images are contained in opened discs in Ω, allows to apply Cauchy's theorem for discs.	It allows to display some topology in complex analysis	The formalization of topological aspects can be a hinderance rather than an advantage in students' learning processes.

The authors note that these four approximations should not be considered as an exhaustive list of possible ways to prove the theorem, but they believe that these are the most common ways to prove it.

Method

Textbook analysis is an important field of research in mathematics education that allows for multiple approaches depending on the various possible research goals (Fan, 2013). We frame our analysis as a textbook content analysis in the sense of Fan et al. (2013), because these authors conceive that this type of analysis "focuses on how a topic or topics are treated, or how a particular idea or aspect of interest is reflected in the textbooks" (p. 636). Moreover, as Schubring and Fan (2018, p. 769) point out: "the essential issue for mathematics textbook research is the assessment of the textbook's practice regarding the relation between academic knowledge and school mathematics knowledge". This makes textbook analysis an important tool in order to address the process of didactic transposition that we already mentioned.

According to Van Dormolen (1986), text analysis can be done *a priori* (if we seek to evaluate the text as a didactic tool without taking actual instruction into account) or a posteriori (if we seek to compare its proposals with the obtained learning outcomes). Gómez (2011) suggests that the analysis can be textual (to analyze a mathematical content in its curricular and methodological dimension) or epistemological (to know how school mathematics has been conceived at different moments in history). Under this point of view, we approach an a priori epistemological analysis.

Scott (1990) proposes four criteria that must be taken into consideration when carrying out documentary research with respect to the selected sources: authenticity, credibility, representativeness, and meaning.

To address the issue of representativeness, we first compiled the list of all textbooks referenced in the teaching guides of all the Spanish public universities that offer a degree in mathematics. These teaching guides are public documents that can be accessed on the websites of the universities. Out of a total of 50 Spanish public universities, 27 offer a degree in mathematics. Below we list the top four textbooks, that are referenced in more than 40% of all these 27 teaching guides:

- Conway, J. (1973). *Functions of one complex variable*. Springer. (24 guides, 88.89%).
- Ahlfors, L. (1979). *Complex analysis, an introduction to the theory of analytic functions of one complex variable*. McGraw-Hill. (17 guides, 62.97%).
- Rudin, W. (1987). *Real and complex analysis*. McGraw-Hill. (16 guides, 59.26%).
- Marsden, J., & Hoffman, M. (1999). *Basic complex analysis*. W. H. Freeman. (12 guides, 44.44%).

Authenticity and credibility rely on the fact that we have analyzed digitized versions of the original texts. Finally, meaning represents the textual analysis of the document. In our case, to analyze how the complex integral concept is presented in the textbook, we first identified the sections of the textbooks that deal with this concept. Then, we determined whether Hanke's (2022) seven categories (Table 2) could describe the way the textbook authors define this concept. On the other hand, given that three of the four textbooks are referenced by Garcia and Ross (2017), with the exception that some of them have different edition numbers, we first identified whether the four different proofs of Cauchy's integral theorem presented by Garcia and Ross (Table 3) were used by the four textbook authors. Next, we decided to analyze the proofs presented by the textbook authors that relied on the use of figures as means of mathematical justification. Finally, we determined which of these proofs could be analyzed by the history-based REM of Piña-Aguirre and Farfán (2023) as described in Table 1.

Results

First, we deal with the concept of complex integration in complex analysis textbooks. As we will see, the four textbooks that we analyzed define the complex integral through the category called *the complex integral as a substitution*.

At the beginning of chapter four, called complex integration, Conway (1973) states the following: “we will begin by defining the Riemann-Stieltjes integral in order to define the integral of a function along a path in $\mathbb{C}$” (p. 58). By defining $\gamma: [a, b] \longrightarrow \mathbb{C}$ to be of bounded variation (which involves an infinite sum), the author argues that this implies that γ is of finite length, which allows him to define the line integral of a continuous function f, defined on γ, via the following expression.

$$\int_{\gamma} f d\gamma = \int_{a}^{b} f\big(\gamma(t)\big) d\gamma$$

Following Conway’s steps, but without previously defining the concept of bounded variation, Ahlfors (1979) and Rudin (1987) directly define the integral of a continuous complex valued function f, defined on a piecewise differentiable arc γ, by a line integral such as the following:

$$\int_{\gamma} f d\gamma = \int_{t_o}^{T} f\big(\gamma(t)\big)\gamma'(t) dt$$

The main difference in the way Ahlfors and Rudin treat this concept is that Ahlfors states that γ is represented by an equation of the form $z = z(t); a \leq t \leq b$, while Rudin conceives that $\gamma = \gamma(t);\ \alpha \leq t \leq \beta$. Apart from this difference in notation, the line integrals presented by Conway, Ahlfors, and Rudin correspond to the same expression, since $d\gamma$ in Conway’s definition amounts to the same as $z'(t)dt$, or alternatively to $\gamma'(t)dt$.

On the other hand, the main difference in Marsden and Hoffman (1999) approximation is that the authors treat the complex integral via the following expression:

$$\int_{\gamma} f = \int_{\gamma} f(z) dz = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f\big(\gamma(t)\big)\gamma'(t) dt$$

That is, the authors emphasize that the integral over a piecewise differentiable arc γ should be obtained by the sum of the integrals over each arc piece γ_i that make up the entire γ integration curve.

It is important to note that some authors derive other results related to the complex integral that can be framed in Hanke’s categories. For example, immediately after presenting their definition for the complex integral concept, Marsden and Hoffman comment that if one expresses the integrand $f\big(\gamma(t)\big)\gamma'(t)$ in terms of its real and imaginary parts, such as $f\big(\gamma(t)\big)\gamma'(t) = \{u[x(t), y(t)] + iv[x(t), y(t)]\}[x'(t) + iy'(t)]$, then

$$\int_{\gamma} f(z) dz = \int_{\gamma} u dx - v dy + i \int_{\gamma} v dx + u dy$$

This last expression coincides with the category called *the complex integral from a vector analysis perspective*. Meanwhile, Conway presents the fundamental theorem of line integrals through the following theorem.

> Let G be open in $\mathbb{C}$ and let γ be a rectifiable path in G with initial and end points α and β respectively. If $f\colon G \longrightarrow \mathbb{C}$ is a continuous function with a primitive $F\colon G \longrightarrow \mathbb{C}$, then
>
> $$\int_\gamma f = F(\beta) - F(\alpha)$$

which corresponds to the category called *the complex integral as an antiderivative*. Nevertheless, the main characteristic that all the authors have in common when dealing with the concept of complex integration is that they define it via an integration path γ. As will be shown below, this is not entirely faithful to how this concept was treated in Cauchy's memoire from 1825.

According to Grattan-Guinness (2004), Cauchy developed his theory of complex functions with the particularity that, at the beginning of his studies, he avoided the use of a geometric scenario because he considered it lacking in rigor. This general vision of the way Cauchy did mathematics around the 1820's is shared by historians and philosophers (Smithies, 1997; Larivière, 2014) who have analyzed Cauchy's mathematical activity in the context of complex analysis.

Following this conception, which describes a way in which Cauchy did mathematics, Piña-Aguirre and Farfán (2023) conjectured that Cauchy attended to the concept of complex integration by relying solely on the first category of the REM. In the sense that they argue that Cauchy, in his memoir *Sur les integrals définies, prises entre des limites imaginaires* (Cauchy, 1825), gives a meaning to the complex integral by extending his definition of real definite integral (presented in his *Cours d'analyse* from 1823) through the introduction of complex quantities of the form $x + iy$. This process of extension allows the complex integral to be defined by the following symbolic expressions:

$$\int_{x_0+y_0\sqrt{-1}}^{X_0+Y_0\sqrt{-1}} f(z)\, dz \coloneqq \sum_{k=0}^{\infty}\left[(x_{k+1}-x_k)+(y_{k+1}-y_k)\sqrt{-1}\right] f\left(x_k + y_k\sqrt{-1}\right)$$

By considering that the points x_k, y_k can be obtained by monotone functions $\varphi, \chi\colon [t_o, T] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, in such a way that they define sequences of numbers of the form $\varphi(t_k) = x_k$ and $\chi(t_k) = y_k$ for $t_k \in [t_o, T] \subseteq \mathbb{R}$, this last expression can be rewritten as follows

$$\int_{x_0+y_0\sqrt{-1}}^{X_0+Y_0\sqrt{-1}} f(z)\,dz = \int_{t_0}^{T}\left[\varphi'(t)+\sqrt{-1}\chi'(t)\right] f\left[\varphi(t)+\sqrt{-1}\chi(t)\right]dt$$

Authors such as Bottazzini and Gray (2013) conceive that the number sequences $\varphi(t_k) = x_k$ and $\chi(t_k) = y_k$ can be interpreted in a geometric scenario by imagining that x_k and y_k can form points in the complex plane of the form $x_k + y_k\sqrt{-1}$. In this way, the complex integral can be attended via integration paths $\gamma(t) = \varphi(t) + \chi(t)\sqrt{-1}$, which connect the initial and final points of the integral via the following pair of equalities.

$$\int_{x_0+y_0\sqrt{-1}}^{X_0+Y_0\sqrt{-1}} f(z)\,dz = \int_{t_0}^{T}\left[\varphi'(t)+\sqrt{-1}\chi'(t)\right] f\left[\varphi(t)+\sqrt{-1}\chi(t)\right]dt = \int_{t_0}^{T}\gamma'(t) f\big(\gamma(t)\big)dt$$

In our view, this type of interpretation is more in line with what Grattan-Guinness (2004) understands as a *heritage approach* of a mathematical concept. In the sense that this type of interpretation seeks to insert the modern complex integral concept into its own historical development. Therefore, we believe that the presentation of the complex integral via integration paths is not necessarily the only way in which Cauchy could have approached this concept.

It is important to note that based on the idea that Cauchy does not make an explicit use of integration paths to deal with complex integrals, Piña-Aguirre and Farfán (2023) argue that the purely algebraic apparatus employed by Cauchy eventually became an obstacle for the production of mathematical knowledge in the field of complex function theory. Since in due course Cauchy had to incorporate figures with a counterpart in an algebraic setting (second category of REM) to further develop complex analysis.

We now turn to an analysis of the proofs of Cauchy's integral theorem presented in the four textbooks. Marsden and Hoffman (1999) prove Cauchy's integral theorem using the first, third, and fourth approximations listed in Table 3, while Conway (1973), Ahlfors (1979), and Rudin (1987) use the third and fourth approximations. We took the decision to analyze only the proofs based on the third approximation because, apart from the fact that it relies on the use of figures, this way of proving the theorem is suitable for analysis by the REM. We claim this because the fourth approximation of Table 3 uses figures based on topological ideas such as homotopy and homology, which are more in line with the proof of the theorem provided by Dixon (1971). That is to say, the REM was configured from five original works spanning from the first quarter of the 19th century to the first half of the 20th century, but it does not take into account topological forms of mathematical activity.

Nevertheless, not all the textbook authors use the same type of figures to prove Cauchy's integral theorem via the third approximation in Table 3. On the one hand, Ahlfors (1979) and

Marsden and Hoffman (1999) use rectangles, while Rudin (1987) and Conway (1973) use triangles. The authors who use rectangles state that their proofs are based on the mathematical work of Goursat. Marsden and Hoffman even mention Goursat's work from 1884 paper, analyzed by Piña-Aguirre and Farfán (2023) from which we recover the REM.

Ahlfors and Hoffman and Marsden prove the theorem for the special case where the integration curve γ is a rectangle. This rectangle is not represented by algebraic expressions, which means that the authors use figures with epistemic value (third category of the REM) to prove the theorem. The use of figures without a homologous representation in an algebraic apparatus is a recurring argument in their proofs, because the authors subdivide the region encompassed by the γ curve via the use of rectangles without algebraic representations (Figure 2). Specifically, by using a sequence of nested γ_i rectangles, Ahlfors and Marsden and Hoffman find upper bounds for the modules of the integrals over the γ_i rectangular curves by finding the length of the diagonal of each γ_i rectangle. This allows them to bound the value of the integral over the γ curve by an arbitrary and positive quantity ε, to conclude that

$$\forall \varepsilon > 0, \left| \int_\gamma f \, d\gamma \right| < \varepsilon$$

As shown below, these types of arguments were used by Goursat in his original 1884 paper.

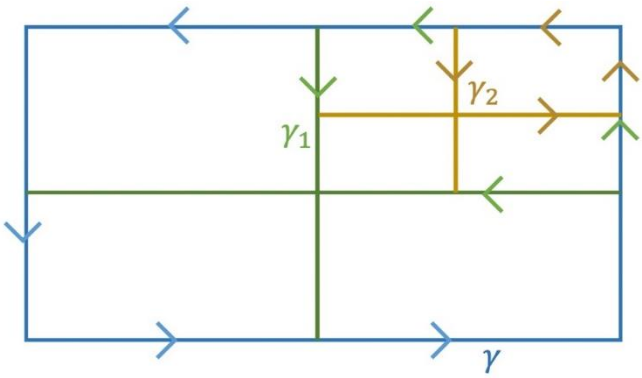

Figure 2.

Subdivision process of the rectangular curve used by Ahlfors and by Marsden and Hoffman (Own elaboration)

According to Piña-Aguirre and Farfán (2023), Goursat proves that the value of the integral $\int_\gamma f \, d\gamma$ is zero by an argument that requires a subdivision of the region bounded by the curve γ by γ_i curves, which are used to traverse the entire γ curve. As shown in Figure 3, the γ_i curves are of two types, but each type of γ_i curve coincides with or can be contained within a square. This allows to obtain an upper bound for the value of the modulus of the integral over γ by finding the longest distance between any two points within or over the γ_i squares that are completely contained within the γ curve, or alternatively, over the squares that encompass the γ_i curves that are not completely contained within the γ curve.

By finding an upper bound for the module of the integrals over the γ_i curves, Goursat bounds the module of the integral over the γ curve by an arbitrary and positive quantity ε, which implies that $\int_\gamma f\, d\gamma = 0$.

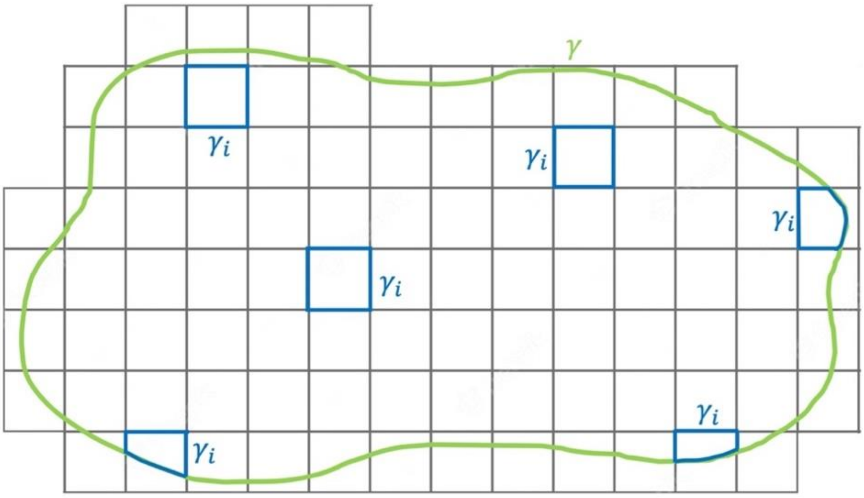

Figure 3.

Type of γ_i curves used by Goursat to traverse the integration curve γ (Own elaboration)

It should be noted that there are other similarities between the proofs of Ahlfors and Marsden or Hoffman and the proof of Goursat. For example, the proofs use linear approximations and the concept of limit as it is known today. However, the main difference between the two proofs is that in the textbooks the authors assume that the integration γ curve is a rectangle, while in Goursat's proof this curve is arbitrary.

The reason why Ahlfors and Marsden and Hoffman prove Cauchy's integral theorem for the case of a rectangle is that the authors follow a series of implications that start with this version of the theorem and end with a proof that relies on arguments based on topological concepts. For example, Ahlfors states that "there are several forms of Cauchy's theorem, but they differ in their topological rather than in their analytical content. It is natural to begin with a case in which the topological considerations are trivial" (p. 109). Meanwhile, the rectangular version of the theorem allows Marsden and Hoffman to prove that the theorem is valid for an open disk, which in turn allows them to extend the result of the theorem for simple and multiple connected regions in which the function f is analytic.

We believe that the textbook authors rely on topological arguments because they draw their ideas from the latest lines of development that allowed the theorem to be proved. We conceive that these latest lines of development occurred throughout history due to various necessities that arose in the development of mathematics as a scientific discipline. For instance, according to Bak and Popvassilev (2017), the proof of the theorem led to the realization that not all continuous curves can be partitioned as Goursat had imagined, which led Pringsheim (1903) to replace the proof presented by Goursat for a different type of integration curves.

In this order of ideas, Gray (2000) states that Pringsheim's proved Cauchy's integral theorem for the case when γ is the boundary of a triangle, which is a case covered in the proofs of Conway (1973) and Rudin (1987). The arguments used by these textbook authors are based on the idea that by using a sequence of nested triangles γ_i (Figure 4), they can prove that the integral over a triangular curve γ is zero. Because the module of the integrals over the γ_i curves allow to bound (by an arbitrary and positive quantity ε) the module of the integral over the triangular curve γ.

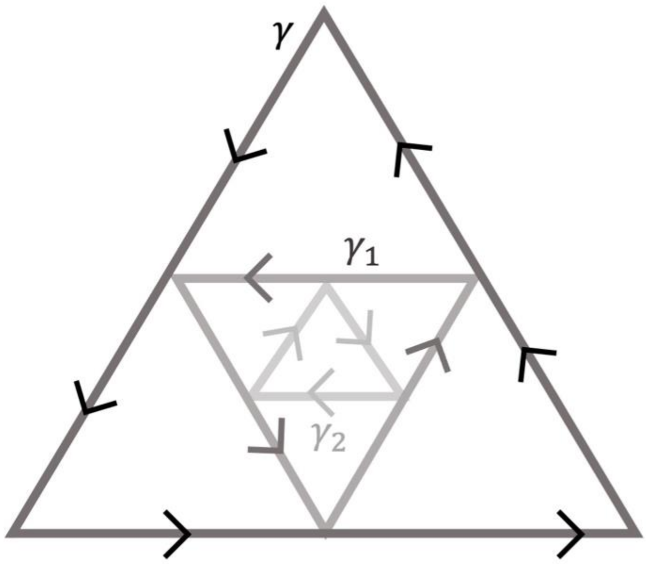

Figure 4.

Subdivision process of a triangular curve used by Conway and Rudin (Own elaboration)

That is, the only difference between how Conway and Rudin prove the theorem and how Ahlfors and Marsden and Hoffman prove it is the type of figures they use. However, they use them in the same way, in the sense that the figures (without a homologous representation in an algebraic setting) allow them to bound the modulus of the integral over the integration curve γ.

Concluding remarks

In this research, a reference epistemological model of complex analysis, configured on the basis of the mathematical activity employed by historical subjects, allowed us to study how figures are used in modern complex analysis textbooks to deal with the complex integral concept and the proof of Cauchy's integral theorem. Clark (2019) points out that studying the “possible parallelism between the historical development and the cognitive development of mathematical ideas” (p. 33) is one of the main topics in the context of research about the relationship between the history of mathematics and mathematics education. We believe that this paper makes a valuable contribution in this line of research.

The aim of our study was to understand differences and similarities between original mathematical works in complex analysis and contemporary textbooks. To do so, we analyzed how the different ways in which historical subjects did mathematics were modified in order to address the concept of complex integration and to prove Cauchy’s integral theorem in complex analysis textbooks. As shown in the analysis of the concept of complex integration, the way textbook authors introduce this concept differs from the way Cauchy treats it in his 1825 memoir. Meanwhile, the main difference between how Goursat proved the theorem in 1884 and how this result is proved in the textbooks is that Goursat considers an arbitrary integration path, while the textbook authors use triangular of rectangular integration paths.

In particular, the textbook analysis of the complex integral concept revealed that figures were used to define this concept via integration paths. We argue that by doing so, the authors are able to circumvent the obstacles that Cauchy may have faced when trying to develop the theory of complex valued functions. In Piña-Aguirre and Farfán (2023), it is conjectured that by facing these obstacles, Cauchy eventually recognized that a purely algebraic apparatus could be complemented by the use of figures as a means of mathematical justification in order to deal with some concepts in complex analysis. On the other hand, the textbook analysis of the proof of Cauchy’s integral theorem made us realize that Goursat’s use of figures is similar to the way the textbook authors prove Cauchy’s integral theorem. In the sense that figures allow them all to find upper bounds for the value of the modulus of some complex integrals.

Therefore, as future work, we would like to understand what didactic phenomena arise when undergraduate mathematics students follow the gradual incorporation of figures depicted in the REM, without the need to instruct them on how to use figures as a means of mathematical justification. To do so, we will configure a series of tasks based on the results of this study. In the sense that some tasks will bring into play the complex integral concept, without the need to bypass the obstacles that Cauchy had to affront in order to develop complex analysis, to study how the students’ transition from the first to the second category of the reference epistemological model. Then, some other tasks will be configured with the aim of understanding how they use figures, as depicted in the third category of the REM to find upper bounds to specific integrals that include a variety of types of integration curves.

We hope that the identification of the didactic phenomena associated with the transition from one category to the next will allow us to enrich the reference epistemological model with empirical data. Specifically, we expect to enrich it by incorporating some methodological tools from the Socioepistemological framework (Cantoral, 2020) for the analysis of the tasks. This framework recognizes the act of doing mathematics as a purely human act, and therefore it allows to study how human beings do and use mathematics in all kinds of contexts, without having to focus only on their final mathematical productions. We consider that by having an enriched reference epistemological model, this model can be considered as an epistemological alternative for the elaboration of didactic material that takes into account how students progressively complement their mathematical justifications in complex analysis with the gradual introduction of figures.

Another route of development that involves the notion of REM as a way to attend to complex analysis in teaching and learning scenarios is the following. To construct a more general epistemological model, that includes the enriched REM previously stated, we can recover results that have been obtained by other researchers in mathematics education who are interested in how figures allow to address different concepts in complex analysis.

All this being said, we would like to emphasize that we are aware of a limitation that we encountered when using the reference epistemological model to analyze the complex analysis textbooks. Since this epistemological model is based on the mathematical activity of historical subjects who did not prove the theorem through topological aspects, we decided not to analyze the textbook proofs that use this type of argument. Nevertheless, this limitation allows us to propose a future study that analyzes the mathematical activity of the original works that first introduced topological concepts in order to prove Cauchy's integral theorem, in order to further enrich the reference epistemological model and consequently analyze the textbook proofs of the theorem that use topological concepts.

As a final comment of this study, we think that no matter how we extend our reference epistemological model, we have to ask ourselves how one real variable calculus can be attended in such a way that when dealing with complex analysis concepts, through the use of figures, textbook readers realize that these two branches of mathematics are related in more ways than just by the similar structuring of their concepts in different textbooks (Figure 1), but without falling into erroneous conceptions like the one about complex and real differentiation that Soto-Johnson and Hancock (2019) disproved.

References

Ahlfors, L. (1979). *Complex Analysis. An introduction to the theory of analytic functions of one complex variable*. McGraw-Hill.

Bak, J., & Popvassilev, S. (2017). The Evolution of Cauchy's Closed Curve Theorem and Newman's Simple Proof. *The American Mathematical Monthly, 124*(3), 217–231. https://doi.org/10.4169/amer.math.monthly.124.3.217

Bosch, M., & Gascón, J. (2006). Twenty-five years of the didactic transposition. *ICMI bulletin*, *58*(58), 51–65.

Bottazzini, U., & Gray, J. (2013). *Hidden Harmony –Geometric Fantasies. The Rise of Complex Function Theory*. Springer.

Cantoral, R. (2020). Socioepistemology in mathematics education. In *Encyclopedia of mathematics education* (pp. 790–797). Cham: Springer International Publishing. https://doi.org/10.1007/978-3-030-15789-0_100041

Cantoral, R., & Farfán, R. (2004). La sensibilité à la contradiction: logarithmes de nombres négatifs et origine de la variable complexe. *Recherches en Didactique des mathématiques*, *24*(2-3), 137–168.

Cauchy, A. (1825). *Mémoire sur les intégrales définies, prises entre des limites imaginaires*. Bure frères.

Clark, C.M. (2019). History and pedagogy of mathematics in mathematics education: History of the field, the potential of current examples, and directions for the future. In U.T. Jankvist, M. Van den Heuvel-Panhuizen, & M. Veldhuis, (Eds.), P*roceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (pp. 29–55). Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Conway, J. (1973). *Functions of one complex variable*. Springer.

D'Azevedo-Breda, A., & Dos Santos, J. (2021). Learning complex functions with GeoGebra. *Revista de la Academia Colombiana de Ciencias Exactas, Físicas y Naturales*, *45*(177), 1262– 1276. https://doi.org/10.18257/raccefyn.1504

Danenhower, P. (2000). *Teaching and learning complex analysis at two British Columbia Universities* [Doctoral dissertation, Simon Fraser University]. https://www.collectionscanada.gc.ca/obj/s4/f2/dsk1/tape3/PQDD_0008/NQ61636.pdf

Dittman, M., Soto-Johnson, H., Dickinson, S., & Harr, T. (2016). Game building with complex-valued functions. *Primus*, *27*(8-9), 869–879. https://doi.org/10.1080/10511970.2016.1234527

Dixon, J. D. (1971). A brief proof of Cauchy's integral theorem. *Proceedings of the American Mathematical Society*, *29*(3), 625–626.

Fan, L. (2013). Textbook research as scientific research: towards a common ground on issues and methods of research on mathematics textbooks. *ZDM Mathematics Education*, *45*, 765–777. https://doi.org/10.1007/s11858-013-0530-6

Fan, L., Zhu, Y., & Miao, Z. (2013). Textbook research in mathematics education: development status and directions. *ZDM Mathematics Education, 45,* 633–646. https://doi.org/10.1007/s11858-013-0539-x

Garcia, S., & Ross, W. (2017). Approaching Cauchy's Theorem. *PRIMUS, 27*(8-9), 758–765. https://doi.org/10.1080/10511970.2016.1234525

Gascón, J. (2014). Los modelos epistemológicos de referencia como instrumentos de emancipación de la didáctica y la historia de las matemáticas. *Educación Matemática, 25 años*, 99–123.

Gómez, B. (2011). El análisis de manuales y la identificación de problemas de investigación en Didáctica de las Matemáticas. *PNA*, *5*(2), 49–65. https://doi.org/10.30827/pna.v5i2.6157

Goursat, E. (1884). Démonstration du théoréme de Cauchy: Extrait d'une lettre adressée à M. Hermite. *Acta Mathematica*, *4*, 197–200. https://doi.org/10.1007/BF02418419

Grattan-Guinness, I. (2004). History or Heritage? An Important Disctinction in Mathematicsa and for Mathematics Education. *The American Mathematical Monthly*, *111* (1), 1–12. https://doi.org/10.1080/00029890.2004.11920041

Gray, J. (2000). Goursat, Pringsheim, Walsh, and the Cauchy integral theorem. *The Mathematical Intelligencer*, *22*(4), 60–66. https://doi.org/10.1007/BF03026773

Hanke, E. (2022). *Aspects and images of complex path integrals* [Doctoral thesis, University of Bremen]. https://doi.org/10.26092/elib/1964

Larivière, G. (2014). On Cauchy's Rigorization of Complex Analysis [Master's thesis, Simon Fraser University]. https://summit.sfu.ca/item/17300

Manning, K. (1975). The emergence of the Weierstrassian approach to complex analysis. *Archive for History of Exact Sciences*, *14*(4), 297–383. https://doi.org/10.1007/BF00327297

Marsden, J., & Hoffman, M. (1999). *Basic complex analysis*. W. H. Freeman.

Nemirovsky, R., Rasmussen, C., Sweeney, G., & Wawro, M. (2012). When the Classroom Floor Becomes the Complex Plane: Addition and Multiplication as Ways of Bodily Navigation. *Journal of the Learning Sciences*, *21*(2), 287–323. https://doi.org/10.1080/10508406.2011.611445

Oehrtman, M., Soto-Johnson, H. & Hancock, B. (2019). Experts' Construction of Mathematical Meaning for Derivatives and Integrals of Complex-Valued Functions. *Int. J. Res. Undergrad. Math. Ed.*, *5*(3), 394–423. https://doi.org/10.1007/s40753-019-00092-7

Panaoura, A., Elia, I., Gagatsis, A., & Giatilis, G.P. (2006). Geometric and algebraic approaches in the concept of complex numbers. *International Journal of Mathematical Education in Science and Technology*, *37*(6), 681–706. https://doi.org/10.1080/00207390600712281

Piña-Aguirre, J. G., & Farfán, R. (2022). Sobre los procesos de demostración y el contexto de producción de la memoria Sur les Intégrales Définies de Augustin-Louis Cauchy. *Revista De História Da Educação Matemática*, *8*, 1–22.

Piña-Aguirre, J. G., & Farfán, R. (2023). What enabled the production of mathematical knowledge in complex analysis?. *International Electronic Journal of Mathematics Education*, *18*(2), em0734. https://doi.org/10.29333/iejme/12996

Ponce, J.C. (2019). The use of phase portraits to visualize and investigate iso-lated singular points of complex functions. *International Journal of Mathematical Educationin Science and Technology*, *50*(7), 999–1010. https://doi.org/10.1080/0020739X.2019.1656829

Pringsheim, A. (1903). Der Cauchy-Goursat'sche Integralsatz und seine Übertragung auf reelle Kurven-lntegrale. Sitzungsberichte der math-phys. *Classe der Königliche Akademie der Wissenschaften zu München*, *33*, 673–682.

Rudin, W. (1987). *Real and complex analysis*. McGraw-Hill.

Schubring, G., & Fan, L. (2018). Recent advances in mathematics textbook research and development: an overview. *ZDM Mathematics Education*, *50*, 765–771. https://doi.org/10.1007/s11858-018-0979-4

Scott, J. (1990). *A matter of record, documentary sources in social research*. Polity Press.

Smithies, F. (1997). *Cauchy and the creation of complex function theory.* Cambridge University press. https://doi.org/10.1017/CBO9780511551697

Soto-Johnson, H., & Hancock, B. (2019). Research to Practice: Developing the Amplitwist Concept. *PRIMUS*, *29*(5), 421–440. https://doi.org/10.1080/10511970.2018.1477889

Soto-Johnson, H., & Oehrtman, M. (2022). Undergraduates' Exploration of contour integration: What is accumulated? *The Journal of Mathematical Behavior, 66*, 100963. https://doi.org/10.1016/j.jmathb.2022.100963

Spivak, M. (1994). *Calculus.* Publish or Perish.

Troup, J., Soto, H. & Kemp, A. (2023). Developing Geometric Reasoning of the Relationship of the Cauchy Riemann Equations and Differentiation. *Int. J. Res. Undergrad. Math. Ed.* Advanced online publication. https://doi.org/10.1007/s40753-023-00223-1

Van Dormolen, J. (1986). Textual analysis. In B. Christiansen, A. G. Howson & M. Otte (Eds.), *Perspectives on Mathematics Education* (pp. 141–171). Reidel. https://doi.org/10.1007/978-94-009-4504-3_4

Zill, D., & Shanaha, P. (2013). *A first course in complex analysis with applications.* Jones and Bartlett Publishers