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Conway's notion of real number and the principle of complementarity, some contributions to the development of epistemological reference models

La noción de número real de Conway y el principio de complementariedad, algunos aportes al desarrollo de modelos epistemológicos de referencia

La notion de nombre réel de Conway et le principe de complémentarité, quelques contributions au développement de modèles épistémologiques de référence

A noção de número real de Conway e o princípio de complementaridade, algumas contribuições para o desenvolvimento de modelos epistemológicos de referência

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Abstract

The objective of this article is to highlight the potential of Conway's theory compared to the classical concept of number with a view to contributing to the development of Epistemological Reference Models for teaching Differential and Integral Calculus. The search for a single answer to the epistemological question "What is a number?" has mobilized Mathematics epistemologists for centuries, considered essential for the foundation of this concept. John Horton Conway, an English mathematician from Princeton University, dedicated himself to researching this issue and resulted in a theory presented in the 1970s. In this article we bring elements about this theory highlighting its contributions to the evolution of the foundation of the concept of number. Conway's definition of number meets the complementarity of the intensional and extensional aspects of this concept, bringing advantages to Mathematics teaching. Scientific investigations and results of teaching practices in the field of teaching have encouraged questions about the importance of the role that the concept of real numbers has for learning Calculus and Real Analysis. Add to this question, and for Mathematics in general, and

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for the formation of analytical thinking, and for mathematical thinking? The reflections carried out in this article aim to raise epistemological and cognitive aspects about the classical construction of number, seeking to have an impact on current epistemology.

Keywords: Real number, Complementarity, Conway number, Epistemological reference model.

Resumen

El objetivo de este artículo es resaltar el potencial de la teoría de Conway frente al concepto clásico de número con miras a contribuir al desarrollo de Modelos de Referencia Epistemológicos para la enseñanza del Cálculo Diferencial e Integral. La búsqueda de una respuesta única a la pregunta epistemológica "¿Qué es un número?" ha movilizado durante siglos a los epistemólogos matemáticos, considerados esenciales para la fundación de este concepto. John Horton Conway, matemático inglés de la Universidad de Princeton, se dedicó a investigar este tema y dio como resultado una teoría presentada en la década de 1970. En este artículo traemos elementos sobre esta teoría destacando sus aportes a la evolución de la fundamentación del concepto de número. . La definición de número de Conway cumple con la complementariedad de los aspectos intensional y extensional de este concepto, aportando ventajas a la enseñanza de las Matemáticas. Las investigaciones científicas y los resultados de las prácticas docentes en el campo de la enseñanza han fomentado cuestionamientos sobre la importancia del papel que tiene el concepto de números reales para el aprendizaje del Cálculo y Análisis Real. ¿Agregar a esta pregunta, y para las Matemáticas en general, y para la formación del pensamiento analítico, y para el pensamiento matemático? Las reflexiones realizadas en este artículo pretenden plantear aspectos epistemológicos y cognitivos sobre la construcción clásica del número, buscando incidir en la epistemología actual.

Palabras clave: Número real, Complementariedad, Número de Conway, Modelo de referencia epistemológica.

Résumé

L'objectif de cet article est de mettre en évidence le potentiel de la théorie de Conway par rapport au concept classique de nombre en vue de contribuer au développement de modèles épistémologiques de référence pour l'enseignement du calcul différentiel et intégral. La recherche d'une réponse unique à la question épistémologique « Qu'est-ce qu'un nombre ? a mobilisé les épistémologues mathématiques pendant des siècles, considéré comme essentiel pour le fondement de ce concept. John Horton Conway, un mathématicien anglais de l'Université de Princeton, s'est consacré à des recherches sur cette question et a abouti à une théorie présentée dans les années 1970. Dans cet article, nous apportons des éléments sur cette théorie soulignant ses contributions à l'évolution du fondement du concept de nombre. . La définition du nombre de Conway répond à la complémentarité des aspects intensionnels et extensionnels de ce concept, apportant des avantages à l'enseignement des mathématiques. Les recherches scientifiques et les résultats des pratiques pédagogiques dans le domaine de l'enseignement ont suscité des interrogations sur l'importance du rôle que joue le concept de nombres réels dans l'apprentissage du calcul et de l'analyse réelle. Ajouter à cette question, et pour les mathématiques en général, et pour la formation de la pensée analytique, et pour la pensée mathématique ? Les réflexions menées dans cet article visent à soulever les aspects épistémologiques et cognitifs de la construction classique du nombre, cherchant à avoir un impact sur l'épistémologie actuelle.

Mots-clés : Nombre réel, Complémentarité, Numéro Conway, Modèle de référence épistémologique.

Resumo

O objetivo deste artigo é destacar potencialidades da teoria de Conway em ralação ao conceito clássico de número, com vistas a contribuir com o desenvolvimento de Modelos Epistemológicos de Referência para o ensino de Cálculo Diferencial e Integral. A busca de resposta única para a questão epistemológica acerca do que é número tem mobilizado epistemólogos da Matemática por séculos, a teoria de John Horton Conway é considerada essencial para a fundamentação desse conceito. Trata-se de um matemático inglês da Universidade de Princeton que se dedicou a pesquisar essa questão e obteve como resultado uma teoria apresentada na década de 1970. Neste artigo serão apresentados elementos sobre essa teoria, bem como as contribuições dos estudos de Conway para a evolução da fundamentação do conceito de número. A definição de Conway para número atende à *complementaridade* dos aspectos intensional e extensional desse conceito trazendo vantagens para a didática da Matemática. Investigações científicas e resultados de práticas docentes no âmbito da didática têm fomentado questionamentos sobre a importância do papel que o conceito de número real tem para a aprendizagem do Cálculo e da Análise Real. Acrescenta-se a essa pergunta, e para a Matemática de um modo geral, e para a formação de um pensamento analítico, e para o pensamento matemático? As reflexões realizadas nesse artigo têm por pretensão levantar aspectos epistemológicos e cognitivos sobre a construção clássica de número, buscando repercutir sobre a epistemologia vigente.

Palavras-chave: Número real, Complementaridade, Número de Conway, Modelo epistemológico de referência.

Conway's notion of real number and the principle of complementarity, some contributions to the development of epistemological reference models

Niels Bohr's term "complementarity" has been used by various authors to capture the essential aspects of the cognitive and epistemic development of mathematical and scientific concepts. Michael Otte (2003, p. 205) conceives of complementarity according to the dual notion of extension and intension of mathematical terms.

The notion of *intension* of mathematical terms is characterized by describing the relations between classes of mathematical objects, as well as their structural relations, but it is important to highlight that it does not describe the mathematical object itself, i.e., axiomatic systems in the sense of Peano and Hilbert or an axiomatic approach to real numbers do not describe the mathematical term number. The *extension* of this concept must be sought.

The *extension* of mathematical terms is characterized by providing the description of mathematical objects, as well as the interpretation and possible applications of axiomatic systems.

The debate on the relationship between the *intensional* and *extensional* views of mathematics particularly and intensely affects the concept of number. The *intensional* view, which implies ordinality and axiomatic descriptions, appears first and receives severe criticism from those who privilege mathematical applications.

The duality between these two views is revealed by Russell in his book *Philosophy of Mathematics*, published in 1919, which deals with numbers and everything related to number.

Peano's approach is insufficient to provide an adequate basis for arithmetic. First, because we are not able to know whether there is any set of terms verifying Peano's axioms, secondly, we want our numbers to count ordinary objects, and this requires that our numbers have a definite meaning, not merely that they have certain formal properties (Russell, 2007, p. 10).

According to Russell, in order to conceptualize a number with some extension, which is real, one must understand "numbers as a number of quantities" and give an application to the concept thus defined by demonstrating the existence of sets of arbitrary cardinality. Obviously, this can only be done axiomatically. In doing so, however, the notion of axiom should not be understood instrumentally in the Peano-Hilbertian sense; the term should rather be conceived in the Euclidean tradition, i.e., as an intuitively evident truth and as a precondition of mathematics. It is for this reason that Russell introduces the "axiom of infinity."

We have to ascertain or make it plausible that there are in fact infinite collections or sets in the world in order to be able to find numbers (Russel, 2007, p. 77).

In Russell's way of thinking, arithmetic intuition must be replaced with the intuition of set theory. This may seem strange, as the axiomatization of arithmetic has been caused by the awareness that we are unable to fully understand number, even more to establish formal laws that numbers satisfy. Russell apparently replaces number with the intuitive concept of set as a foundation for these formal laws.

Nearly half a century later, mathematics education worldwide attempted to repeat this deed, with little success. Mathematics is not a quasi-empirical science, which establishes its methods by the property meanings of its objects; rather, the objects have to be constructed simultaneously with the rules and methods of reason.

Dedekind was also not ready to imagine an axiomatic definition of number, because after recognizing the essential characteristics of such a system, he still asks: "does such a system exist in the reality of our ideas?" (Dedekind in his letter to Keferstein in 1890). He considered an infinite totality of things attributing to us, human subjects, the ability to infinitely repeat certain ideas or mental actions, as if we were adding them one to another. Dedekind considered his thought experience as a proof of its logical existence and, he was not concerned, like Russell, with the meaning of the individual symbols of number. Unlike Dedekind, Russell thought that one can never reach infinite totalities by mere enumeration, and he considered it an empirical fact "that the mind is not capable of repeating the same act infinitely". A *complementarian* approach is induced by the impossibility of defining mathematical reality independently of cognitive activity itself.

For Thom (1972 apud Otte, 2003, p. 203) "the real problem facing the teaching of mathematics is not rigor, but rather the problem of developing a 'meaning' of the 'existence' of mathematical objects". For Otte, a modern axiomatic theory has become, to a certain extent, a dual theory, in the sense that this set of axioms and postulates does not only determine the intension of the theoretical terms, but also constitutes the extensions or references and applicability of this theory.

For example, the objects of Euclidean geometry seem to be given by intuition, being, in a certain way, independent of the theory. In Hilbert's geometry the situation is completely different, because to answer the following questions "what is a point?" or "what is a number?", a axiomatic description of relations or laws by which these entities are governed is necessary.

This *complementarity* becomes visible, and distinguishable from mere duality, only from a genetic perspective, which focuses on the mathematical character of our knowledge. Only from this perspective is the relationship between matter and object, beyond the object itself, focused on. According to Otte (2003), the notion of complementarity is particularly relevant for any study of the epistemological foundations of mathematics education.

The conceptualization of number proposed by Conway guarantees this complementarity, since it is a formally rigorous theory and can be interpreted by several classes of games.

It is these reflections on the notion of number, especially on real number, that interest us in this article. The historical and epistemological aspects of the definition of numbers reverberate in Conway's proposal, therefore it has advantages over the classical ones. It is expected that, with this potential, it will bring essential elements for the construction of Epistemological Reference Models (ERM), aiming at the teaching of Differential and Integral Calculus.

The notion of real number is one of the essential pillars of Mathematical Analysis, as well as of Differential and Integral Calculus. The teaching and learning of these areas of Mathematics require the treatment and study of properties of real numbers, such as density, order and completeness, which are essential for the rigorous demonstration of theorems necessary for the understanding of concepts, for example, limit, continuity and derivative.

We believe that Epistemological Reference Models for the teaching of Differential and Integral Calculus can play a fundamental role in learning, as they provide conceptual and methodological structures that guide the pedagogical approach and assist in the monitoring and questioning of established knowledge. This is because, like Gascón (2014), we admit that one of the important roles of Didactics in Mathematics is to question mathematical knowledge. And for this reason, we focus this text on questioning the classically established concept of real number.

This questioning is based on the principle of *complementarity*, as presented by Otte (2003), since it allows the analysis of epistemological and cognitive aspects related to mathematical objects, and can contribute to the development of Epistemological Reference Models, mainly by favoring questioning and epistemological surveillance of established mathematical knowledge.

We admit, as Gascón (2014, p. 100), that:

> [...] to take the processes of didactic transposition as an object of study, the teacher needs to critically analyze the epistemological models of mathematics that are dominant in the institutions involved and, thus, free himself from the uncritical assumption of such models. This is what epistemological emancipation consists of, while institutional emancipation refers to the need for the teacher (and the science of didactics) to free himself from the dependencies that accompany the position of "teacher" (subject of a

> given school institution), that of “noosphere” (subject of the noosphere, that is, author of textbooks, study plans, curricular documents, teacher training texts, etc.) and, also, that of “mathematician guardian of orthodoxy” (subject of the institution that produces and preserves knowledge). Obviously, epistemological emancipation constitutes a particular aspect, an essential first step, of institutional emancipation that could be defined, in general, as the liberation from subjection to the dominant ideology in the institutions that are part of its object of study, i.e., the emancipation not only from epistemological provincialism, but also from all didactic, pedagogical and cultural provincialism (Gascón, 2014, p. 100, translation by the author).

The most common (or classical) approaches to real numbers, especially in textbooks on Mathematical Analysis or Differential and Integral Calculus, raise discomforts and inconveniences and have generated debates and discussions of a historical and epistemological nature.

At the heart of these discussions is, for example, the lack of a single answer to the question “What is a number?”, and also the impossibility of definitions of the concept of number considering the dual condition of *intensionality* and *extensionality*.

We indicate as classical the approaches to real numbers: as a set of equivalence classes of Cauchy sequences of rational numbers (completion of the set of rationals), as the set of cuts of rationals (Dedekind cuts) or as a complete ordered field with respect to the operations of addition and multiplication (axiomatic conceptualization).

The work of mathematicians in different areas of study presupposes the use of numbers, from natural to transfinite[3], and the inconvenience of the lack of a single answer to the question “what is a number?” gains strength in the epistemological debate (Fonseca, 2010). Here we defend the possibility of envisioning the evolution of mathematical ideas of the different types of numbers, through history, in addition to the epistemological emancipation of more traditional or classical approaches.

The study of the historical evolution of mathematical concepts is not synonymous with harmony, but rather with conflicts and complexity, with the questioning of established knowledge.

Our considerations regarding the conceptualization of number are based on the principle of *complementarity*, since, according to this principle, mathematical objects have a dual nature, that is, on the one hand they can be characterized axiomatically (*intensionality*), on the other they must be complemented by possible applications, i.e, models that translate their logical processes (*extensionality*), as indicated by Otte (2003). Considering this reference, when

[3] A transfinite number is one whose cardinality is greater than C (continuous). “one of the most striking results of Cantor's Mengenlehre is that there are such numbers” (Boyer, 1949, p.297)

analyzing a mathematical object, we seek to identify and explain the non-dissociation of the aspects that make up this duality.

The conceptualization of real number, proposed by Conway (2001), highlights historical and epistemological potentialities in relation to classical conceptualizations.

The text below is divided into sections that aim to contemplate, respectively, considerations about the nature of numbers, a brief introduction to Conway's theory, the principle of *complementarity*, the questioning of classical approaches to real numbers in light of the precepts of *complementarity*, as well as Conway's own proposal and, finally, the final considerations.

Considerations on the nature of numbers

Historically philosophers and mathematicians have made several criticisms of the conceptions about the nature of numbers, as exposed by Barker (1969) and Russell (2007).

> The question "should a definition for the mathematical object 'number' start from the assumption that it is purely an object of thought or should it be based on external things that are part of our sensible reality?" has always been involved in philosophical or epistemological debates about the nature of numbers (Fonseca, 2010, p. 16).

In this article, this issue is considered taking into account the principle of *complementarity*, as established by Otte (2003), and a possible reformulation may encompass other mathematical notions, highlighting the potential that this principle presents to analyze mathematical notions from an epistemological and cognitive point of view.

Historically, we can observe that when trying to answer questions about the nature of numbers, the arguments used by mathematicians and philosophers suggest a debate between the hierarchy involving Pure and Applied Mathematics (Barker, 1969).

In fact, the development of the concept of number was not harmonious at all, for example, negative numbers and complex numbers were not accepted and considered doubtful for a long time, acquiring the status of number only in the 19th century.

Frege (1992, p. 30) was one of the mathematicians who argued that negative numbers and irrational numbers should be analyzed and subjected to a number credential; this defense involves discussions about the nature and definition of such numbers.

From the 19th century onwards, real numbers were logically well founded by some mathematicians such as Richard Dedekind, Karl Weierstrass, Charles Méray and Georg Cantor. Since then, it has been widely accepted that the system of real numbers is constructed starting from the natural numbers, and, through successive constructions, the integers, the rational numbers and finally the real numbers are obtained (Fonseca, 2010).

> Dedekind, for example, can legitimately be named as the first to have constructed real numbers from rational numbers. However, when confronted with the question, "What is a number?", he responded with a general theory of ordinals that gives status to integers, but that cannot be applied directly to real numbers, as we can see in his text The Nature and Meaning of Numbers (DEDEKIND, 1901, p. 21). How can we say that a real number is a "number"? (Fonseca, 2010, pp. 136-137)

According to Fonseca (2010, p.18), the classical approaches to real numbers (Dedekind cuts, equivalence classes of Cauchy sequences of rational numbers and the axiomatic approach) present epistemological and philosophical drawbacks, such as the impossibility of answering the question "What is a number?", and the construction of numbers in a unique way, in addition to not providing *complementarity* between the *intensional* and *extensional* aspects in the conceptualization of real numbers.

Considering the drawbacks previously mentioned, we point to the theory developed by Conway (2001) that allows the construction of numbers in a unique way, from natural to transfinite numbers and that can be carried out through sets (guaranteeing their *intensional* character) and some classes of games (guaranteeing their *extensional* character). This theory can be conceived through a duality, with an axiomatic characterization and models (games) that provide the interpretation of its terms and explain properties that constitute the conceptualization of numbers (Fonseca, 2010).

We present below a brief introduction to the main ideas of Conway's theory (2001) that permeate its construction. Later, we will make considerations regarding its potential when confronted with the classical approaches to real numbers, guided by the principle of *complementarity* (Kuyk, 1977; Otte, 2003). According to this principle, mathematical objects have a dual nature, i.e., they can be characterized axiomatically, but they must be complemented by interpretations or applications, models that translate their properties. We argue that analyzing a mathematical object from the perspective of complementarity means seeking to identify its capacity to make inseparable the aspects that make up this duality (Fonseca, 2010).

A brief introduction to Conway's ideas

Conway's notion of number, developed in the 1970s, is a generalization of Dedekind's cuts and "deserves the qualification of new not only because of the time in which it was presented, but because of the epistemological advances that it enables" (Fonseca, 2010, p. 21).

This notion of number makes it possible to address epistemological questions, for example, "what is a number?" and, in its construction, encompasses the *intensional* and *extensional* aspects of the concept of number, in addition to enabling the construction of natural numbers to real numbers with a single procedure, overcoming classical approaches.

Conway conceptualizes number using the notion of cut, in addition to a specific class of games and set theory.

> Conway's notion of cut is a generalization of Dedekind's notion insofar as it does not require the set of rationals as a starting point, encompassing all "large" and "small" numbers: real numbers such as 0, 1, ..., n, -1, 1/2, $\sqrt{2}$, π, ...; transfinite numbers such as ω (the first infinite ordinal); and also infinitesimal numbers such as $1/\omega$. The definition of order, in the set of cuts, is achieved by taking a special class of games as models of "generalized cuts" (Fonseca, 2010, p. 22).

We emphasize that Conway criticizes the construction of real numbers from rational numbers through Dedekind's cuts, claiming that the distinction between the "old" and the "new" rational seems artificial, but is essential (Conway, 2001, p. 4).

Although Conway uses a generalization of Dedekind's method, what is important and new is that he does not presuppose rational numbers. At the beginning, he uses empty sets and constructs a broader class of numbers, called 'Surreal Numbers', including real numbers, transfinite numbers and infinitesimal numbers, in addition to complex numbers, i.e., 'Surreal Numbers' encompass all numbers, according to Fonseca (2010).

Conway generalizes Dedekind's method by considering two classes of numbers E (left class) and D (right class), such that no element of class E is greater than or equal to any element of class D. He then defines number as the set whose elements are the two classes E and D, i.e., the set $\{E \mid D\}$.

Conway's definition for a number $x = \{E \mid D\}$ assumes that classes E and D are classes of numbers defined before x. In other words, the construction of numbers occurs by recurrence. Let's see how this occurs.

The empty set is used to construct the first number $\{\emptyset \mid \emptyset\}$. This number is zero, i.e., $\{\emptyset \mid \emptyset\} = 0$. From it, other numbers are obtained by finding their two classes: the one on the left and the one on the right. The number 1, for example, will be the number $\{\{0\} \mid \emptyset\}$, the number 2, the number $\{\{0,1\} \mid \emptyset\}$, the number 3, the number $\{\{0,1,2\} \mid \emptyset\}$, and thus all the whole numbers are obtained. The representation of rational and irrational numbers in Conway (2001, p. 4).

Number is game

The number/game association developed by Conway (2001) considers certain classes of games, those in which: a) there are only two players; b) one of them is the winner; c) only a finite number of moves are allowed. The Hackenbush game class is one that fits into Conway's game classes. This class is derived from the well-known NIM game governed by the mathematical theory developed by Bouton (1901). In our research, we chose a version of the Hackenbush game class.

This version of the Hackenbush game is composed of colored pieces, blue and white, and its rules are as follows: Player A must remove blue pieces, while player B removes white pieces. The configuration of a game must be such that the pieces are overlapped and one of them connected to a horizontal line.

The players play alternately. Each player must remove only one piece of the color assigned to him/her. If a piece is removed, the pieces that are overlapping will be automatically erased. The player who first runs out of pieces of his/her color to remove will lose the game. Figure 1 shows four examples of Hackenbush games.

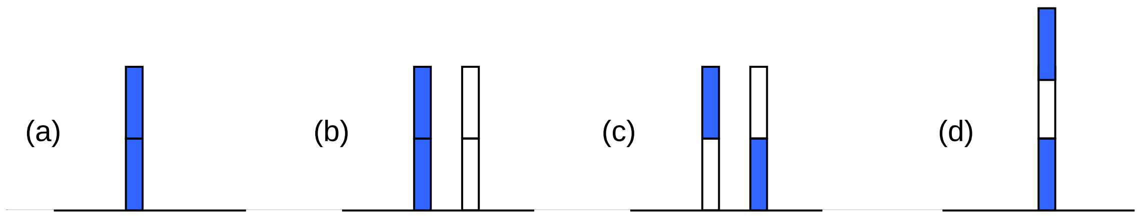


Figure 1.

Examples of Hackenbush games.

For example, in game (a) in Figure 1, only player A has the possibility of a move; therefore, he wins regardless of who starts the game.

In game (b), as we can see, the situation is different from the previous one, since both players have the possibility of a move, as follows: if player A starts the game by removing the blue piece furthest from the line, B can remove a white piece also furthest from the line. Then A will have only one piece to remove and B will win the game. If player B starts the game, he can use a strategy analogous to the previous one, and in these conditions player A will win regardless of B's moves. In this case, the player who starts loses.

In game (c), the same situation described in the previous paragraph will occur, i.e., the player who starts loses. In example (d), the following happens: if player A starts the game, he removes the blue piece that is connected to the line, automatically erasing the pieces that are overlapping it. In this case, player A will immediately win the game, since B will not be able to remove any pieces. If player B starts the game, he will remove the only white piece and consequently erase the blue piece above it. And player A will also win, as he will still have one chance to play. In other words, player A always wins. Below, we will indicate some specific games and the numbers associated with them.

Zero game/zero number

A zero game is one in which the player who starts the game loses, that is, the game in which the player who starts loses. Below are some zero games.

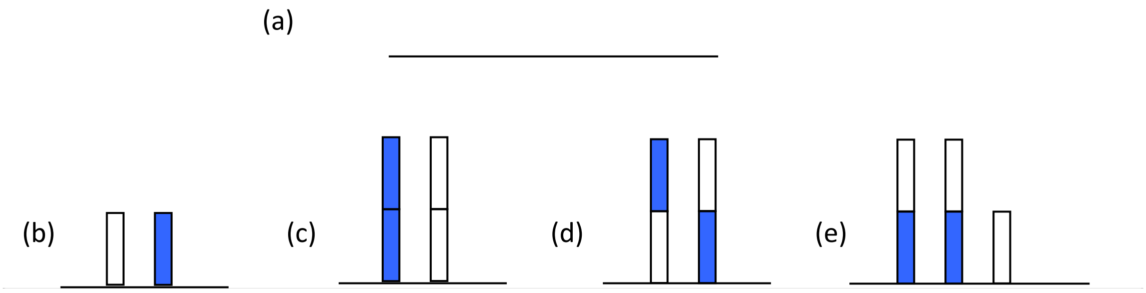


Figure 2.

Zero-value Hackenbush games.

Conway associates the zero game with the number zero. The zero game according to the configuration (a) indicated in Figure 2 is associated with the number zero represented by $\{\varnothing | \varnothing\}$. This is the first number constructed by Conway and one of the representations of the number zero. The empty set on the left side of the bar indicates the absence of moves for player A, while the empty set on the right side represents the absence of moves for player B.

To construct new numbers, it is necessary to define the following order relation in the set of games: a game is positive (or greater than zero) if player A wins regardless of who starts the game. Similarly, when the advantage is for player B, that is, when B wins regardless of who starts the game, the game is negative (or less than zero).

Thus, the association between game zero and number zero is effective. Let us move on to other numbers.

First, we will indicate how a given game J is associated with an integer number x, pointing out how the elements of classes E and D that define the number x associated with it are obtained. This is done by recursion, as follows: each time a player removes one of the pieces, game J is reduced to another game J' whose associated number is x'. This number x' will be an element of class E or D of x, depending on whether the player who removed the piece is player A or B, respectively.

In Figure 3 below, there are some examples of games and the respective integers associated with each of them. The games are indicated by the configuration of the pieces and the number associated with it is indicated below the horizontal line. The numbers 0, 1 and -1 arranged next to each piece (vertically) are the numbers associated with the games to which they are reduced when the respective piece is removed. The numbers 0, 1 and -1 are, as previously stated, in each case, elements of classes E and D of the numbers 1, 2, -1 and -2.

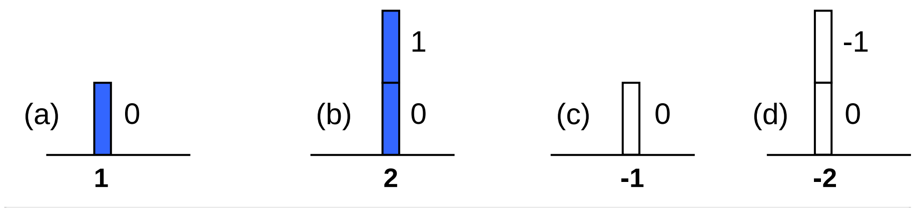


Figure 3.

Games/numbers 1, 2, -1 and -2.

Let us now detail each case: Game (a): the game consists of only one blue piece. In this game, only player A has a piece to remove. When this piece is removed, the game is reduced to game zero. The player who removed the piece was player A, so the number zero will be an element of the left class E defining the number x associated with game (a). Player B has no pieces to remove and so the right class D defining x is the empty class. Therefore, the number x associated with game (a) is the number $\{\{0\}|\varnothing\} = 1$. In other words, the game with only one blue piece (a) is game 1 and the number associated with it is the number 1.

Game (b) is game 2, and the number associated with it is the number $\{\{0,1\}|\varnothing\} = 2$.

Game (c) consists of only one white piece. There is, therefore, no piece for player A to remove, which implies that E is empty. And class D will be composed of zero, because, when player B removes the white piece, game (c) is reduced to a null game. The number x associated with game (c) is $\{\varnothing|\{0\}\}$.

Let us show that $\{\varnothing|\{0\}\}$ is the number -1. In fact: it is negative, because it is associated with a negative game (Player B always wins). And, defining the sum of two games J and J' as a game J" (J+J'=J"), such that the pieces of J are placed next to the pieces of J' and supported on the horizontal line, we have that 1+ (c) = 0, as in game (a) in Figure 4 below. And so $\{\varnothing|\{0\}\} = -1$.

Similarly, we obtain the number -2 = {∅|{–1, 0}}. Two numbers x and –x whose sum is zero are said to be opposite numbers. In general, the positive integer n is defined as n={{n-1}|∅}. All integers are constructed in a similar way.

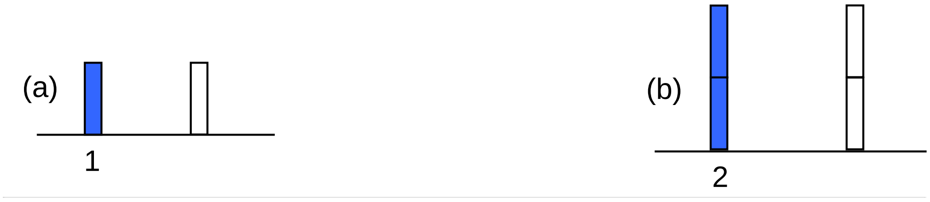


Figure 4.

Numbers 1, -1, 2 and -2.

Let us now analyze a new game indicated in Figure 5.

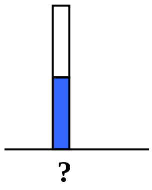


Figure 5.

A new number.

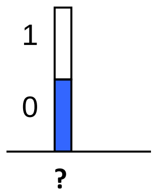
Let us check what number is associated with it. In fact, it turns out that:

(1) There are possible moves for both players;

(2) It is a positive game, since player A has an advantage, and consequently the number associated with it is a positive number.

(3) In this game, although the advantage is with player A, player B has a possible move. Its representation by means of sets is {{0}|{1}}, since:

(4)



(5) Using the sum of games, we obtain game (b) in Figure 6. However, it turns out that this is not a zero game, since in this case the advantage is with player B. We try a new possibility by playing with (c). We conclude that game (c) in Figure 7 is a zero game.

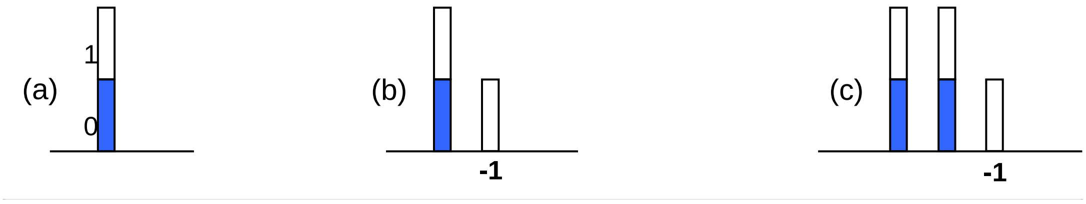


Figure 6.

Construction of the number $\frac{1}{2}$.

The game configuration (c) can be represented by the equation 2x + (–1) = 0, whose solution is $\frac{1}{2}$..

That is, game (a) in Figure 6 corresponds to the number $\{\{0\}|\{1\}\}$, which is the number $\frac{1}{2}$. Its opposite is obtained by a game resulting from the inversion of the moves of A by B. The result is the number $-\frac{1}{2}$. = $\{\{-1\}|\{0\}\}$.

Other rational numbers can be constructed from these and others constructed previously.

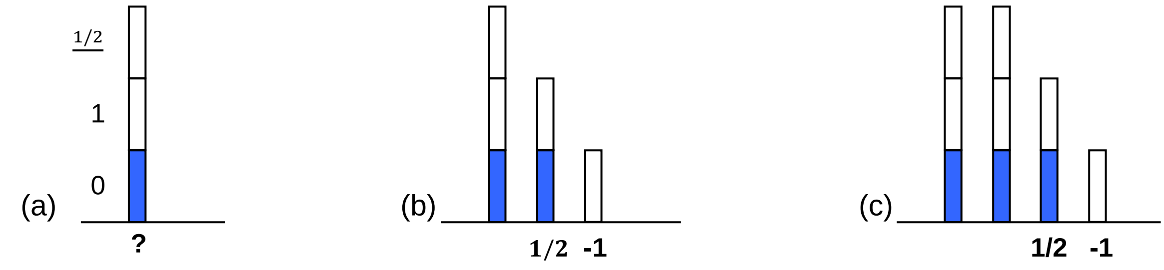


Figure 7.

Construction of the number $\frac{1}{4}$.

In game (a) in Figure 7, the advantage is for player A, so this game is associated with a positive number. Player B has two possible moves, and can be represented by $\{\{0\}|\{\frac{1}{2}, 1\}\}$ or $\{\{0\}|\{\frac{1}{2}\}\}$.

A new attempt leads us to construct the sum of games, as in (b), obtaining a game in which the advantage is for player B, and therefore not zero. A new attempt can be made with game (c), whose sum results in a game in which the first player loses, that is, a zero game. Thus, the new number associated with game (a) in Figure 7 is the solution to the equation:

$2x + \frac{1}{2} + (-1) = 0$, i.e., $x = \frac{1}{4}$. Therefore $\frac{1}{4}$. $= \{\{0\} | \{\frac{1}{2}, 1\}\}$ or simply

$\frac{1}{4} = \{\{0\} | \{\frac{1}{2}\}\}$. And $-\frac{1}{4} = \{\{-\frac{1}{2}\} | \{0\}\}$.

So far, we have not associated any game with rational numbers such as $\frac{1}{3}$ or irrational numbers. One of the reasons for this is that in order to associate games with such numbers, we will admit Hackenbush games that have an infinite configuration of pieces. It is important to emphasize that such games also satisfy the initial rules. Even considering infinite pieces for the players, it is a game that can be played finitely, since, from the first movement made by one of the players, all the pieces overlapping the removed piece will be erased, making the game finite.

We also emphasize that in Conway's theory such numbers are constructed from dyadic numbers, and have an infinite process.

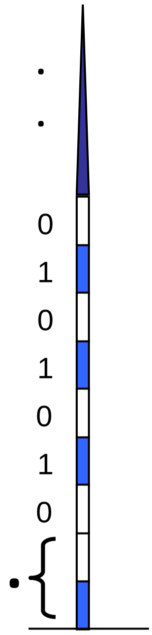
According to Conway and Guy (1999, p. 299), the set "{a, b, c,... | d, e, f,...} defines the simplest number strictly superior to all numbers a, b, c,... and strictly inferior to all numbers d, e, f,...", this definition, associated with the rule developed by **Elwin Berlekamp**[4], allows establishing the correspondence between real numbers and the Hackenbush game.

The rule developed by Berlekamp, Conway and Guy (2001) is as follows: the first pair of pieces of different colors that appear counting from bottom to top will represent the "binary comma", the blue and white pieces that follow this pair are the digits 1 and 0, respectively, which appear to the right of the comma, with a last 1 being added in the case where the configuration of pieces that make up the game is finite. The whole part is equal to the number of pieces that appear before the pair that represents the comma.

Let us look at some examples. The game associated with the rational number $\frac{1}{3}$ has an infinite and periodic configuration, as shown in Figure 8. In binary notation, the number $\frac{1}{3}$ is represented by 0.010101..., and by means of sets as follows:

$\frac{1}{3} = \{0.01; 0.0101; 0.010101; \ldots | \ldots; 0.0101011; 0.0111; 0.011; 0.1\}$

[4] Elwyn Berlekamp was born in Dover, Ohio, on September 6, 1940. He has been Professor Emeritus of Mathematics, Electrical Engineering, and Computer Science at the University of California, Berkeley, since 1971. He is known for his work in information theory and combinatorial game theory. With John Horton Conway and Richard K. Guy, he wrote the book Winning Ways for Your Mathematical Plays.



$\frac{1}{3}$

Figure 8.

Hackenbush game corresponding to the number $\frac{1}{3}$.

As we know, irrational numbers have an infinite and non-periodic binary representation, and we can use Berlekamp's rule to associate sets with irrational numbers. Let us look at an example. The irrational number *e*, whose representation in binary notation is 10.101101..., and through sets we have:

e = {10.101; 10.1011; 10.101101; ... | ...; 10.11001; 10.1101; 10.11}

This number corresponds to the set indicated in Figure 9.

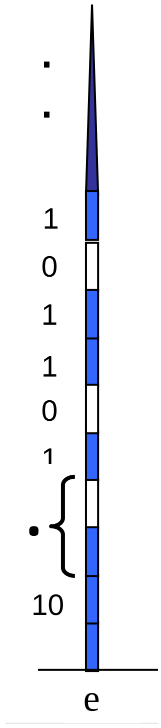


Figure 9.

Game associated with the irrational number e.

So far, we have presented the ideas behind Conway's theory, shown the interpretation of some numbers through a specific class of games and their representations through sets. Next, we discuss the principle of *complementarity*.

The notion of *complementarity*

The article Complementarity, Sets and Numbers by Otte (2003) raises several questions about numbers. Otte uses the principle of *complementarity* to analyze the epistemological and cognitive development of these concepts, highlighting the necessary inseparability of the *intensional* and *extensional* aspects in the conceptualization of number. This treatment considers "complementary two opposing concepts which, however, correct each other reciprocally and are integrated in the description of a phenomenon" (Abbagnano, 1982, p. 144).

According to Otte (2003), the *complementarity* between the *intensional* and *extensional* conceptions is intrinsically related to the concept of number, but is not restricted to it, and can be used to analyze the historical and epistemological development of other mathematical objects, as was also defended by Kuyk (1977). Both argue that *complementarity* occurs naturally, and there is no hierarchy between the *intensional* and *extensional* aspects.

For us, the principle of *complementarity* defended by Kuyk (1977) and Otte (2003) can contribute to mathematical praxiology in the development of Epistemological Reference Models, since it allows the analysis of established mathematical knowledge, based on *intensional* and *extensional* aspects.

The distinction between the terms *intensional* and *extensional* has been debated in contemporary philosophy and logic:

> This pair of terms was introduced by Leibniz to express the distinction that the Logic of Port-Royal had expressed with the pair comprehension-extension and the logic of Stuart Mill had expressed with the pair connotation-denotation [...]. The use of these two terms was adopted by Hamilton: "The internal quantity of a notion, its intensionality or comprehension, is constituted by different attributes of which the concept is the sum, i.e., of the several connected characters of the concept itself into one whole thought. The external quantity of a notion or its extension is constituted by the number of objects which are thought mediately through the concept" (Lectures on Logic, 2nd ed., 1866, I, p. 142). [...] The intension of a term is defined by Lewis as "the conjunction of all other terms each of which must be applicable to that to which the term is rightly applicable". In this sense, the intension (or connotation) is delimited by every correct definition of the term and represents the intention of the person who uses it, hence the primary meaning of "meaning". The extension, however, or denotation of a term is the class of real things to which the term applies (Lewis, Analysis of Knowledge and Valuation, 1950, p. 39-41). The same determinations are given by Quine: the intension is the meaning, the extension is the class of entities to which the term can be truly attributed. The adjectives *intensional* and *extensional* are used analogously [...]. (Abbagnano, 1982, p. 549)

In mathematical objects, the notion of intension characterizes the relations between classes, as well as their structural relations, but this does not exhaust the conceptualization. We can take axiomatic systems as an example, such as those used by Peano and Hilbert, or even an axiomatic approach to real numbers (complete ordered field). Normally, an axiomatic approach does not deal with objects that exist concretely, but rather with general relations or ideal objects.

Philosopher and mathematician Bertrand Russell (2007) severely criticized the axiomatic method, because for him axioms as non-specific terms need to be interpreted and specified, establishing connections with certain applications. He argues: "First, Peano's three primitive ideas – namely, '0', 'number' and 'successor' – are susceptible to infinitely many different interpretations, all of which will satisfy the five primitive propositions" (Russell, 2007, p. 23).

Based on Russell's arguments, one can infer the search for a definition for number that contemplates the mathematical nature, considering the way it is conceived by man, its applications and the description of the object itself, aspects that are not considered only with the notion of *intension* in the axiomatic method, for example, in the axiomatic conceptualization of real number.

Considering the impossibility of conceiving mathematical objects independently of their representations and of the cognitive activity itself, the notion of *extension* becomes essential, since it concerns the interpretation of such objects, as well as the applications, characterizing models of the theory.

For Otte (2003), an axiomatic theory must be conceived according to the principle of *complementarity*, that is, as a pair, satisfying the *intensional* aspect, which describes the relations between its theoretical terms through axioms, and the *extensional* aspect with references or extensions of such terms, explaining applications, interpretations or models of the theory.

We emphasize that we should not conceive complementarity as a simple duality between the two aspects mentioned, but rather as complementary within the construction of the theoretical framework (Otte, 2003, p. 205). For Bachelard (2004, p.14),

> "[...] purely deductive knowledge is, in our view, nothing more than the mere organization of schemes, at least until the root of abstract notions is established in reality. In fact, the very advancement of deduction, by creating abstractions, requires a continuous reference to the data that essentially surpasses the logical".

The debate on the relationship between the *intensional* aspect and the *extensional* views of Mathematics was particularly intense regarding the concept of number, as it can be seen in Russell (2007) and Barker (1969).

In *complementarity*, a constituent part of mathematical activity is the constructive procedure based on basic qualities as "building material"; and that the second constituent part of mathematical activity is knowledge about mathematical constructions (including basic qualities), as well as knowledge about the world, the formulation of this knowledge then happens in deducible models (Kuyk, 1977, p. 156).

Real numbers and the notion of complementarity

In this section, we consider the classical approaches to real numbers (axiomatics, equivalence classes of Cauchy sequences of rational numbers and Dedekind cut) based on the principle of *complementarity* and highlight theoretical potentialities in relation to the proposed conceptualization of number developed by Conway.

As we know, the axiomatic approach to real numbers results from the presentation of a list containing elementary facts accepted as axioms, explaining how these mathematical objects relate, so that the theorems that constitute the theory can be demonstrated from them.

These axioms make the set of real numbers equipped with the operations of addition and multiplication into a complete ordered field. Within this axiomatic approach, there are no type of description, interpretation or application for the mathematical object (real number); only the relations between the objects (real numbers) are emphasized, unilaterally characterizing the *intensional* aspect of these objects (Fonseca, 2010).

The notion of *intension* only establishes relations between classes of mathematical objects (structural relations). The axiomatic approach to real numbers does not describe the mathematical object itself, but only shows how operations should be performed with these numbers, treating them as ideal objects. In other words, the exclusively axiomatic method does not guarantee the *extensional* aspect of the concept of number (Fonseca, 2010).

Considering the principle of *complementarity*, the axiomatic approach to real numbers will always be incomplete, as it does not encompass the *extensional* aspect of these numbers.

Richard Dedekind's proposal for the construction of real numbers presupposes rational numbers and their properties. It develops the concept of real numbers based on a purely logical framework, the essence of which lies in ordinality.

> Traditionally, to obtain numbers, from natural numbers to real numbers, the following path can be used: natural numbers can be characterized by Peano's axioms; then, the set of integers is constructed by means of equivalence classes of ordered pairs of natural numbers; the next step is to construct rational numbers by means of equivalence classes of ordered pairs of integers, and finally the real numbers by means of Dedekind cuts or by equivalence classes of Cauchy sequences (of rational numbers) (Fonseca, 2010, p.126).

In the process of constructing numbers, from natural numbers to real numbers, whether through Dedekind cuts or equivalence classes of Cauchy sequences, it must also be considered that with each extension from one set to another, all properties must be demonstrated again.

We note that, in such a construction, from the point of view of *complementarity*, only logical deductions are considered, without interpretations or reference models for the numbers (Fonseca, 2010).

In addition, there is a certain type of rupture in the transition from rational numbers to real numbers, characterized by a change in method: operations with ordered pairs (of natural numbers or integers) are abandoned in order to use "new" objects, Dedekind cuts or equivalence classes of Cauchy sequences (Fonseca, 2010).

In relation to cuts, Dedekind postulated that every cut has a separating element (supreme of class A or infimum of class B). The effect of such a postulate is the creation of irrational numbers, which leads to the completion of the field of real numbers.

“Philosophically, Dedekind’s definition of irrational numbers involves a very high degree of abstraction, since it does not place any restrictions on the nature of the mathematical law that defines the two classes A and B” (Courant; Robbins, 2000, p. 86).

We consider that in such approaches possible models, applications or interpretations of numbers are not explored, therefore, *extensional* aspects are not contemplated and the desired *complementarity* between the *intensional* and *extensional* aspects of the concept of number does not occur.

In *complementarity,* a constituent part of mathematical activity is the constructive procedure based on elementary facts (which can be given axiomatically). Another part is the knowledge about mathematical constructions (including elementary facts) and knowledge about the world, which is linked to the applications of the concepts involved in mathematical activity (Kuyk, 1977, p. 156).

According to Otte (1993, p. 226), the object of Mathematics or the content of mathematical activity can in no way be defined absolutely and independently of the means of mathematical activity.

As Courant and Robbins (2000, p. 106) state, “in one way or another, explicitly or implicitly, even under the most intransigent formalist, logical or axiomatic aspect, constructive intuition will always remain the vital element in Mathematics”.

According to Fonseca (2010, p. 158), “a complementary approach between the *intensional* and *extensional* character of mathematical concepts is necessary because it considers mathematical reality as intrinsically linked to cognitive activity itself.

Mathematician George Cantor proposed a construction for real numbers, based on rational numbers and their properties. He used convergent sequences of rational numbers to construct real numbers. In this construction, a real number is an equivalence class of Cauchy sequences of rational numbers. We confirm here that our considerations about the construction of numbers step by step, from natural numbers to real numbers by Dedekind cuts, apply to the construction of real numbers through the equivalence classes of Cauchy sequences.

> If we construct the real numbers by means of Dedekind cuts, we will obtain a complete ordered field, whose elements are sets of rational numbers. If we use the Cantor process, the complete ordered field we obtain is formed by equivalence classes of Cauchy sequences of rational numbers. They are two complete ordered fields that differ in the

> nature of their elements, but not in the way their elements behave. In other words, they are isomorphic (Fonseca, 2010, p. 129).

We note that in these two constructions of real numbers, only the *intensional* aspects of numbers are considered, there is no mention of the *extensional* aspects.

We defend here the relevance of considering new ways to approach numbers, new references and models. Following this defense, we point to the approach proposed by Conway (2001), since it provides in its core some axioms and definitions based on set theories, which allows exploring the *intensional* aspect of the concept of number, and guarantees the interpretation of such numbers by a specific class of games, that is, it provides models for the interpretation of numbers, considering the extensional aspect.

"In this theory, games do not simply play the role of applying axiomatics; they provide an interpretation and an intrinsic model to the theory itself, since the ordering of numbers is inspired by games" (Fonseca, 2010, p. 130). We emphasize that in this theory, there is no hierarchy between the *intensional* and *extensional* aspects. An approach to numbers, from natural to real, through Conway's theory can be carried out without breaking procedures, contrasting with what we saw in the proposals of Dedekind and Cantor.

In Conway's theory (2001), we can construct numbers, simultaneously, through sets and through games, which are an empirical model, enhancing creativity, conjectures, motivation and experimentation, characteristics that are related to mathematical activity through investigation processes.

This construction involving the *extensional* aspects of the concept of number can serve to support the understanding of the logical apparatus, involving definitions, deductions, theorems and their respective demonstrations, contemplating the *intensional* aspects.

Here we are signaling the potential that Conway's theory (2001) has for conceptualizing numbers, from natural to transfinite, in a unique way and guaranteeing the principle of *complementarity* between the *intensional* and *extensional* aspects. We could use Conway's ideas to present a new foundation for real numbers, since he stated that he taught his theory, in undergraduate courses, as the theory of real numbers (Conway, 2001, p. 27).

Final Considerations

With Conway's (2001) proposal for the construction of numbers, we can present an answer to the question "what is a number?", which encompasses natural numbers to transfinite numbers. He himself answered this question as follows: "number is a game" (Conway, 1999, p. 300).

Here, we seek to show the potential of his theory for the construction of numbers, highlighting that it guarantees *complementarity* in the conceptualization of number, simultaneously contemplating the *intensional* and *extensional* aspects, which brings epistemological, philosophical and cognitive advantages.

Another aspect defended here deals with the possibility of using the principle of *complementarity* in the development of Epistemological Reference Models, since this principle is a powerful theory for questioning established mathematical knowledge, allowing us to analyze whether the *intensional* and *extensional* aspects are contemplated.

In this article, we use the case of real numbers as an example of analysis, questioning the classical approaches to the principle of *complementarity* and indicating a new theory that presents epistemological advantages. In this questioning, we also seek support in the historical, epistemological and philosophical developments of the concept of number. Thus, we propose the use of such a principle to question knowledge, maintain vigilance and allow the emancipation advocated by Gascón (2014, p. 100) and we intuit that its use in Theoretical Reference Models may be promising.

This article aims to contribute to Mathematics Education in general, and in particular, to the development of Epistemological Reference Models, under two distinct perspectives: theoretical and practical. The first, of a theoretical nature, involves the epistemological context of Mathematics. In the text, we illustrate how to question the nature and criteria of truth that mathematicians use, considering a rigorous analysis of the diversity of conceptual forms involved in mathematical notions, with particular emphasis on the concept of real number.

The second is of a more practical nature and aims to support reflections on the conceptualization of real number. In this sense, the study can provide valuable insights for the development of new pedagogical approaches, especially in Higher Education. By exploring innovative ways of presenting the concept of real number, we intend to motivate empirical research to investigate its suitability in practice.

We aim to contribute both to theoretical advancement and to practical application in Mathematics Education. By addressing the diversity of conceptual forms in the theoretical field and proposing new pedagogical approaches in the practical field, we believe that it will be possible to enrich the teaching and learning of mathematical notions.

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