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Os *Principia* de Isaac Newton: uma proposta de modelo epistemológico para o Ensino de integral nas licenciaturas em matemática

Isaac Newton's Principia: a proposal for an epistemological model for teaching integral in mathematics degrees

Principia de Isaac Newton: una propuesta de modelo epistemológico para la enseñanza integral en las carreras de matemáticas

Principia d'Isaac Newton : une proposition de modèle épistémologique pour l'enseignement intégral dans les licenciaturas en de mathématiques

Everaldo Roberto Monteiro dos Santos[1]
Universidade Federal Pará (UFPA)
Secretaria de Estado de Educação do Pará (SEDUC-PA)
Mestre em Educação Matemática
https://orcid.org/0009-0002-5818-2313

Lucélia Valda de Matos Cardoso[2]
Universidade Federal do Pará (UFPA)
Mestre em Educação Matemática
https://orcid.org/0000-0002-3482-2489

Reginaldo da Silva[3]
Instituto Federal de Educação, Ciência e Tecnologia do Pará (IFPA)
Doutor em Educação Matemática
https://orcid.org/0000-0003-0724-166x

Abstract

The article aims to propose an alternative epistemological model (MEA) for teaching calculation in Mathematics degrees, using Isaac Newton's Principia. To achieve our objective, we start from the following research question: What are the mathematical objects or historical artifacts present in Principia that will provide support for the creation of this Alternative Epistemological Model? In order to answer the question and achieve our objective, a historical, epistemological and contextual analysis of the aforementioned work was carried out, making it possible from there, in a process of didactic transposition and using the Anthropological Theory of Didactics (TAD), the elaboration of a MEA for the teaching of calculation for degrees in Mathematics.

[1] profroberto2009@gmail.com
[2] luceliamatosmat@gmail.com
[3] reginaldo.jamacaru@ifpa.edu.br

Keywords: Alternative epistemological model, Teaching calculus, Mathematics didactics.

Resumen

El artículo tiene como objetivo proponer un modelo epistemológico alternativo (MEA) para la enseñanza del cálculo en las carreras de Matemáticas, utilizando los Principia de Isaac Newton. Para lograr nuestro objetivo, partimos de la siguiente pregunta de investigación: ¿Cuáles son los objetos matemáticos o artefactos históricos presentes en Principia que brindarán apoyo para la creación de este Modelo Epistemológico Alternativo? Para dar respuesta a la pregunta y lograr nuestro objetivo, se realizó un análisis histórico, epistemológico y contextual del citado trabajo, posibilitando a partir de allí, en un proceso de transposición didáctica y utilizando la teoría antropológica de lo didáctico, la elaboración de MEA para la enseñanza del cálculo para licenciaturas en Matemáticas.

Palabras clave: Modelo epistemológico alternativo, Enseñanza del cálculo, Didáctica de las matemáticas.

Résumé

L'article vise à proposer un modèle épistémologique alternatif (MEA) pour l'enseignement du calcul dans les cours de licence (licenciatura) de mathématiques, en utilisant les Principia d'Isaac Newton. Pour atteindre notre objectif, nous partons de la question de recherche suivante : Quels sont les objets mathématiques ou les artefacts historiques présents dans Principia qui serviront de support à la création de ce modèle épistémologique alternatif ? Afin de répondre à la question et d'atteindre notre objectif, une analyse historique, épistémologique et contextuelle du travail précité a été réalisée, permettant à partir de là, dans un processus de transposition didactique et en utilisant la théorie anthropologique de la didactique, l´élaboration de MEA pour l'enseignement du calcul pour les diplômes en Mathématiques.

Mots-clés : Modèle épistémologique alternatif, Enseignement du calcul, Didactique des Mathématiques.

Resumo

O artigo tem como objetivo propor um Modelo Epistemológico Alternativo (MEA) para o ensino de cálculo nas licenciaturas em Matemática, usando os *Principia* de Isaac Newton. Para lograr nosso objetivo, partimos da seguinte questão de pesquisa: *Quais são os objetos matemáticos ou artefatos históricos presentes nos Principia que darão subsídios para a criação* *desse Modelo Epistemológico Alternativo?* Para respondermos à questão e atingirmos nosso objetivo, foi realizada uma análise histórica, epistemológica e contextual da obra citada, sendo possível a partir daí, em processo de transposição didática e utilizando a Teoria Antropológica do Didático (TAD), a elaboração de um MEA para o ensino de cálculo para as licenciaturas em Matemática.

Palavras-chave: Modelo epistemológico alternativo, Ensino de cálculo, Didática da matemática.

Principia by Isaac Newton: A proposal for an epistemological reference model for comprehensive teaching in mathematics teaching degree courses.

This article uses the theoretical assumptions of didactics of mathematics, such as the anthropological theory of the didactic (ATD) and the reference epistemological model (REM), to propose an alternative for calculus classes. Thus, the objective of our article is to propose an alternative epistemological model (AEM) for teaching calculus in mathematics teaching degrees, using the *Principia*[4] by Isaac Newton.

Starting from the conjecture that, despite calculus, especially the teaching of integrals, being a very present object in mathematics education research[5], these works do not exhaust the range of possible methodological approaches for classes on this mathematical object. Therefore, it becomes relevant for the field of study of didactics of mathematics and mathematics education to propose an EAM for teaching the introduction of integrals, more precisely, the calculation of the area under a curve, for mathematics teaching degrees, using the mathematical organizations present in the *Principia*.

To construct the EAM, excerpts from the work under study, especially the one referring to the mathematical object, which is the focus of our research, must undergo a process of didactic transposition. According to Chevallard (1985), didactic transposition is a process in which scientific knowledge is transformed into knowledge to teach, i.e., transforming a mathematical object of knowing, produced by a mathematician, into an object of school knowing, that is, into a didactic organization.

Our research problem emerges when addressing the idea of constructing an epistemological model for teaching the notion of integrals to prospective mathematics teachers. We recall, based on Corazza's (2003) words, that "... to constitute a research problem is... to ask whether that element of the world –of reality, things, practices, reality– is so natural in the meanings that are its own..." (p. 118), to ask:

What are the mathematical objects or historical artifacts present in the Principia that will provide support for the creation of this alternative epistemological model?

In an attempt to answer this question, we formulate the following hypothesis: *In Isaac Newton's work Mathematical Principles of Natural Philosophy, one can find mathematical objects or historical artifacts that could subsidize the creation of an alternative epistemological model for teaching calculus in mathematics teaching degree courses.*

4The work *Philosophiæ Naturalis Principia Mathematica* [Mathematical Principles of Natural Philosophy] published by Isaac Newton (1643-1727) in 1686, also known as *Principia*, the plural of *principium*, comprises three books.

5 Validated by the Capes Thesis Bank.

Due to the nature of this research, the methodological procedures can be divided into two moments, the first related to the history of mathematics when seeking to construct a text based on mathematical objects or historical artifacts contained in the book *Mathematical Principles of Natural Philosophy*, and the second, related to the construction of the alternative epistemological model, using these mathematical objects or historical artifacts.

About the first moment: when seeking subsidies in history, the research is classified in the list of qualitative research of the documentary bibliographic type under a historical-descriptive approach since it proposes to investigate the various historical contexts that led to the epistemological development of calculus in the 17th century.

In this research, we argue that these subsidies are present in the book *Mathematical Principles of Natural Philosophy* as the book brings some of the foundations of Newtonian calculus. However, as it is a book originally published in 1686, its language and epistemology, the truths that validate it, are no longer current. Therefore, a didactic transposition is necessary.

About the second moment: we will use some concepts from didactics of mathematics, more specifically from the anthropological theory of the didactic (ATD), such as mathematical organization (MO) and didactic organization (DO), as explained below:

> Chevallard (1999) defines didactic organization as the set of types of tasks, techniques, technologies, etc., mobilized for concrete study in a concrete institution, while Bosch (2001) calls mathematical organization an entity composed of types of problems or problematic tasks, kinds of techniques that allow solving the types of problems, technologies, or discourses ("logos") that describe and explain the techniques. This theory underpins and organizes technological discourses (Ordem & Almouloud, 2010, p.70).

Thus, according to what was explained by Chevallard (1991), Bosch (2001), and Ordem and Almouloud (2010), we can say in general that an MO is mathematical knowing produced by a university institution, in which there is no teaching intention, and that a DO can originate from a MO that, when undergoing a process of didactic transposition (Chevallard, 1991) in an institution, assumes a teaching intention.

Therefore, to meet the objective of this research, which is to *develop an alternative epistemological model for teaching notions of calculus for mathematics teaching degree courses using mathematical organizations present in* the *Principia* and through a process of didactic transposition that can be understood as "... the passage from an object of knowing to an object of teaching..." (Chevallard, 1991 *apud* Almouloud, 2007, p.113), these organizations will be mobilized in teaching activity, with an intention in an institution, that is, they will be didactic organizations (Dos).

The issue of modeling mathematical objects

The didactic transposition developed by Chevallard (1991) in his first reflections on mathematics teaching aimed to distinguish the different knowings involved in the teaching and learning process. According to the mathematician, it was necessary to distinguish between the mathematics of the teacher, the mathematics of the student, and the mathematics of the researcher, as each of these individuals uses a mathematics with its own characteristics

For this reason, the mathematics to teach in schools or colleges is necessarily the result of other mathematics that has undergone a process of didactic treatment. These mechanisms that allow the transition from an object of knowing to an object of teaching are grouped under the name of didactic transposition.

> The theory of didactic transposition categorizes mathematical objects into:
> **paramathematical**: Tools that are used to describe and study other mathematical objects;
> **mathematical**: In addition to being valuable instruments for studying other mathematical objects, they become objects of study in themselves;
> **proto-mathematical**: They have properties used to solve some problems without acquiring the status as an object of study or as a tool for studying other objects. (Almouloud, 2007)

However, the insufficiency of this classification in the process of reflection on phenomena related to didactic processes gave rise to a new theory, the ATD. In other words, this theory emerges to expand the relationship between mathematics and individuals because:

> Chevallard emphasizes that mathematical knowing organizes a particular form of knowledge, the product of human action in an institution characterized by anything that is produced, used, and taught, besides being able to eventually transpose the institutions. (Almouloud, 2007)

From this perspective, Chevallard (1999) lays the foundations for the development of a didactic anthropology in which the object of study is the relationship between the teacher and/or student and mathematical knowledge. For example, the teacher and the student facing a theorem.

Therefore, from the development of this theory, some theoretical precepts emerge, which guide those who use it. We will present the most necessary ones to achieve the objective of this work.

Some notions of the ATD and REM:

Yves Chevallard's TAD (1991, 1992, 1999), resulting from the problem of didactic transposition between institutions, is considered a fundamental analysis tool in the didactics of mathematics[6]. This theory follows the guiding line of the Programa Epistemológico de Investigação em Didática da Matemática [Epistemological Research Program in Didactics of Mathematics] created in the 1970s from Guy Brousseau's studies, which gave rise to the theory of didactical situations (TDS).

The ATD situates mathematical activity and, consequently, the activity of studying mathematics within human activities and social institutions. According to Almouloud (2007), "The ATD studies the conditions of possibility and functioning of didactic systems, understood as subject-institution-knowing relationships (about the didactic system addressed by Brousseau, student-teacher-knowing)" (p.111).

Therefore, for this theory, mathematical activity and mathematics teaching are considered anthropological phenomena, according to the following reflection:

> The ATD is based on the understanding that human beings act by coming together in groups—institutions—that impose a particular way of doing and thinking on the development of their activities. In this sense, a teacher's doing, when solving an equation in class or correcting their students' exams, take as reference constructions developed in institutions, resulting from a collective production in which that teacher participated and participates, but which they assume as their own (Andrade, 2012).

In this sense, the ATD ensures that such actions can be described in their accomplishment by a model that Chevallard (1991) summarizes in the word *praxeology*. So, to support this process of analysis, study, and explanation of such didactic actions, the ATD employs three primitive elements as follows:

*Institutions (**I**)*: the social instances that guide the individual in the way they act and think.

*Individuals (**X**)*: subjects who become active when they occupy the place that people occupy in institutions. By occupying certain positions in institutions, individuals make institutions exist.

6 The didactics of mathematics was born approximately 40 years ago, and, although it is certainly a human science –a science of human activities in society– it carries the ambition of constructing rigorous theories that can constitute models for the analysis of teaching and learning phenomena in mathematics. Mathematics in a didactic environment: a social environment designed for teaching (Almouloud, 2007 p.13).

And the *object* (**O**): fundamental element of the ATD, which, in turn, postulates that "everything is an object" and that it only exists from the moment in which an individual (***X***) or an institution (***I***) recognizes it as existing.

According to this theory, an individual's relationship with an object of knowing is only established when the person enters an institution in which that object exists. Likewise, the institutional activities of a teaching environment (school, study group, classrooms) are linked to the institutional activities requested of individuals. In this way, the praxeological relations of individuals (**X**) with the objects (**O**) in institutions (***I***) are made through four notions.

These notions are the task (**T**), the technique (**τ**), technology (**ɵ**), and theory (**Θ**); notions that allow us to model mathematical activities as a social practice and which will be briefly presented below:

First: Task (**T**), which is nothing more than an action with a well-defined objective, for example, finding the GCD, measuring the height, and drawing the quadrilateral. These tasks become routine when they are no longer challenging to execute.

There is one or a certain number of techniques to carry out a given task. The second notion, technique (**τ**), must be recognized by the institution that problematized the task (**T**). According to Almouloud (2007), "... alternative techniques may exist in other institutions..." (p.115). The techniques correspond to the way of doing/solving/performing the corresponding task.

In turn, the technique that is used to perform a certain task requires the individual to use a particular technology (**ɵ**), which is the third notion. Technology (ɵ) justifies the technique used; that is, the technology will provide logical and rational support to the technique. In this sense, it will be more linked to the discourse so that the technique can be understood and justified in carrying out the task. The last notion is theory (**Θ**), which justifies technology through scientific arguments.

The four notions: types of tasks (**T**), technique (**τ**), technology (**ɵ**), and theory (**Θ**) make up a complete praxeological organization [**T**/**τ**/**ɵ**/**Θ**] that is subdivided into two blocks: (1) the practical-technical block [**T**/**τ**], formed by certain types of tasks and a technique corresponding to the knowing-how-to-do; (2) the technological-theoretical block [**ɵ**/**Θ**], containing a theory that justifies a technology.

To illustrate these fundamental notions, we will present an example in which the objects of calculation are related to the ATD:

> Calculate the derivative of a function *f* on point x_0 of its domain is a type of task for which there is the technique of calculating the limit of a function at a point, with a technological-theoretical environment on functions, their graphical representations, and function limits (Mateus, 2007).

However, the theoretical principles that guide those who research and/or apply the ATD in their studies go beyond these ideas. In this sense, this study uses what is known in the ATD as epistemological models, which can be classified as dominant, of reference, or alternative. The first type is the most common and serves as an epistemological basis for the knowledge studied in an educational context. However, this model is not always suitable for establishing a necessary link between teaching and learning. Therefore, according to the didactics of mathematics, the demand for alternative epistemological models arises, according to Pérez (2013):

> […] I propose to characterize the approaches or didactic theories that form part of the epistemological program as those that question the epistemological models of dominant mathematics in the various institutions (for example, institutions and schools) and, what is most important, as those that explicitly elaborate alternative epistemological models of different mathematics areas and use them as a reference system to formulate and approach didactic problems. (p.71)

In this context, the didactic scope of a DEM becomes a target for reflection when restrictions of this model are observed regarding its didactic use, thus raising the need to develop new models, that is, an AEM, as "it can be useful to guide us toward the type of didactic problems that the different approaches pose and address, and also about what is considered in each case as an acceptable response to said problems" (Gascón, 2013, p.72).

A good example of AEM related to calculus can be found in Figueroa and Almouloud (2018), where the authors aim to "contribute to the teacher formative process based on reflections on a REM, which considers the incompleteness of institutional work related to the mathematical object function limit of a real variable" (Figueroa & Almouloud, 2018, p.1). Based on the analysis of mathematical organizations, in light of the ATD in textbooks and student notebooks, and a process of epistemological construction, the researchers propose a REM for this mathematical object.

Still about REM, Bolea's research (2010) questions the dominant epistemological model that is used to introduce algebra as being generalized arithmetic and discusses the scope of this model:

> According to Gascón (1993; see also Bolea, 2003), the usual epistemological model of school algebra highlights the similarities between arithmetic and algebra and tries to

present the second as a continuation of the first, as generalized arithmetic. This model does not fit the vision of an algebra whose objectives and techniques radically differ from arithmetic. (Bolea, 2010, p.582)

In her research, the author proposes an epistemological model for teaching whole numbers, arguing that the reference epistemological model is insufficient to support all phenomena related to teaching these numbers.

Regarding the teaching of calculus, the dominant epistemological model taught in the vast majority of teaching degree courses[7] is what follows the direct line of the calculus by Gottfried Wilhelm Leibniz (1646-1716), which uses the idea of infinitesimals in a predominantly algebraic approach and employs the idea of limits and functions "incorporated" by Augustin-Louis Cauchy (1789-1857). On the other hand, for several reasons, the explanation of which goes beyond the objectives of this work, Newton's calculus developed using the method of the first and last ratios of quantities in a geometric approach was practically abandoned.

Assuming that some DEMs do not support all phenomena related to a mathematical object, as is the case with calculus, and that the construction of an alternative epistemological model can help in understanding them, we will launch the proposal for the construction of a REM for teaching calculus in mathematics teaching degree courses based on the calculus developed by Newton.

To construct this REM, we developed two tasks (T) based on the MOs found in *Principia*. These MOs, in turn, undergo a process of didactic transposition to modify some mathematical structures and make them DOs, that is, "... the passage from an object of knowing to an object of teaching..." (Chevallard,1991 *apud* Almouloud, 2007, p.112-113).

Some brief considerations regarding Newton's calculus in *Principia*

This section is dedicated to making a brief historical, contextual, and epistemological analysis of *Principia.* It is important to note that when presenting the method of the first and last ratios of quantities in his work, Newton demonstrates it in an epistemology valid for his time. Nevertheless, the fact that Newtonian calculus is presented shily in this work lies in the fact that the main objective in *Principia* was not to publicize his method of calculating areas but to demonstrate the law of gravity based on the unification of the laws that govern the

7 This statement requires further investigation. It is dangerous from a methodological point of view because it claims that no calculus course uses alternative epistemological models. For this statement, we were based on Matos and Almouloud's (2010) work, which uses the reference model when analyzing the praxeologies of textbooks used in calculus courses.

movement of bodies on Earth –Newton's laws– with the laws that govern bodies in space –Kepler's laws– which were already known, but not rigorously proven. Throughout his work, Newton enunciates his three laws, proves Kepler's laws, and finally demonstrates the law of universal gravitation. In this context, calculus is a tool for calculating moving areas.

Principia

The work entitled *Mathematical Principles of Natural Philosophy*[8] published by Isaac Newton (1643-1727) in 1686, also known as *Principia* –the plural form of *principium*–, comprises three books, the first two of which lay the foundations of the basic principles of movement and the third applies these principles to the solar system (Cohen & Westfall, 2002).

We can say that *Principia* presents mathematical principles to prove the laws of natural philosophy. In this regard, it is worth noting, as emphasized by Alfonso-Goldfarb (1994), that, at that time, there were no areas of knowledge as we understand them today. At that time, natural scholars referred to the investigation of the whole of nature as natural philosophy. In this way, individuals like Newton, considered natural philosophers, analyzed different natural phenomena. Thus, we must understand what Newton said in his *Principia* when he referred to the purposes of his work:

> Newton (2008) [in this work] examines above all things that relate to gravity, lightness, elastic force, the resistance of liquids, and similar forces, whether attractive or impulsive; thus, I offer this work as constituting the mathematical principles of philosophy, since the whole task of philosophy seems to consist in this: to investigate, from the phenomena of movements, the forces of nature, and from these forces to demonstrate other phenomena, and it is to this objective that the general propositions of books I and II are directed. In Book III, I give an example of this in the explanation of the system of the world, for, from the propositions mathematically demonstrated in the two preceding books, I deduce in the third, from the celestial phenomena, the forces of gravity with which bodies tend towards the Sun and the various planets... (p.14)

In his preface, Newton clarifies that the first two books deal with general propositions that are "mathematically demonstrated." Those propositions are later mobilized to demonstrate celestial phenomena, bringing mathematics closer to natural philosophy. In the third book, as Newton observes, he deduces "from celestial phenomena the forces of gravity with which bodies tend towards the Sun and the several planets." In other words, in the first two books, Newton lays a solid foundation so that, in the third book, he can enunciate and prove the law of universal gravitation.

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8 Work originally written in Latin: Philosophiae Naturalis Principia Mathematica.

The first two books of the *Principia* bring several mathematical propositions organized axiomatically. Those propositions served to justify his demonstrations, which were geometric in nature and basically followed the same axiomatic structure found in Euclid’s *Elements* (360-295 BC).

Euclid’s *Elements* brings an organization the author used to present and justify his arguments, which were definitions and axioms, which are truths accepted without proof; theorems, which are truths proven with the help of axioms and definitions; corollaries, which are statements resulting from theorems; lemmas, which are theorems that serve to help prove a theorem of greater importance; and propositions, which are sentences associated with another theorem of lesser mathematical importance. Newton uses this same argumentative rigor in his work, developing his proofs on the movement of bodies. For Newton, only geometry had the fundamental elements to demonstrate the phenomena of nature (Cohen & Westfall, 2002), because geometry was based on a system of accepted truths, which led to demonstrations of other truths.

Indeed, as in the *Elements*, Newton organized his propositions by initially presenting definitions and axioms, followed by theorems (propositions). Each theorem is demonstrated geometrically from the axioms and definitions. Some corollaries and scholia follow the theorems.

Textually, the first book of the *Principia* (Book I) is divided as follows: Newton presents a preface, followed by eight definitions and three axioms or laws of motion, currently called Newton’s three laws. The work then presents fourteen sections comprised of propositions or theorems dealing with the movement of bodies. We will now present one of the definitions.

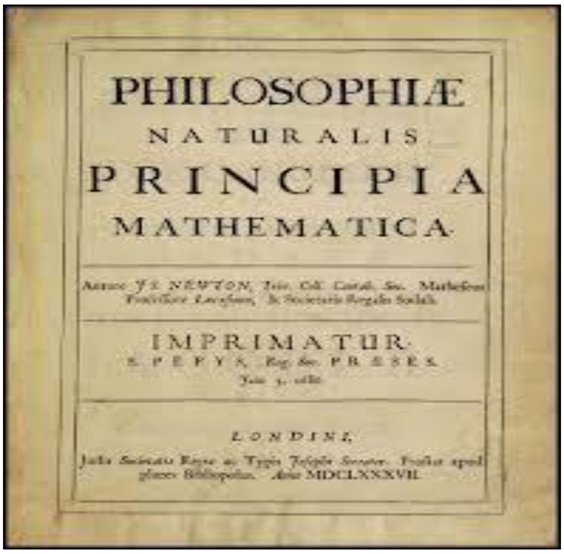

Figure 1.

Cover of the Principia published in 1686 (istockphoto, 2022)

Definitions

Principia initially displays eight definitions related to the laws of motion. In the same way that Euclid defined the objects of geometry, Newton sought to define the objects of his dynamics. We can say that the definitions underpin and form the basis of the *Principia.* We will only mention the fifth definition and then present a brief explanation:

> **Definition V** -A centripetal force is by which bodies are directed or impelled, or tend in any manner, toward a point as a center.

It is important to emphasize that Newton, by making clear what centripetal force is, introduces the idea that, according to Cohen and Westfall (2002), is central to *Principia*:

> The concept of centripetal force expressed the perception that circular or orbital movement is an accelerated movement and that a body will only continue to move in a closed orbit as long as a force that prevents it from reaching the center (centripetal means "that which seeks the center") sustains it on this trajectory. (p.273)

Based on these definitions, Newton stated that due to this force, celestial bodies would move in circular or orbital trajectories in an accelerated motion and would remain in this state as long as the force that impels them towards the center continues to act. He called this force centripetal force, which is the opposite of the already-known centrifugal force ("which moves away from the center"). However, he had not arrived at the generalized conception of force, which would only happen in what is known today as Newton's second law.

We believe that centripetal force has a special place in *Principia* because Newton sought to understand the movement of celestial bodies. As we have seen, this force would be responsible for keeping one body rotating around another.

After the definitions and axioms had been set out, eleven lemmas were presented, demonstrated with the assistance of the first and last ratios method. This method represents the contemporary concept of the limit of a function, although our focus is on integrals. We must epistemologically understand how Newton conceived the idea of limit in his work since, from a modern perspective, the definite integral is linked to the sum of infinitesimals. Newton named "Book I: The Motion of Bodies" the part of the *Principia* that brings together the eleven lemmas in section I and the other ninety-eight propositions distributed across thirteen more sections.

According to the objective of our article, we will analyze only a small part of section I, which includes Lemma I, which deals with the infinitely small; Lemma II, which refers to areas of curves; and we only state Lemma III, in addition to some corollaries.

Section I is titled: The method of first and last ratios of quantities, with the help of which we demonstrate the propositions below.

The lemmas presented in this section correspond to elementary notions of differential calculus. Even though there is no explicit reference to calculus, such as symbols or other terminology, we can see that the lemmas deal with the limits of areas, lines, and arcs of curves, as observed by the mathematician and author of calculus books:

> Delachet (1967) must give thanks to the genius of Newton, who knew how to explain in his *Philosophiae Naturalis Principia Mathematica* (published in 1687) the rules of his infinitesimal calculus without using the unique terminology or symbols he had invented in this regard. (p.35)

In this sense, the quote corroborates our hypothesis that in the work *Mathematical Principles of Natural Philosophy* by Isaac Newton, one can find mathematical objects or historical artifacts that could support the creation of an alternative epistemological model for teaching calculus in mathematics teaching degree courses.

In the following lemmas, Newton presents the method of calculating infinitely small quantities, which will serve to demonstrate his propositions about the movement of bodies. This method, which he calls the method of first and last ratios, consists of proving that if there are two infinitely small quantities and if these quantities move toward each other, they will become equal at the end of a specific time. The method developed by Newton is naturally intuitive, as we will see below:

The lemmas

> **Lemma I** – Quantities, and ratios of quantities, that in any finite time continually converge to equality, and before the end of that time approach nearer to each other than for any given difference, becoming finally equal9.

This lemma presents one of Newton's main ideas of calculus. He discusses infinitely small quantities and the movement of these quantities, as Baron (1985) emphasizes:

> What Newton seems to be saying here is that if we have two quantities, say Q1 and Q2, which vary in time, and if the difference between Q1 and Q2 continually decreases in

9 Ibid., p. 71.

> such a way that within a finite interval of time, they come closer and closer to each other, then we eventually have Q1=Q2 (p.31).

The idea that quantities approach each other to the point of becoming equal in a finite time interval leads us to consider the importance of movement and the notion of velocity in formulating a limit.

It seems to us that, in some way, Newton was looking for a satisfactory intuitive basis to support his calculation. The key notion in this process seems to be instantaneous velocity. Newton was uncomfortable with infinitesimals, which he felt had dubious geometrical credentials. He tried to eliminate them by proposing the idea of instantaneous velocity (Cohen & Westfall, 2002, p.453).

Indeed, this seems to be reinforced by another passage where Newton observes that:

> I will not consider mathematical quantities here as composed of *extremely small* parts but as *generated* by a *continuous movement*. Lines are described, and by describing them they are generated, not by an alignment of parts but by a continuous movement of points. Surfaces are generated by the movement of lines, solids by the movement of surfaces, angles by the rotation of their sides, time by a continuous flow, etc. This genesis is based on nature and can be seen every day in the movement of bodies. (Baron, 1985)

However, although Newton denied infinitesimals, they are present in his study since the idea of instantaneous velocity implicitly implies the idea of an infinitesimal distance traveled by a body in a finite time (Cohen & Westfall, 2002, p.453). Newton's denial of the infinitesimals was due to philosophical rather than mathematical reasons.

We will not delve into these issues; we simply want to note here that issues about mathematics and nature were intertwined with many others linked to the very mathematical procedure. Roughly speaking, we can say that, in the 17th century, two major philosophical currents conflicted with each other (Meneghetti & Bicudo, 2002). On the one hand, a group of people argued that mathematical truths should be subjected to arithmetic to be achieved. This is because the objects of arithmetic would be more abstract than those of geometry. In turn, another group of mathematical scholars argued that man would only reach mathematical truths through observation and experimentation and that, therefore, the geometric method would be preferable to the arithmetic one because geometry, unlike arithmetic, was much closer to sensible reality.

In other words, when treating nature mathematically, it would be more prudent to seek mathematical procedures that are more appropriate to it.

As is well known in *Principia*, Newton founded the basis of calculus on geometry. According to Meneghetti and Bicudo (2002, p.109), he would have been influenced by the ideas of Isaac Barrow (1630-77), who was his professor at Cambridge. Barrow criticized the arithmetization of calculation and analytical symbolism, seeking to value sensory evidence.

The above leads us to agree with Meneghetti and Bicudo (2002) that Newton's work had a more intuitive character since the treatment he gave to the "infinitely small" sought to avoid the idea of "infinitesimals":

> His view of limits, for example, especially in his early work, was based on geometric intuitions [...]. Influenced by the 17th-century thought, he was led to think about ultimate geometric indivisibles, and, in his theory, he uses terms such as ratios and ultimate forms, expressions that stem from rigorously correct abstract interpretations, but which strongly suggest others, in terms of an intuitively more attractive vision produced by infinitesimals. Newton's concept of limit was heavily dependent on the idea of the infinitely small. This dependence can be seen in his *Principia*, when he speaks of the nature of ultimate ratios… (p.111-112)

Thus, it is in this sense that we must understand Lemma I. If we can refer to Newton's idea of limit, it was based on infinitely small quantities that, when approaching each other to the point where there was no longer any difference between them, would become equal -instead of the modern notion of infinitesimal. This principle, which Newton called the method of the first and last ratios of quantities[10], underpins the calculation of the area under a curve.

Reference Epistemological Model

Finding the area under a curve is one of the pillars of calculus, more precisely, of the study of integrals. Newton, by applying his method to parallelograms, proved that it is possible to find such areas, as shown in lemmas II and III, which we present below, together with some corollaries.

> **Lemma II -** Suppose that in any figure AacE, bounded by the straight lines Aa, AE, and the curve acE, there is any number of parallelograms Ab, Bc, Cd etc., of equal bases AB, BC, CD etc., parallel to a side Aa of the figure; and the parallelograms aKbl, bLcm, cMdn etc., is completed; then, if we suppose that the width of those parallelograms was progressively diminished and their number increased *ad infinitum*, I affirm that the final ratios that the inscribed figure AKbLcMdD, the circumscribed figure AalbmcndoE and the curvilinear figure AabcdE, will have for each other, are ratios of equality.[11]

10 Ibid., p. 71.
11 Ibid., p.71-72.

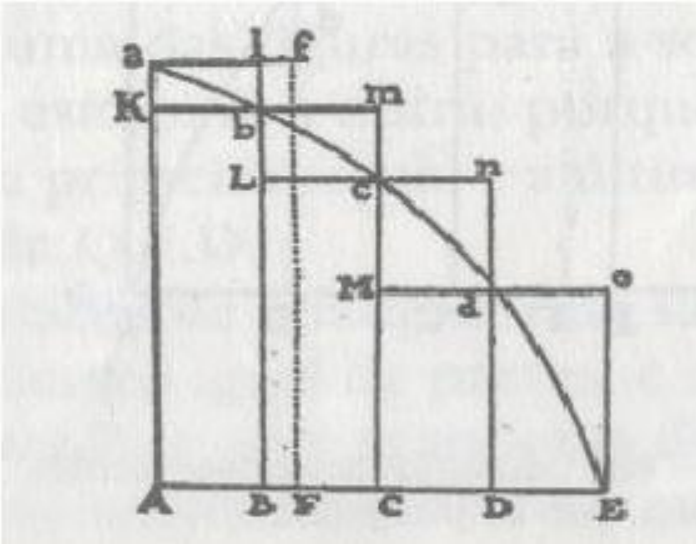

Figure 2.

Geometric representation of Lemma II (Newton, 2008 p.72)

Lemma II demonstrates that we can determine the area of a curve using the method of first and last ratios of quantities.

Newton sought to show that to determine the area under a curve with center A, delimited by points E, c, a, as shown in the figure, we should proceed as follows:

1-Draw any number of parallelograms inscribed in this curve. In the case of the figure, there are three bases: $\overline{AB}$, $\overline{BC}$, $\overline{CD}$.

2- With the help of other smaller parallelograms, subscribe to the curvilinear figure. In this case, the base parallelograms: $\overline{Kb}$; $\overline{Lc}$, $\overline{Md}$; $\overline{DE}$ will have this role.

Newton argues that if the width of the parallelograms is made smaller and smaller, it will result in a higher number of them and that, in this case, if the width reduces progressively, the number of parallelograms will increase at the same rate until it reaches an infinite number.

Consequently, the area of the curvilinear figure will be the area of the inscribed and circumscribed parallelograms.

Newton extends this idea to parallelograms of unequal widths, stating that if these widths decrease *ad infinitum,* the area of the plane figure will also be that of the parallelograms (Lemma III). From these lemmas, he presents some propositions, which are:

Corollary I – Thus, the final sum of those evanescent parallelograms will coincide in all parts with the curvilinear figure[12].

According to our chosen translation[13], the word evanescent is understood by quantities as small as one wants, i.e., infinitesimal.

As the width of the parallelograms decreases, and when parallelograms of infinitesimal width are added, the area of the curvilinear figure coincides.

Corollary II – The rectilinear figure, limited by the chords of the evanescent arcs ab, bc, cd, etc., will finally coincide even more with the curvilinear figure[14].

Thus, as the width of the parallelograms aKbl, bLcm, cMdn etc., which complete the curvilinear figure, is reduced, the arcs that form their diagonals are reduced infinitely until they become rectilinear, contributing to the figures becoming equal.

Corollary III – In the same way, the circumscribed rectilinear figure is limited by the tangents of the same arcs[15].

Likewise, when reduced to infinity, the upper bases of the circumscribed parallelograms will coincide with the curvilinear figure.

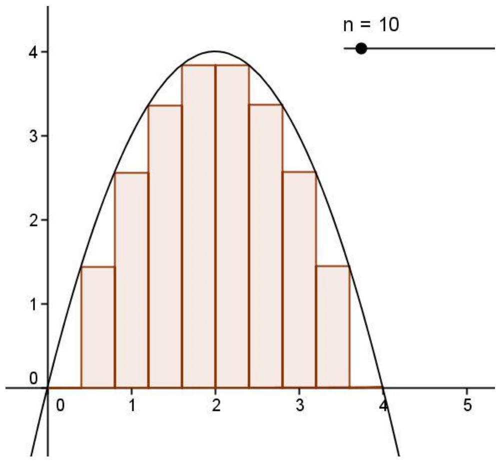

Figure 3.

Parallelograms inscribed in the function graph (Geogebra.org)

12 Ibid. p.72
13 Translation by edusp - several translators.
14 Ibid. p.73
15 Ibid. p.73

Corollary IV – Therefore, these final figures (regarding their perimeters acE) are not rectilinear but curvilinear boundaries of rectilinear figures[16].

The segments of the figures, straight at the beginning and apparently curved at the end, ceasing to be rectilinear, are actually curvilinear limits of the rectilinear figures.

Based on the analysis of the *Principia,* , we will present two tasks that will be solved using some of Newton's arguments to find the area below the graph. At the end, we will briefly analyze the tasks and mobilize the ATD concepts. We will use Geogebra, a dynamic geometry software, to better visualize the resolution of these tasks.

Task (**T**): Using Newton's method of first and last ratios of quantities, calculate the approximate area under the graph of the function $f(x) = -x^2 + 4x$ in the interval [0,4].

To solve **T,** we will use **Lemma II,** which demonstrates that the method of first and last ratios of quantities determines the area of a curve.

Thus, according to Newton, we should proceed as follows:

1-Draw any number of parallelograms inscribed in this curve.

Using the slider button, we can see that if the number of inscribed parallelograms is equal to 8, and, adding the areas, we have:

S_1 = 8.96 units of units of area (ua)

2- With the help of other parallelograms with the same base as item 1, circumscribe the curvilinear figure.

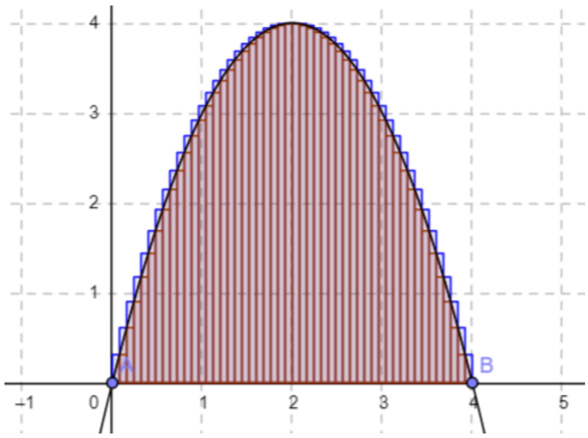

Figure 4.

Parallelograms circumscribed around the graph of the function (Geogebra.org)

16 Ibid. p.73

Repeating the procedure with the slider button, when the number of circumscribed parallelograms is equal, it is equal to 10, and, adding the areas, we have:

S_2 = 12.16 ua

Repeating the procedures of **Lemma II**, but in this case, with an increased number of parallelograms, we have:

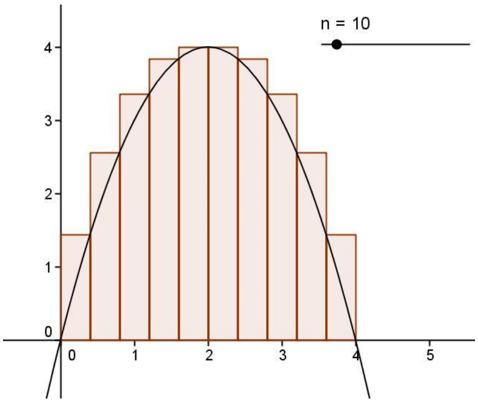

Figure 5.

Increasing the number of parallelograms to 50 (Geogebra.org)

Increasing the number of parallelograms to 50, we have the following sums of areas:

S_1= 10.34 ua

S_2= 10.98 ua

Increasing the number of rectangles to 1000, we have the following sums of the areas:

S_1= 10.66 ua

S_2= 10.66 ua

Corollary I – Thus, the final sum of those evanescent parallelograms will coincide in all parts with the curvilinear figure.

Therefore, the approximate area under the graph of the function $f(x) = -x^2 + 4x$, in the interval [0,4], will be 10.66 ua

Although the method of first and last ratios of quantities is used in *Principia* to calculate the area under a curve, we can extend the idea and apply it to calculate the area under the graph of any function. In the following task, we will use the method to calculate the area below a line.

Task (**T) -** Using the method of equality ratios, calculate the approximate area below the graph of the function f(x)= x+1 in the interval [-1,3]

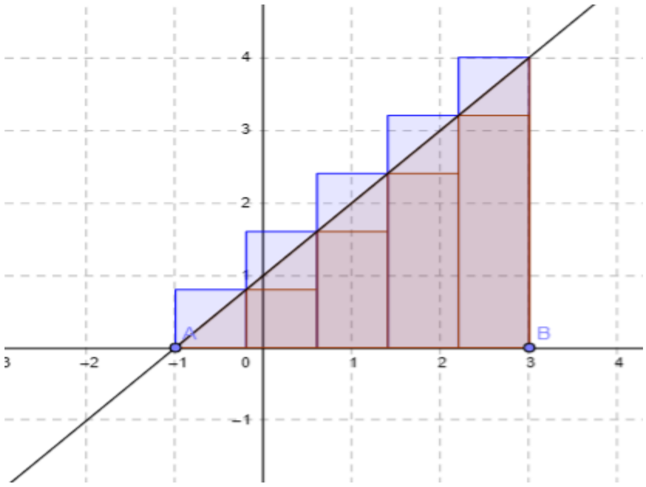

Figure 6.

Calculating the area in the interval [-1,3] (Geogebra.org)

Applying **Lemma II** and adding the areas of the circumscribed and inscribed parallelograms, we have:

S_1 = 9.6 ua

S2= 6.4 ua

Increasing the number of parallelograms to 50 and adding the areas of the circumscribed and inscribed parallelograms, we have:

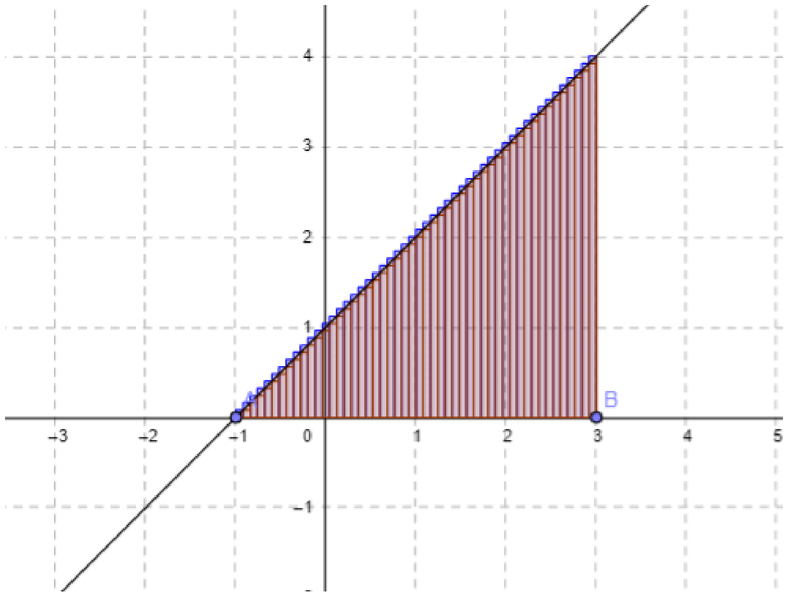

Figure 7.

The sum of the areas of the circumscribed and inscribed parallelogram (Geogebra.org)

S_1 = 8.1 ua

S_2 = 7.8 ua

Increasing the number of rectangles to 1000, we have the following sums of the areas:

$S_1 = S_2$ = 8 ua

Applying **Corollary I**, we have:

S = 8 ua (area below the graph)

Task analysis: (T): Calculate the area under the graph of a quadratic polynomial function on a given interval. (T): Calculate the area under the graph of a first-degree polynomial function on a given interval. Technique (**τ**): Method of first and last ratios of quantities Theoretical/technological discourse [**ɵ/Θ**]: Following Lemma II of the *Principia*, to calculate the area under a curve, we must add the area of parallelograms registered and circumscribed in it. Using the method of first and last ratios of quantities, we found that the greater the number of parallelograms, the more precise the area will be. And that the sum of the area of the inscribed and circumscribed parallelograms tends to be equal as the number of parallelograms increases, and their upper bases begin to be interpreted as part of the graph curve, meeting Corollaries I, II, III, and IV. Therefore, we can conclude that the area of any graph is the area of the sum of the parallelograms, whether they are inscribed or circumscribed in the graph.

Figure 8.

*Analysis of **T** from the ATD perspective*

Final considerations

Solving the task using Newton's calculus shows that it allows solving problems involving the area under a curve, as with polynomial functions. When Newton published his *Principia*, the idea of a modern function did not exist, and his primary purpose was to calculate areas of moving bodies. However, task (**T**) showed that his procedures can be used to find areas below graphs of functions.

Task (**T**) showed us that procedures used by Newton to calculate area, such as the use of the "methods of the first and last ratios of quantities," Lemma II, and the corollaries, are tools that, when undergoing a process of didactic transposition, are an alternative to the current DEM, which uses Leibniz's calculus.

Finally, when presenting the solution to task (**T**) through theoretical-technological techniques and discourses using Newton's calculus in light of the ATD, we propose an AEM for teaching calculus for mathematics teaching degrees.

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