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Transpositional connections from the perspective of developing reference epistemological models based on school mathematics objects

Conexiones transposicionales desde la perspectiva del desarrollo de modelos epistemológicos de referencia basados en objetos matemáticos escolares

Les connexions transpositionnelles dans la perspective de l'élaboration de modèles de référence épistémologiques à partir d'objets mathématiques scolaires

Conexões transpositivas na perspectiva da elaboração de modelos epistemológicos de referência a partir de objetos da matemática escolar

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Resumo

Nosso objetivo neste artigo é expor ideias vinculadas a alguns objetos da matemática escolar que revelem conexões transpositivas pertinentes à elaboração de modelos epistemológicos de referência, vinculando-os às noções de objetos matemáticos do cálculo diferencial e integral. O aporte teórico centra-se na transposição didática e teoria antropológica do didático. Mostramos como alguns objetos da matemática escolar estão no foco das noções dos objetos do cálculo diferencial e integral, além de evidenciarmos conexões transpositivas existente entre a matemática escolar e a matemática do ensino superior. Nessas conexões transpositivas residem possibilidades para a elaboração de modelos epistemológicos de referência que tornem mais compreensíveis o ensino dos objetos do cálculo diferencial e integral.

Palavras-chave: Teoria antropológica do didático, Matemática escolar, Cálculo diferencial e integral, Modelo epistemológico de referência.

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Abstract

Our aim in this article is to present ideas linked to some objects of school mathematics that reveal transpositional connections pertinent to the development of epistemological reference models, linking them to the notions of mathematical objects of Differential and Integral Calculus. The theoretical framework centers on didactic transposition and the anthropological theory of the didactic. We show how some of the objects of school mathematics are the focus of the notions of the objects of differential and integral calculus, as well as highlighting the transpositional connections between school mathematics and higher education mathematics. In these transpositive connections lie possibilities for the development of reference epistemological models that make the teaching of the objects of differential and integral calculus more comprehensible.

Keywords: Anthropological theory of the didactic, School mathematics, Differential and integral calculus, Reference epistemological model.

Resumen

Nuestro objetivo en este artículo es presentar ideas vinculadas a algunos objetos matemáticos escolares que revelan conexiones transposicionales pertinentes para el desarrollo de modelos epistemológicos de referencia, vinculándolos a las nociones de objetos matemáticos en cálculo diferencial e integral. El marco teórico se centra en la transposición didáctica y en la teoría antropológica de lo didáctico. Mostramos cómo algunos objetos matemáticos escolares son el foco de las nociones de los objetos del cálculo diferencial e integral, además de destacar las conexiones transpositivas entre las matemáticas escolares y las matemáticas de la enseñanza superior. En estas conexiones transpositivas residen posibilidades para el desarrollo de modelos epistemológicos de referencia que hagan más comprensible la enseñanza de los objetos del cálculo diferencial e integral.

Palabras clave: Teoría antropológica de lo didáctico, Matemáticas escolares, Cálculo diferencial e integral, Modelo epistemológico de referencia.

Résumé

Notre objectif dans cet article est d'exposer les idées liées à certains objets mathématiques scolaires qui révèlent des connexions transpositionnelles pertinentes pour le développement de modèles épistémologiques de référence, en les reliant aux notions d'objets mathématiques du calcul différentiel et intégral. Le cadre théorique est centré sur la transposition didactique et la théorie anthropologique du didactique. Nous montrons comment certains objets des mathématiques scolaires sont au centre des notions d'objets du calcul différentiel et intégral, et nous soulignons les liens transpositifs qui existent entre les mathématiques scolaires et les mathématiques de l'enseignement supérieur. Dans ces connexions transpositives se trouvent des possibilités de développement de modèles épistémologiques de référence qui rendent l'enseignement des objets du calcul différentiel et intégral plus compréhensible.

Mots-clés : Théorie anthropologique du didactique, Mathématiques scolaires, Calcul différentiel et intégral, Modèle épistémologique de référence.

Transpositive connections from the perspective of developing reference epistemological models based on school mathematics objects

School mathematics[4] (Moreira & David, 2003; Valente, 2012) has transpositive aspects regarding wise (academic) mathematical knowing (Chevallard, 1982, 1991). We add to this idea that the teaching of school mathematics objects involves mathematical modeling, a type of written discourse that becomes the content of didactic books or textbooks aimed at initial or continuing teacher education (Nacarato, 2006; Pereira, 2012, 2017; Valente, 2022).

The personalization of knowledge to teach, we believe, occurs through a non-banal transpositive phenomenon and requires attention to the principle of epistemological surveillance (Chevallard, 1991). This epistemological surveillance serves as a "magnifying glass" to visualize the text of wise knowing and the text of knowing rewritten for school mathematics or the text prepared by the mathematics teachers themselves, aiming at teaching a particular mathematical object (Almouloud, 2022).

The cultural production of mathematical knowing –let us call it the rustic epistemological model– is associated with a Stone Age timeline (Eves, 2011). Mathematical knowing changes according to the social transformations of human civilization, especially when it comes to Greek civilization (Eves, 2011; Mendes, 2022). However, this article does not aim to discuss the historical evolution of mathematics. Instead, we want to outline ideas that dialogue with the didactic transposition (Chevallard, 1982, 1991) and the anthropological theory of the didactic (ATD) (Chevallard, 1988, 1992, 1997, 1999, 2002; Bosch & Chevallard, 1999; Gascón, 2011, 2018; Delgado, 2006; Pereira, 2012, 2017).

Thus, we want to expose ideas linked to some objects of school mathematics that reveal transpositive connections pertinent to the elaboration of reference epistemological models, linking them to the notions of mathematical objects of differential and integral calculus.

Mesogenesis and Praxeological Organizations

Our foray into praxeological organizations sets off in the mesogenesis of the didactic *milieu.*[5] Mesogenesis is the "genesis of didactic *milieu*, that is, of the system of resources used in the process of praxeological construction" (Chevallard, 2009a, p. 3, our translation), which means that didactic transposition delves into the anthropological theory of the didactic (ATD),

[4] We will assume, in part, in this article, what is described by Moreira and David (2003, p. 72): "Thus, school mathematics, seen as a result of the teacher's practice at school and not as a list of contents to be taught, must also incorporate this critical retranslation of knowledge operated by the teacher [...]".
[5] The word *milieu* is translated into English as **medium**, but in this article, we prefer the French word, as it has a broader meaning.

and the transpositive phenomena are structured based on the fundamental elements of the ATD: types of tasks, techniques, technologies, and theories (Chevallard, 1999, 2009a, 2009b). Those fundamental elements are grouped into two blocks: praxis or know-how-to-do and logos or knowing.

The praxis block has at least one type of task T (capital tau) and a technique τ (lowercase tau), a block denoted as [T/τ] (Chevallard, 1999; Bosch & Chevallard, 1999). By extension, the logos block is formed by technology θ (lowercase theta) and by theory Θ (capital theta), denoted as follows: [θ/Θ] (Chevallard, 1999; Bosch & Chevallard, 1999). The combination of these two blocks constitutes the unique praxeological organization[6] (UPO) or unique mathematical organization (UMO), represented by the quartet [T / τ / θ / Θ][7] (Chevallard, 1999). The essence of this type of UMO is tasks t belonging to T ($t \in T$). Tasks t require a technique τ that solves them, but this technique needs a discourse that makes it intelligible or rational, showing how technique τ operates to obtain the solutions of tasks t (Chevallard, 1999). However, for technology $\theta$to exist, it needs a more elaborate justifying and intelligible discourse, i.e., theory Θ (Chevallard, 1999).

A UMO is a starting point for developing a reference epistemological model (REM) (Delgado, 2006; Pereira, 2012). However, this REM is not limited to one UMO; several must exist, which gives rise to local mathematical organizations. An LMO has at least two UMOs; that is, there is more than one type of task T (task types) T_i = 1, 2, 3, 4, 5, ..., n), which requires more than one technique τ to solve t_i (i = 1, 2, 3, 4, ..., n). An LMO is denoted by [T_i / τ_i / θ /Θ] (Chevallard, 1999).

We understand that the UMOs and LMOs are the bases for developing reference epistemological models. From Gascón (2011, p. 208, our translation), we abstract that the REM "[...] has an *always temporary* nature. Based on the REM, teaching can *deconstruct* and *reconstruct* the praxeologies whose intra-institutional and inter-institutional diffusion it intends to analyze". Thus, we see that deconstructing and reconstructing UMOs and LMOs is uncomplicated, and the didactic transposition goes through mesogenesis.

The sophistication of a REM is embodied from the mesogenesis of regional mathematical organizations, which begins to exist when the LMOs expand and group together around technologies θ_j (j = 1, 2, 3, 4, 5, ..., n). Simply put, two LMOs already indicate the

[6] In a PMO, the applicability of the technique is restricted to the set of tasks of the same type; that is, it is not satisfactory for tasks that require another way of resolution.
[7] They are letters of the Greek alphabet.

possible embryonic existence of a regional mathematical organization (RMO), but the consolidation of this idea becomes explicit when the praxeological analysis shows the existence of groupings of several LMOs, and, for each LMO, there is a justification discourse for techniques τ_{ij}. The representation $[T_{ij} / \tau_{ij} / \theta_j / \Theta]$ denotes an RMO (Chevallard, 1999).

The highest degree of a praxeological organization occurs at the level of global mathematical organizations (GMO). For this to happen, several RMOs are grouped around theories Θ_k (k = 1, 2, 3, 4, 5, ..., n). A GMO is denoted by $[T_{ijk} / \tau_{ijk} / \theta_{jk} / \Theta_k]$ (Chevallard, 1999). GMOs, in general, are in higher education, such as the praxeological organizations of differential and integral calculus, linear algebra, and real analysis. If a REM reaches the praxeological structure of a GMO, we see that the mesogenesis is highly sophisticated and the potential of this REM turns to the analysis of the current dominant model of mathematics taught in higher education (Matos, 2017).

Praxeological Organizations and School Mathematics from the Perspective of the Development of Reference Epistemological Models

In the previous section, we presented a discourse that characterizes the types of praxeological organizations (POs) according to the ATD theory. Furthermore, we intersperse ideas that converge toward developing reference epistemological models. In this section, we will describe ideas that connect to the purpose of the objective announced in the introductory text and complement what we explained in the previous section. Thus, our focus in this section is on some objects of school mathematics that reveal praxeological possibilities for the elaboration of reference epistemological models with specific transpositive connections to the objects of differential and integral calculus.

In this assumption of transpositive connections, we have the basic arithmetic of the base-10 positional numbering system or Hindu-Arabic positional numbering system (we will adopt the name decimal numbering system (DNS) in this text). This numbering system is the basis of mathematical objects for the initial years of elementary school and continues until the final years of high school (Brasil, 2018). The DNS has several epistemological developments in school mathematics (Pereira, 2012) and a dominant epistemological model (DEM)[8] (Gascón, 2011, 2014; Pereira, 2017; Almouloud, 2022) in force in mathematics textbooks of the National Book and Teaching Material Program (PNLD), from 2023, for the initial years, linked to the

[8] [...] the *dominant epistemological model* of a certain branch of mathematical knowledge taught (in a given institution) strongly conditions not only the type of mathematical activities that will be possible to carry out in that institution around the mathematical branch in question, but also the corresponding didactic activities that materialize in a teaching model [...] (Gascón, 2014, p. 108, our translation, author's emphasis).

most manageable notions of natural numbers (Giovanni Junior, 2021; Centurión, Teixeira & Rodrigues, 2021).

In this discourse, we ask: What is the relationship between the DNS and differential and integral calculus teaching? To answer this question, we do not need to delve into the epistemological historicity of the DNS, but we must understand that this numbering system has a *habitus*[9] from practice (Bourdieu, 2013) with a particular didactic transposition in school mathematics itself. What we mean by this is that from the initial years, the development of the student's cognitive ability to learn to calculate is encouraged, so it is expected that when they reach higher education, they will be able to perform much more complex calculations. However, higher education newcomers do not express this expectation and have difficulty learning mathematics (Masola & Allevato, 2016).

We understand that calculations in school mathematics are very different from calculus as a branch of mathematics in higher education. In school mathematics, there is a practice of praxeological organizations often associated with arithmetic operations and resolutions of equations, whereas, in higher education, calculus transcends to complex praxeological organizations that require analytical study of functions, linear algebra, analytical geometry, differential geometry, among others. However, as in the DNS, we see that the basic notions of school mathematics are relevant when teaching the calculation of limits, derivatives, and integrals.

When viewed in a sophisticated REM, the DNS allows us to understand the epistemological potential for other mathematical objects, including single-variable polynomials (Pereira, 2012; Pereira & Nunes, 2017). Polynomials are essential to learning differential and integral calculus, especially in the study of functions (Meneghetti, Rodriguez & Pofffal, 2017). To illustrate the ideas of a REM based on the DNS, we quote Pereira and Nunes (2017, p. 262, authors' emphasis):

> Let us take the integer – 456. If we wanted to generate a polynomial from it, we would first have to rewrite it like this: **(–1) (456)**. From this writing, other representations then emerge: $\mathbf{(-1)\ (400 + 50 + 6) = (-1)\ (4 \cdot 10^2 + 5 \cdot 10 + 6) = -\,4 \cdot 10^2 - 5 \cdot 10 - 6}$. Replacing 10 with x, we then have: $\boldsymbol{-\,4x^2 - 5x - 6}$. The technology θ_k to represent an integer N_k (k = 1, 2, 3, ..., n), in the base ten power polynomial writing is $N_k = (\pm\ 1)\ (a_0 + a_1 \cdot 10 + a_2 \cdot 10^2 + a_3 \cdot 10^3 + ... + a_n \cdot 10^n)$, $0 \leq a_i < 10$, i = 0, 1, 2, ..., n. This technology includes natural numbers since the set of natural numbers is contained in the set of integers ($\mathbb{N} \subset \mathbb{Z}$). We assume that in the polynomial $4x^4 - 6x^3 + 5x^2 - 8x - 2$, the technology θ_k allows us to

[9][...] systems of durable and transposable provisions, structured structures predisposed to function as structuring structures, i.e., as generating and organizing principles of practices and representations [...] (Bourdieu, 2013, p. 87).

think of this polynomial as the result of $(4x^4 – 6x^3 + 5x^2 + 0x + 0) + (0x^4 + 0x^3 + 0x^2 – 8x –2) = (4 +0) x^4 + (– 6 + 0) x^3 + (5 + 0) x^2 + (0 –8) x + (0 –2)$.

When we bring the DNS to the praxeologies of differential and integral calculus, we can think of the situation of having the formation law function $f(x) = x^4 + 1$, and we want the first derivative of this function. If we associate N_k technology, which is more complete, to the polynomial of the function defined by $f(x)$, i.e., $N_k = (\pm 1) (a_0 \cdot 10^0 + a_1 \cdot 10^1 + a_2 \cdot 10^2 + a_3 \cdot 10^3 + ... + a_n \cdot 10^n)$, $0 \leq a_i < 10$, $i = 0, 1, 2, ..., n$; we obtain $f(x) = 1 \cdot x^0 + 0 \cdot x^1 + 0 \cdot x^2 + 0 \cdot 10^3 + 1 \cdot x^4$. In short, we have $f(x) = 1 \cdot x^0 + 1 \cdot x^4$. Applying Lagrange's notation (1806) and the practical technique for the first derivative, we have that $f'(x) = (0 \cdot 1) \cdot x^{0-1} + (4 \cdot 1) \cdot x^{4-1} = 0 \cdot x^{-1} + 4 \cdot x^3 = 0 + 4x^3 = 4x^3$. Therefore, $f'(x) = 4x^3$. This praxeological modeling explains why the derivative of a constant equals zero: $f(x) = c = cx^0$, which gives rise to $f'(x) = 0$.

Evolving the ideas, we moved from the DNS to the DEM of polynomials in school mathematics, i.e., the DEM in textbooks and taught in schools. The first praxeological model taught concerns algebraic expressions with first-degree polynomials (Bianchini, 2022). In general, the tasks involve calculating numerical values or reducing similar terms to arrive at the reduced form $ax + b$, and then comes the types of tasks with first-degree polynomial equations (Bianchini, 2022).

The development of the reference epistemological models (REMs) that deal with the object of first-degree polynomial equations, as Almeida (2017) did, has diverse praxeological potential, both for teaching school mathematics objects and for teaching differential and integral calculus objects. The REM to which we refer is supported by the anthropological theory of the didactic and Gascón's (2014, p. 106, our translation, author's emphasis) perceptions:

> [...] the reference epistemological models (from now on, REM) produced within the scope of the ATD can be considered as ideal types that have allowed the emancipation of the didactics of mathematics regarding dominant epistemological models in the various institutions that are part of its object of study and, thanks to its phenomenotechnical function[10], the REMs have made new didactic phenomena visible.

The reduced form $ax + b$ is used to compose the polynomial equation $ax + b = c$ ($a, b,$ *and* $c \in \mathbb{R}; a \neq 0$) (Briant, 2013). This equation in the DEM of school mathematics is modeled by making c = 0, which results in $ax + b = 0$ ($a, b \in \mathbb{R}; a \neq 0$). Thus, we ask: What is the most appropriate model in a REM to articulate it with the teaching of differential and integral

[10] [...] in the sense of "manufacturing" objects of knowledge [...] (Silva, 2017, p. 125).

calculus? Our perception of this question falls under the ostensive[11] $ax + b = c$ and its algebraic variations because b and c can assume not only numerical values but also other polynomials; for example, the ostensive $b = dx + e$ and $c = fx + g$. Thus,

$ax + b = c \Rightarrow ax + (dx + e) = fx + g \Leftrightarrow ax + dx - fx = g - e \Leftrightarrow x\,(a + d - f) = (g - e) \Leftrightarrow x = \frac{(g-e)}{(a+d-f)}$.

In an ostensive manipulation $ax + (dx + e) = fx + g$, we had to resort to one of the cases of polynomial factoring –factoring with a common factor. Polynomial factoring in school mathematics can assist with tasks such as calculating function limits. To illustrate this, let us look at the task of calculating limits at infinity, shown in Figure 1.

$$\boxed{\text{Encontre } \lim_{x\to\infty} \frac{x^2+x}{3-x}.}$$

Figure 1.

Example 11 (Stewart, Clegg, & Watson, 2022, p. 121)

The task in Figure 1 has both ostensive and non-ostensive objects ([12]Chevallard, 1994) that must be well understood in teaching differential and integral calculus. Still, our perception is that the basis begins in school mathematics because $x^2 + x$ is a second-degree polynomial that can be factorized by a common factor: $x\,(x + 1)$. This means that in Figure 1, we have the ostensive representation of the quadratic polynomial function $f(x) = x^2 + x = x(x + 1)$. Furthermore, the first-degree polynomial, $3 - x = -x + 3$, is of the form $ax + b$ of the dominant epistemological model (Almouloud, 2022) of school mathematics. Thus, in Figure 1, we also have the first-degree polynomial function $g(x) = 3 - x = -x + 3$. These two types of polynomial functions constitute praxeological organizations (Chevallard, 1999), which we see as necessary for teaching mathematics in higher education, based on aspects of didactic transposition, i.e., the "*'Objets de Savoir' et Autres Objets*" (Chevallard, 1991, p. 49).

[11] Objects that have a material, sensitive form for us, whatever it may be, are called ostensive. A material object (a pen, a compass, etc.) is an ostensible [...]. The characteristic of ostensives is that they can be *manipulated*. This word must be understood in a broad sense: manipulation in the strict sense (with the compass or the pen), but also through the voice, the look, etc. (Chevallard, 1994, p. 4-5, our translation).

[12] [...] the *non-ostensives* – usually called a notion, concept, idea, etc. – cannot, strictly speaking, be manipulated: they can only be evoked through the associated ostensives. Thus, when we say that, to solve equation $2^x = 10$ "we take the logarithm of both members", it is convenient that the *non-ostensive* concept of logarithm exists [...] (Chevallard, 1994, p. 5, our translation, authors' emphasis).

The task in Figure 1 motivates the development of reference epistemological models that deal with objects related to the study of polynomial functions, an example of which is shown by Almeida (2017) and Figueiredo et al. 2023).

Let us look at another differential and integral calculus task from Anton, Bivens, and Davis (2014), shown in Figure 2.

$$\text{Encontre } \lim_{x \to 1} \frac{x-1}{\sqrt{x}-1}.$$

Figure 2.

Example 10 (Anton, Bivens, & Davis, 2014, p. 85)

The task shown in Figure 2 appears to be simple, but it has certain non-ostensive objects (Chevallard, 1994, Bosch & Chevallard, 1999) that are part of the technique and technology that allows solving such a task. Perhaps one might think that it is just a matter of doing $x = 1$ and calculating the value in $x – 1$ and $\sqrt{x} – 1$. Therefore, the numerical value obtained is divided or simplified by $x – 1$ by the numerical value of $\sqrt{x} – 1$. However, using the ostensiveness and non-ostensiveness of the calculation of the numerical value of an algebraic expression, which is taught in school mathematics, we have that $x – 1 = 1 – 1 = 0$ and $\sqrt{x} – 1 = \sqrt{1} – 1 = 1 – 1 = 0$. Thus, what is the result of $\frac{0}{0}$? In this regard, Anton, Bivens, and Davis (2014, p. 85, authors' emphasis) write:

> A quotient $f(x)/g(x)$ where the numerator and denominator both have a limit of zero when $x \to a$ is called an ***indeterminate form of the type* 0/0**. The problem with these limits is that it is difficult to tell by inspection whether the limit exists, and if it does, it is difficult to tell its value. Informally speaking, this is because there are two conflicting influences at play: the value of $f(x)/g(x)$ would tend to zero when $f(x)$ tends to 0 if $g(x)$ remained fixed at some non-zero value, while the value of this quotient would tend to increase or decrease without quota when $g(x)$ tends to 0 if $f(x)$ remained fixed at some non-zero value. However, with $f(x)$ and $g(x)$ tending to zero, the behavior of this quotient depends on precisely how these conflicting tendencies cancel each other out for the particular functions f and g under consideration.

We will not dwell on the mathematical discussions of higher education pertinent to the task in Figure 2 but on what can be linked to a possible REM on rationalization of denominators and calculation of limits, which reveals ostensive and non-ostensive objects of school mathematics. At this point, we can think about rationalizing the denominators of a fraction in the set of real numbers. That said, we can think of a praxeological organization that discusses operations with radicals in the form of a fraction, for example, "[...] when should one write in the form $u + v\sqrt{e}$ an expression of the type $\frac{a+b\sqrt{e}}{c+d\sqrt{e}}$ (where $a, b, c, d, u, v \in \mathbb{Q}$ and where *and* $\in$

$\mathbb{N}$ is a perfect non-square integer) [...] (Chevallard, 1998, p. 27, our translation). To simplify Chevallard's idea, we can think of the type of tasks T: Rationalize the denominator of $\frac{a}{\sqrt{b}-c}$. This type of task has a technique τ subjugated to a technology θ associated with the case of factoring in school mathematics, called the product of the sum and the difference: $(\sqrt{b}+c)(\sqrt{b}-c) = (\sqrt{b})^2 - c^2 = b - c^2$. However, this is just one part of the technique that solves the T-type tasks. Its complete form is described as follows: $\frac{a}{\sqrt{b}-c} \to \frac{a}{\sqrt{b}-c} \cdot \frac{\sqrt{b}+c}{\sqrt{b}+c} = \frac{a(\sqrt{b}+c)}{(\sqrt{b}-c)(\sqrt{b}+c)} = \frac{a(\sqrt{b}+c)}{b-c^2}$. Thus, fraction $\frac{a}{\sqrt{b}-c}$ has its rationalized name in the representation $\frac{a(\sqrt{b}+c)}{b-c^2}$. By applying the technique of rationalizing denominators, we can understand the resolution of the task in Figure 2 and one of the ways used to eliminate indeterminacy $\frac{0}{0}$, in the calculation of limits: $\frac{x-1}{\sqrt{x}-1} \to \frac{x-1}{\sqrt{x}-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1} = \frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)} = \frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x})^2-1^2} = \frac{(x-1)(\sqrt{x}+1)}{x-1} = = \frac{(x-1)}{(x-1)} \cdot \frac{(\sqrt{x}+1)}{1} = 1 \cdot (\sqrt{x}+1) = \sqrt{x}+1$.Figure 3 shows the final part of the solution to the task in Figure 2.

$$\lim_{x \to 1} \frac{x-1}{\sqrt{x}-1} = \lim_{x \to 1} \sqrt{x} + 1 = 2$$

Figure 3.

The final part of the solution to Example 10 (Anton, Bivens, & Davis, 2014, p. 86)

What we show from the task in Figure 2 –in this case, a minimal part of the didactic transposition of the rooting subject from school mathematics, included in the basic education curriculum (Brasil, 2018)– helps us to instigate ideas regarding the elaboration of reference epistemological models for this subject, for example, on exponentiation and rooting in the set of real numbers, rational and irrational functions, among others.

Let us look at some ideas from the study of integrals connected to objects from school mathematics. We focus on the calculus of definite integrals, particularly, the ideas associated with the fundamental theorem of calculus, shown in Part I of Figure 4.

5.6.1 TEOREMA (*Teorema Fundamental do Cálculo, Parte 1*) *Se f for contínua em* $[a, b]$ *e F for uma antiderivada de f em* $[a, b]$, *então*

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad (2)$$

Figure 4.

Fundamental theorem of calculus – Part I (Anton, Bivens & Davis, 2014, p. 363)

The theorem in Figure 4 seems like a puzzle to those who have not studied differential and integral calculus. However, this theorem has many obvious mathematical notions from school mathematics, especially in high school.

When we say that function f is continuous on the interval $[a, b]$, it means that the values of a and b are on the graph of this function. In high school, these intervals appear in the real intervals component (Bonjorno, Giovanni Junior, & Sousa, 2020). In notation $[a, b]$, in which $a < b$, we have the non-ostensive notion of lower extreme and upper extreme, meaning that the values assigned to a will be lower and the values for b, upper extremes. If we have the interval [2, 3] in the set of real numbers, then $a = 2$ and $b = 3$. The unfolding of the dialectic between ostensive and non-ostensive objects (Bosch & Chevallard, 1999) falls on the elementary study of functions that begins in basic education and becomes more sophisticated in higher education, such as the fundamental theorem of calculus.

To illustrate the dialectic between ostensive and non-ostensive functions, from Figure 4, in school mathematics, let us take the quadratic function defined by $f(x) = x^2$, with dominance in the set of real numbers. That way, x can assume any value in the interval [2, 3], including the extremes. Carrying out the calculations of f(2) and f(3): $f(2) = 2^2 = 4$ and $f(3) = 3^2 = 9$. If we want to approximate the calculation of F(a) – F(b) without considering the mathematical theory of the fundamental theorem of calculus, we do f(b) – f(a), which results in 9 – 4 = 5. Figure 5 shows part of the graph of the formation law function $f(x) = x^2$.

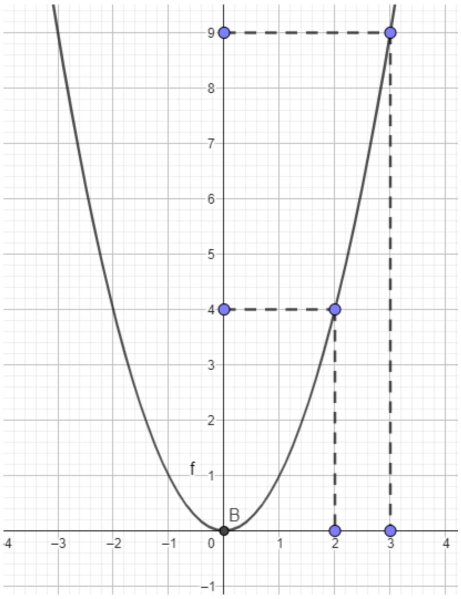

Figure 5.

Cutout of part of the graph of the function defined by f(x) = x^2 *generated in Geogebra (Authors, 2024)*

Figure 5 shows the continuity of the graph of the formation law function. $f(x) = x^2$, in the range [a, b] = [2, 3]. A plausible question: Is the graph of the function defined by $f(x) = x^2$ the same as the antiderivative *F(x)*? This question presents some complexity related to technologies θ that justify the techniques that apply to the types of definite integrals tasks. However, let us stick to task t: Calculate $\int_2^3 x^2\,dx$. The technique τ to obtain the antiderivative *F(x)* of $f(x) = x^2$ and: $F(x) = \frac{x^{n+1}}{n+1}$. Hence: $\int_2^3 x^2\,dx \rightarrow F(x) = \frac{x^{2+1}}{2+1} = \frac{x^3}{3}$. Calculating *F(b) – F(a) = F(3) – F(2)*: $\frac{3^3}{3} - \frac{2^3}{3} = \frac{27}{3} - \frac{8}{3} = \frac{27-8}{3} = \frac{19}{3}$.

When observing the calculations performed to obtain the antiderivative *F(x)* and in F(b) – F(a), one can see the elementary praxeological connections of the mathematical organizations belonging to school mathematics (exponentiation, algebraic expressions, polynomial functions, operations with fractions, etc.) with those of differential and integral calculus. To make these transpositional connections more explicit, we display the graph of the function defined by $F(x) = \frac{x^3}{3}$, shown in Figure 6.

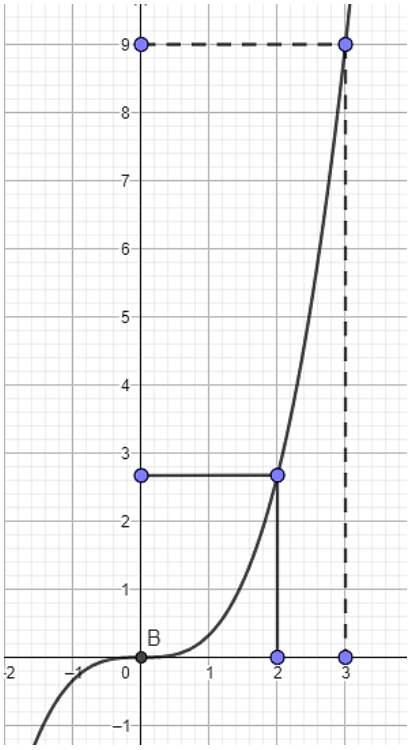

Figure 6.

Cutout of part of the graph of the formation law function F(x) = $\frac{x^3}{3}$ *generated in Geogebra*

(Authors, 2024)

The graph in Figure 6 shows the characteristics of the function defined by *F(x)* = $\frac{x^3}{3}$, in the interval [2, 3]. This is because the graph is an ostensive object (Chevallard, 1994) that displays the behavior of the curve on the coordinate axis of the abscissas and ordinates. This axis is the subject of school mathematics under the name of the Cartesian plane (Bonjorno, Giovanni Junior, & Sousa, 2020). We thus have further evidence of transpositive connections for elaborating reference epistemological models, which show how the praxeological organizations of school mathematics can dialogue with mathematical objects of differential and integral calculus. Figure 7 shows a schematic summary of praxeological organizations of school mathematics and the possible transpositive connections between these mathematical organizations from the perspective of modeling reference epistemological models.

Each praxeological organization of school mathematics in Figure 7 has several ostensive and non-ostensive objects (Bosch & Chevallard, 1999) necessary for the techniques τ that solve specific types of T tasks of the praxeological organizations of differential and integral calculus.

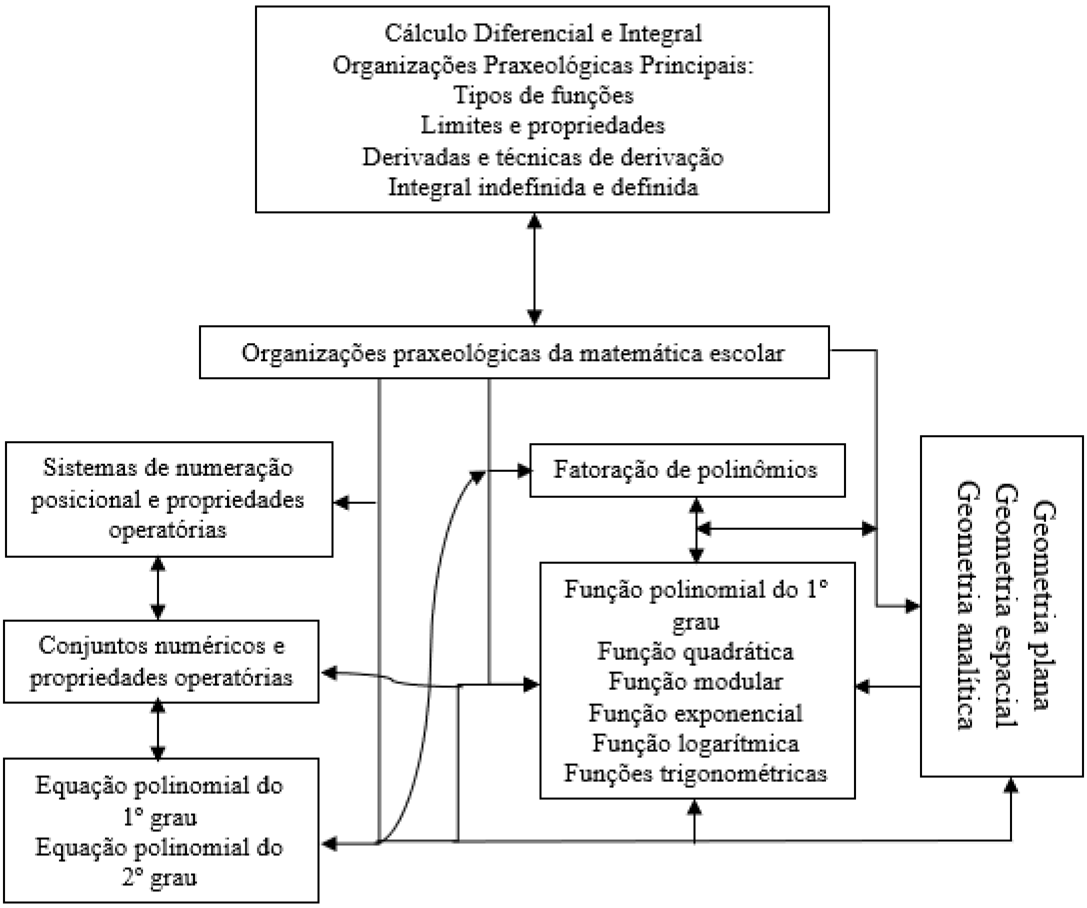

Figure 7.

Schematic summary to think about elaborations of reference epistemological models from the perspective of transpositive connections of school mathematics with the teaching of differential and integral calculus (Authors, 2024)

The ideas contained in Figure 7 can be contemplated in the phenomenological technique of the elaboration of the REM (Gascón, 2014; Silva, 2017), because the praxeological organizations shown in this figure are from the DEM of school institutions and are in the praxeological basis of mathematics teachers in basic education.

Final Considerations

There are multiple possibilities in the development of epistemological reference models that link school mathematics to differential and integral calculus teaching, which is reflected in the types of praxeological, mathematical, and didactic organizations, whether particular, local, regional, or global. In other words, if the idea of a REM is to understand the functionality of the cases of polynomial factorizing in school mathematics when calculating limits, then the work of producing this REM goes through stages that begin with specific praxeological organizations (type of T tasks of factorization by common factor) and attempts to achieve a regional praxeological organization[T_{ij} / τ_{ij} / θ_j / Θ], where technologies θ_j must contain justifications of the techniques τ_{ij} applied in solving types of tasks $T_{ij}$of much more complex limit calculation methods, as shown in Figure 2.

The transpositive connections between various objects of school mathematics and those of differential and integral calculus are noticeable, as we show in the ideas of the decimal numbering system (DNS), in the study of polynomials, equations, and types of functions, mainly, first-degree polynomial and quadratic functions. Using ideas from the DNS, we show, in a straightforward manner, why the derivative of a constant function is zero.

In the task in Figure 1, we saw two objects in school mathematics, the first degree polynomial function and the second degree polynomial function or quadratic function. These two objects are part of the mathematics curriculum for middle school and high school. These objects have a transversal praxeological nature, ranging from basic to higher education. Furthermore, the first degree polynomial equation object is modeled on the first degree polynomial function and has a necessary geometric representation (tangent line to a curve) to understand the relationship between the calculation of derivatives and limits.

We saw ostensive and non-ostensive objects that can help solve some differential and integral calculus tasks. In this way, the transpositive coherence between the teaching of school mathematics objects and various objects of higher education mathematics becomes clear. We see this coherence in the task in Figure 1, in which we have the calculation of limits that relates two functions in the form of a quotient between them, i.e., $\frac{f(x)}{g(x)}\,(g(x) \neq 0)$. Function f has the formation law $f(x) = x^2 + x$ and the function g is defined by $g(x) = 3 - x$. These two objects, the quadratic function and the first degree polynomial function, are in the praxeological organizations of school mathematics.

It is noteworthy that the study of the praxeological organizations of differential and integral calculus enhances the perspectives for the elaboration of reference epistemological models, which highlight the didactic transposition in the objects of school mathematics. This perception of ours is illustrated in the technique underlying the fundamental theorem of calculus for definite integrals, which was applied in calculating the definite integral $\int_2^3 x^2\,dx$. The process of obtaining the antiderivative reveals ostensive objects pertinent to school mathematics (exponentiation, operations with rational numbers, construction of the graph of the quadratic function with the formation law $f(x) = x^2$ etc.).

Differential and integral calculus mathematical organizations are of the regional and global types. These two types of praxeological organizations have very pronounced abstractions that compose refined demonstrations of technologies θ_j and theories Θ_k, which support the mathematical basis of differential and integral calculus, for example, the confrontation theorem, the theorem of continuous functions, demonstrations of the rules of derivation, calculation of areas of curves, the fundamental theorem of calculus, among others. However, these praxeological organizations have elementary connections of ostensive and non-ostensive objects that lack the transpositive versions of the particular, local, and regional mathematical organizations of school mathematics. Furthermore, these three types of school mathematical organizations, mainly particular and local, demand transpositive re-elaborations, which can be carried out when elaborating a reference epistemological model (REM).

It is palpable that we conjecture that the objects of differential and integral calculus lack reference epistemological models developed from studies of the objects of school mathematics, shown in the tasks of Figures 1, 2, and 3, which bring the calculation of function limits.

The tasks in Figures 5 and 6, covering the definite integral and the fundamental theorem of calculus, show that the preliminary basis is in school mathematics, in the study of the set of real numbers. Add to this study the obvious graph to show the continuity of a function in a closed interval. Other ostensive and non-ostensive objects are used to obtain the antiderivative and apply the fundamental theorem of calculus and its more refined extensions.

Thus, we want to expose ideas linked to some objects of school mathematics that reveal transpositive connections pertinent to the elaboration of reference epistemological models, linking them to the notions of mathematical objects of differential and integral calculus.

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