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Understanding of derivative concepts by undergraduate mathematics students from three inland institutions in the state of Paraná

Comprensión de conceptos derivados de la licenciatura en matemáticas estudiantes de tres instituciones del interior del estado de Paraná

Compréhension des concepts dérivés des étudiants en mathématiques de trois institutions de l'intérieur de l'état du Paraná

Compreensão de conceitos de derivada de estudantes de licenciatura em matemática de três instituições interioranas do estado do Paraná

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Abstract

Learning mathematics is intrinsically related to understanding, i. e. the apprehension and elaboration of meanings concerning mathematical objects, without neglecting their applications. Several official Brazilian documents emphasize that the focus on understanding should permeate the "initial and continuing training" of teachers or educators, indicating that training should have studies and practices on the subject on its agenda. This led us to ask: "What is the understanding of the concepts of derivative of a variable among mathematics undergraduate students at universities in western Paraná?" The subjects investigated come from three university campuses in the interior of Paraná. Richard Skemp's theoretical framework was used to produce the evaluations and study the research data. Analytical tables were drawn up based on the answers to the questionnaires and individual interviews. The analysis showed, according to established criteria, failures and shortcomings in the understanding of these concepts, indicating strong signs of non-lasting learning.

Keywords: Instrumental understanding, Relational understanding, Logical understanding, Initial formation of the teacher, Derivatives.

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Resumen

El aprendizaje de las matemáticas está intrínsecamente relacionado con la comprensión, es decir, con la aprehensión y elaboración de significados relativos a los objetos matemáticos, sin descuidar sus aplicaciones. Varios documentos oficiales brasileños destacan que el enfoque en la comprensión debe impregnar la "formación inicial y continua" de los profesores o educadores, indicando que la formación debe tener en su agenda estudios y prácticas sobre el tema. Esto nos llevó a preguntarnos: ¿Cuál es la comprensión de los conceptos de derivada de una variable entre los estudiantes de pregrado en Matemáticas de las universidades del oeste de Paraná? Los sujetos investigados provienen de tres campus universitarios del interior de Paraná. Se utilizó el marco teórico de Richard Skemp para elaborar las evaluaciones y estudiar los datos de la investigación. Se elaboraron cuadros analíticos a partir de las respuestas a los cuestionarios y de las entrevistas individuales. De acuerdo con los criterios establecidos, el análisis reveló fallas y deficiencias en la comprensión de estos conceptos, indicando fuertes señales de aprendizaje no duradero.

Palabras clave: Comprensión instrumental, Comprensión relacional, Comprensión lógica, Formación inicial del profesorado, Derivada.

Résumé

L'apprentissage des mathématiques est intrinsèquement lié à la compréhension, c'est-à-dire à l'appréhension et à l'élaboration de significations concernant les objets mathématiques, sans négliger leurs applications. Plusieurs documents officiels brésiliens soulignent que l'accent mis sur la compréhension devrait imprégner la "formation initiale et continue" des enseignants ou des éducateurs, ce qui indique que la formation devrait mettre à l'ordre du jour des études et des pratiques sur le sujet. Cela nous a amenés à poser la question suivante : « Quelle est la compréhension des concepts de la dérivée d'une variable parmi les étudiants de premier cycle en mathématiques dans les universités de l'ouest de l'État du Paraná ?» Les sujets étudiés proviennent de trois campus universitaires de l'intérieur du Paraná. Le cadre théorique de Richard Skemp a été utilisé pour réaliser les évaluations et étudier les données de la recherche. Des tableaux analytiques ont été élaborés à partir des réponses aux questionnaires et des entretiens individuels. Selon les critères établis, l'analyse a révélé des échecs et des lacunes dans la compréhension de ces concepts, indiquant des signes forts d'apprentissage non durable.

Mots-clés : Compréhension instrumentale, Compréhension relationnelle, Compréhension logique, Formation initiale de l'enseignant, Dérivées.

Resumo

A aprendizagem em Matemática está intrinsecamente relacionada à compreensão, ou seja, à apreensão e elaboração de significados concernentes aos objetos matemáticos, sem deixar de lado suas aplicações. Vários documentos oficiais brasileiros enfatizam que o enfoque à compreensão deverá permear a “formação inicial e continuada” de professores ou educadores, indicando que formação deve ter em sua pauta estudos e práticas sobre o tema. Isso nos levou interrogar: “qual a compreensão dos conceitos de derivada de uma variável de estudantes de Licenciatura em Matemática de universidades do oeste[3] do Paraná?” Os sujeitos investigados são oriundos de três *campi* universitários do interior do Paraná. O referencial teórico de Richard Skemp foi assumido para a produção das avaliações e estudo dos dados da pesquisa. Foram elaborados quadros analíticos próprios a partir das respostas aos questionários e entrevistas individuais. A análise evidenciou, segundo critérios estabelecidos, falhas e fracassos na compreensão desses conceitos, indicando fortes indícios de uma aprendizagem não duradoura.

Palavras-chave: Compreensão instrumental, Compreensão relacional, Compreensão lógica, Formação inicial de professores, Derivadas.

[3]O primeiro autor estava afastado para pós-doutorado na Unioeste sob supervisão do segundo autor, cujo projeto de pesquisa resultou neste artigo.

Understanding of derivative concepts by undergraduate mathematics students from three inland institutions in the state of Paraná

Brazil has undergone many changes in education since 1996, the year in which the Law of Guidelines and Bases for National Education (LDB) was enacted (Lei nº 9.394, de 20 de dezembro de 1996). These changes have been motivated by several documents issued by the Ministry of Education (MEC), guided by the LDB, such as the National Curricular Parameters (PCN) (Brasil, 1998); more than 30 resolutions issued by the National Council of Education (CNE); and, recently, the National Common Curricular Base (BNCC) (Brasil, 2018).

These regulations and guidelines based on the LDB, in addition to regulating the right to education guaranteed by the 1988 constitution in schools, education departments and society, also serve to guide the teacher's actions in the classroom.

The PCN “Aims to build a framework that guides school practice […]”(Brasil, 1998, p. 15), and the BNCC “[...] defines the organic and progressive set of essential learning that all students must develop [...] throughout Basic Education” (Brasil, 2018, p. 7). These documents differ from other CNE resolutions that generally deal with administrative-political issues, as they are a reference for the teacher's practice in the classroom. As a result, both documents emphasize that this requested practice will influence the initial and continuing training of teachers (Brasil, 1998) or, in more recent terminology, educators (Brasil, 2018). One of the relevant topics for teacher or educator training emphasized in official documents is understanding, because according to the BNCC “Mathematics is intrinsically related to understanding, that is, to the apprehension of the meanings of mathematical objects, without leaving aside their applications” (Brasil, 2018, p. 276). The lack of understanding had already been cited in an official MEC document as one of the reasons for student retention: “In our country, the teaching of Mathematics is still marked by high retention rates, early formalization of concepts, excessive concern with skill training and mechanization of processes without understanding (Brasil, 1998, p. 19).

The term understanding, linked to the learning of mathematical concepts, appears more than 90 times in the PCN of Mathematics (Brasil, 1998) (Brasil, 2006) (Brasil, 2000) and more than 30 times in the mathematics chapters of the BNCC (Brasil, 1998), according to our analysis of the document. This indicates to us that understanding is a central theme in initial training courses for mathematics teachers, therefore, it should be on the agenda both in pedagogical practices and in research.

Therefore, when a mathematics teacher leaves university, he or she must understand the concepts which he or she will address with their students. This presentation of official documents and the reflection on their impact on the initial training of undergraduate teachers leads us to think and reflect on the understanding of undergraduate students regarding the mathematical content learned during the mathematics course.

However, talking about understanding is something broad, since there are different theories that study it, such as Anderson et al (2001, p. 31), who reviewed Bloom's Taxonomy (1956), and assume that "UNDERSTAND – Construct meaning from instructional messages, including oral, written and graphic communication". Therefore, in addition to more general theories, it is necessary to think about mathematical understanding in a situated way. In this article, because of the first author's post-doctoral study, we assume Skemp's theory (1987) that states there are three types: **instrumental understanding**, **relational understanding** and **logical understanding**. This theory has been assumed in international research in Higher Education, such as Weber (2002) and Keene et al (2011).

For Skemp (1987), relational understanding is the most important among them, since, as we can infer from his theory, it gives meaning to instrumental understanding for the proper operation and to logical understanding for the adequate recording of mathematical concepts.

Since the relevance of this topic, both nationally and internationally, we considered it pertinent to pursue the phenomenon of mathematical understanding among undergraduate students in Mathematics. However, it was necessary to delimit the scope, both in terms of the time to carry out the research and the significant subjects participating in the research, as well as geographic coverage.

Avoiding emptying the data production, we took into account the high dropout rate of undergraduate courses for training[4] teachers (Brasil, 2022, p. 86), since final subjects generally have few students, and we focused on a course (subject) that, according to MEC guidelines (Brasil, 2001), is mandatory content, and that, in general, undergraduate students must take in their first year of university, although there are exceptions. We chose the course that, according to national guidelines, contains in its syllabus the concepts of derivatives (Brasil, 2001), (Brasil, 2003).

Considering the legal, theoretical and experiential aspects, since we work in the undergraduate course, it became pertinent to ask: what is the understanding of the concepts of derivative of a variable by undergraduate students in Mathematics at universities in western Paraná?

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[4]In Brazil, undergraduate course for training teachers is named "Licenciatura".

It is important to highlight that the research did neither seek to evaluate teaching methodologies, teaching performance or curricula, nor to identify educational policies or socioeconomic conditions that lead undergraduate students to understand or not understand the concepts. What we want is to explain their understanding, linking it to the concepts of the authors cited above, under the guidance of the official curricula in force in our country, and to explain the meaning of these students' understanding and reflect on it in the context in which the data were produced.

This work does not end here. It is just the beginning; of course, we can investigate the understanding of other content that is part of the initial training of mathematics teachers. So there is room for replicating this project for other important subjects in the training of mathematics teachers.

Problematical: from failure to studies to understanding

Much has been published on the problems of teaching and learning in Brazilian University Education. Some of these, such as Baruffi (1999), Rezende (2003), Torres and Giraffa (2009), Trevisan and Mendes (2018), Pinheiro (2022), were generally concerned with problems in the teaching of calculus; or Pinheiro and Boscarioli (2022) who carried out a literature review looking for methodologies that would reduce failure rates in the teaching of calculus in engineering courses. All of them observed failure rates as a motivation for their investigation and then proposed changes in teaching approaches in the classroom with promising results; but all these works report in their circles of problems “the lack of understanding of concepts”.

Here we will not dwell on the failure rates, as the importance given to the understanding of concepts by various authors, since the 1970s and the emphasis of more than 40 years of Brazilian government documents since the 1980s, encouraged us to clarify, questioning “in loco”, through this research, what our possible future mathematics teachers (those in undergraduate courses) understood about the concepts of Calculus, particularly derivatives.

In this sense, it is important to mention the Tulane conference (1986 – USA), exclusively to address the problems of teaching Calculus, which set as one of the objectives for teaching Calculus “To develop students’ understanding of concepts” (DOUGLAS, 1986, p. xvi). There were even authors of textbooks who followed the recommendations of this conference, such as (Hughes-Hallett et al., 1997) and (Stewart, 2006); the latter becoming a “best-seller” in the USA, number 1 in Brazil (www.amazon.com.br) and reproduced in several countries.

The problematic undergraduate students' understanding of calculus concepts led us to the question that we have already explained: what understanding of derivative concepts do undergraduate students in Mathematics acquire during their course? And from this, we have explained the objectives of the research.

From our research question, another question arises: why is learning based on understanding so important? In addition to what is advocated in the official documents of the understanding already explained by Skemp (1976, 1987), and by Anderson et al. (2001), Pozo asserts that:

> Perhaps, if repeated many times, it may be possible to condense and automate such a tongue twister (something that I frankly do not recommend). But such learning will not be long-lasting and not very transferable, that is, not very effective [...] A distinct, constructive learning is needed, based on understanding the meaning of the material and not just trying to "copy" it literally with more or less success. This constructive learning will be directed towards extracting the meaning of the text, which is why it is also called meaningful learning (Pozo, 2002, p. 125).

Specifically, regarding the training of Mathematics teachers in Higher Education, Trevisan and Mendes (2018, p. 213) highlighted that, in general, students, when entering different undergraduate courses, present a study dynamic developed in Basic Education, prioritizing aspects related to memorization and mechanization of procedures, instead of understanding and attribution of meaning. Research, over two decades old, has already discussed this didactic-pedagogical aspect, as indicated in Barufi (1999):

> In order to minimize failure in the construction of meaningful knowledge, the solution often adopted is to prioritize the application of calculus, presenting a large number of problems and exercises, often repetitive, where the student ends up memorizing, in some way, resolution processes. In this sense, the idea, the concept, is reduced to the algorithm, and that eternal question of students, unanswered and "hated" by teachers, remains: What is this for? (BARUFI, 1999, p. 162).

Skemp (1987), unlike Trevisan and Mendes (2018) and Barufi (1999), considered memorization, mechanization of procedures and resolution processes as a type of understanding; for him, it is a type of understanding, however, instrumental. Certainly, it is a level of understanding that, if isolated, is minimal and often inert, as it is attached to the rules and procedures for solving specific problems, therefore, without attribution of a broader meaning, as it is only "operational".

For Anderson et al. (2001), who reviewed Bloom's Taxonomy (1956), understanding is "constructing meaning from instructional messages, including oral, written and graphic communication". In a way, the concepts of understanding of Skemp (1987) and Anderson et al.

(2001) have an intersection in their spheres of meaning, since the terms (instructional messages; remembering rules and applying them; constructing meaning; ability to deduce), present in their respective concepts, occur mathematically in a dimension of meaning common to both.

It is worth noting that Pozo (2002) also does not dismiss copying for the purpose of memorizing, but emphasizes that we must go beyond this to obtain understanding, as it is insufficient to remain at this level. He says that "The more deeply or significantly a material is processed and learned, the more lasting and generalizable its results will be. Understanding is the best alternative to repetition" (Pozo, 2002, p. 210). In the same way, we understand that "the goals of relational learning are long-term" [...] (Skemp, 1987, p. 169).

Specifically regarding our theoretical framework, according to Skemp (1987, p. 164-166), understanding in mathematics is divided into three types that are interrelated:

> **Instrumental understanding** is the ability to apply an appropriate remembered rule to the solution of a problem without knowing why the rule works.
> **Relational understanding** is the ability to deduce specific rules or procedures from more general mathematical relationships.
> **Logical understanding** is the ability to connect mathematical symbolism and notation with relevant mathematical ideas and to combine these ideas into chains of logical reasoning.

These ideas have sparked interest among researchers in university mathematics education. Weber (2002), concerned about the difficulty his undergraduate students had in constructing proofs of propositions about group isomorphisms, knowing that this skill is important in pure mathematics, investigated a small group of undergraduate and doctoral students. He adapted for his research the first two types of understanding given by Skemp (1987) to assess the understanding of group theory concepts and concluded that the undergraduates, even having the logical and instrumental understanding of mathematics necessary to construct such proofs, made few of them, while the doctoral students demonstrated all the propositions. When analyzing the proofs, he concluded that doctoral students regularly used their relational understanding to guide their attempts at proofs, while undergraduates rarely used it. He observed that only instrumental understanding leaves them limited as to what they can prove and perhaps relational understanding is necessary for a proof without difficulties.

Keene et al (2011), already aware that the relational understanding of concepts of Ordinary Differential Equations (ODE) is very important for lasting learning, believed that students can learn to calculate solutions of an ODE with deep conceptual knowledge (instrumental and relational understanding). They took into account that there is a significant tension between learning based on the (relational) understanding of ODE concepts and the apprehension of a set of algorithms to solve standard ODE exercises (instrumental), since students tend to spend more time trying to grasp the algorithms. So they developed a teaching structure for understanding ordinary differential equations (ODE) and their solutions that combines teaching for conceptual understanding and teaching of procedures or algorithms. At the same time, they also developed a set of assessment items that helped them understand what students are learning. Based on this assessment, they incorporated new questions and hints with the assessment items to help students deepen their relational understanding.

Although we do not draw on Keene's ideas and use some of Weber's methodological insights in this article, we cite them because they indicate that there is concern about teaching without an emphasis on understanding in higher education. Our research assessed student understanding considering Skemp's definitions and so we need to heed his warning:

> [...] we are unable to observe our pupils' feelings and schemas directly, and look for confirmatory behavior. As evidence that A understands X, we accept the fact that A applies X in situations different (in greater or less degree) from that in which it was learned. (Skemp, 1987, p. 166).

Since the National Curricular Guidelines for Mathematics courses (DCN-M) (BRASIL, 2003) states that curricula should promote the acquisition of various skills and abilities by students, we agree that it is necessary to develop the "*ability to express oneself in writing and orally with clarity and precision*" (Brasil, 2001, p. 3). Under this understanding and also to include gaps (partial learning) in the scope of the analysis, we decided to classify their responses not only as success or failure, but in three levels (Table 1), in which we combined the alert, the types of understanding, both from Skemp, with the skill required in the DCN-M.

Table 1

Classification of levels of understanding

Classification of levels of understanding (evidence of**)**	**Descriptor**
Success	When the student "expresses him/herself clearly and precisely in writing or orally" according to current Calculus literature[5].
Shortcoming	When the student "expresses him/herself in writing or orally, but without clarity or precision" according to the current Calculus literature.

[5] There is a vast literature on Calculus; we chose Stewart (2013) e Hughes-Hallet et al. (2013).

Failure	When there is no answer; otherwise, when the student expresses him/herself either a random[6], an evasive, or an incompatible[7] answer.

Methodology: About the *modus operandi*

The research data were produced by applying questionnaires (mathematics assessments) and interviewing undergraduate mathematics students from three university campuses in western Paraná. We previously asked the course coordinators for groups of students who had taken the first course involving content on "derivatives of functions of a real variable" (hereinafter referred to as Calculus), even if there were students who had failed. Each coordinator of each campus chose a group among the largest and negotiated with their respective professors one of their classes to conduct the research. Thus, we applied the questionnaires and conducted the interviews with groups provided by professors, at their respective times, on dates scheduled by the coordinators. It is important to note that the questionnaires were not made available to the students in advance, to avoid contamination of the results.

Once the date was set, on the day of the interview we began by explaining what the event consisted of and that their participation was voluntary. We asked them not to talk while the questionnaires were being administered and, as soon as they had answered them, to come with us for an interview. Some students dropped out. The interview was conducted in a separate room where no one could hear the exclusive conversation between each student and their interviewer.

We were unable to estimate the number of dropouts, as only those willing to participate in the event attended the class; the event was announced in advance; even so, after the explanation, there were dropouts before the questionnaires were administered and before the interviews. We believe that we administered the questionnaire and the interview to a significant number of students on each of the three campuses visited (7+9+9 questionnaires and 5+7+9 interviews), as the dropout rate in undergraduate courses is very high in the semester after the Calculus disciplines (BRASIL, 2022).

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[6] Student declared that he or she guessed the answer.
[7] Incompatible with the current Calculus literature.

Methodology: About the references for formulating the questionnaire

We observed that the syllabuses of the disciplines with derivatives content present in most of the Pedagogical Projects of the undergraduate courses in Mathematics, where we investigated the understanding of their students, follow the same sequence of content for teaching calculus, that is, functions, limits, continuity, derivatives and integrals; This is still the sequence suggested by Augutin Cachy in his book "Calcul Infinitésimal" published in 1823 (Grabine, 2005, p. 78) which was widely disseminated only from 1850 onwards by Karl Weierstrass (Grabiner, 1983, p. 205) in his classes, this time with the symbology of "epsilons" and "deltas" that we know today. So, we carried out extensive search, in the physical library (and some of the virtual ones) of the institution where we developed the post-doctorate[8], for textbooks with derivatives content that provided us with questions whose objective was to evaluate the students' understanding. We chose to observe the preface of each one in search of clues that would point to a focus on comprehension, since there was not enough time for an in-depth analysis of each title and, also, this was not our focus of the research. We read all the prefaces of the titles found in the Unioeste library in Cascavel, (Ávila, 2011), (Boulos, 2019), (Guidorizzi, 2018), (Hughes-Hallett et el., 2013), (Leithold, 1994), (Stewart, 2013), (Swokowski, 1995), among others and found only two books in Portuguese that contained explicit objectives concerning understanding, namely, "Cálculo" (Stewart, 2013) and "Cálculo e aplicações" (Hughes-Hallett et al., 2009)[9]. Both indicated an explicit theoretical basis for the construction and organization of their contents, which we describe in the next subsection of the methodology "About the authors".

The questions developed involved two dimensions: one with objective data related to socioeconomic data (Appendix A) and the conceptual part related to derivatives (Appendix B). For the second dimension, we looked for questions whose answers would give us evidence of the level of both a single understanding (instrumental or relational) and of these combined understandings. We chose questions that allowed us to assess only the level of instrumental understanding, others only the relational understanding, and some that allowed us to assess the levels of instrumental and relational understanding together; in the latter, the student could mobilize his relational or instrumental understanding (or both) of the content to answer them

[8] The first author is a professor at UFAM and was on sabbatical year to postdoctoral at Unioeste under the supervision of the second author, whose project resulted in this article.

[9]Theoretical basis for the construction and organization of their contents was explained better on edition (Hughes-Hallet, et all, 2013).

(Table 3 — column 2). The student's logical understanding, in each question, was assessed according to “the ability to connect mathematical symbolism and notation with relevant mathematical ideas and to combine these ideas in chains of logical reasoning” (Skemp, 1987, p. 166).

Although all the Calculus course syllabuses on the campuses we visited mentioned applications, it was not clear in which area of science we should address them; therefore, we chose and formulated essay and objective questions that did not involve extra-mathematical applications. We followed the ideas of the literature (Hughes-Hallett, et al., Cálculo de uma Variável, 2013) and (Stewart, Cálculo, 2013), since, as mentioned above, they indicated an explicit theoretical basis for the construction and organization of their contents with a focus on understanding. Two of their principles[10-11] and Skemp's ideas[12] on concept apprehension guided us in the choice of questions, so that we could assess the level of each type of understanding of the students in a broad aspect of presenting questions involving derivatives (Table 3 – column 3). It is worth noting that we assessed the verbal responses through interviews, as explained in the section “Methodology: About the *modus operandi*”. This way we connect the requirements of the DCN-M with our theoretical framework.

About the authors of the sources that helped us formulate the questions about derivatives James Drewry Stewart (1941–2014) was a professional violinist and professor emeritus of mathematics at McMaster University in Canada. He received his master's degree from Stanford University, his PhD from the University of Toronto, and his postdoctoral degree from the University of London. Although he supervised several dissertations in his pure research area, harmonic analysis, his other passion besides music was teaching. His sympathy for the principles of the Calculus Reform, as developed at the Tulane conference in 1986 (Douglas, 1986, p. viii) and the influence of George Polya during his undergraduate years at Stanford, established in him some philosophical principles, which, together with his classroom experience, were transformed into a new edition of his early Calculus textbooks. The sales success made these principles the standard for new editions, as stated in the preface to the Portuguese translation of the 5th edition in 2006 (the first in Brazil) of his book “Calculus”,

[10]Both the authors on similar way stated in the preface of their books that, where appropriate, topics should be presented geometrically, numerically, algebraically (analytically) and descriptive (verbally or orally).
[11]A mathematical problem should be difficult in order to entice us, yet not inaccessible lest it mock our efforts.
[12]Once the concept is formed, we may (retrospectively and prospectively) talk about example of concepts (SKEMP, 1987, p.11).

> The emphasis is on understanding concepts. [...] I came from the Tulane Conference in 1986, which formulated as their first recommendation: Focus on conceptual understanding. I have tried to implement this goal through the Rule of Three: "Topics should be presented geometrically, numerically, and algebraically." [...] The Rule of Three has been expanded to become the Rule of Four by emphasizing the verbal, or descriptive, point of view as well. (Stewart, Cálculo, 2006, p. vii)

Stewart, still in the preface, presents some characteristics of his book; some that are clearly philosophical influences of George Polya and David Hilbert, such as "the four steps to problem solving" (Polya, 1995, pp. XII-XIII) and an excerpt from the opening speech of the II ICM[13] in 1900, "A problem should be difficult enough to challenge us, but not so difficult as to mock our efforts" (Hilbert, 1902); others, cited below, without sources, but which refer us to learning theories, meeting the following characteristics: Conceptual exercises [...] Graded Exercise Sets [...] Real-World data [...] Projects [...] Problem solving [...] technology.

Deborah Hughes-Hallet (1944 –) is a professor of mathematics at the University of Arizona and an adjunct professor of public policy at Harvard (Kennedy School). Her main interest is higher education. Deborah has already received more than 11 national (US) and international awards and honors for her significant contributions to Mathematics Education (Hughes-Hallett, Faculty Profile, 2024), the most recent in 2022, "Award for Impact on the Teaching and Learning of Mathematics" from the American Mathematical Society – AMS (AMS, 2022).

She, together with Andrew Gleason from Harvard University, founded, around 1988, the "Calculus Consortium for Higher Education" which promised to innovate the curriculum and pedagogy of calculus teaching in the USA. They, together with 13 other professors from several North American universities, funded by the National Science Foundation, created a calculus teaching program. His program, in addition to proposing a new approach to teaching, produced a Calculus book with a focus on understanding (Hughes-Hallett, et al., Cálculo, 1997).

Its preface states that the book follows two principles, the first is the "rule of three", later transformed into the "rule of four", although without reference, it is one of the suggestions contained in the annals of the conference for the reform of calculus that took place in Tulane (Douglas, 1986); the second was inspired by Archimedes: (1) "The rule of four": Every subject must be presented geometrically, numerically, algebraically and verbally; (2) [...] The Archimedes way: Formal definitions and procedures arise from the study of practical problems.

[13] Second International Congress of Mathematics - II ICM, took place in Paris, 1900

The book was also severely criticized by traditionalist Calculus teachers (Wilson, 1997) (Klein and Rosen, 1997) (Mac Lane, 1997) (Knill, 2004), and praised or cautiously accepted by reformists (Wu, 1997) (Mumford, 1997) (Liu, et al., 2009) (Smoryński, 2017). The "reduction in rigor" was the main complaint, as it treats many concepts, such as "limit and continuity" only verbally and graphically, and, in its first edition, does not even mention theorems, such as the mean value theorem (added in later editions). Reformists defend it mainly because it improves the understanding, and grades, of students with a weak mathematical background. Although we do not have data on how widely the "Harvard Calculus Consortium" book is adopted as a textbook in US universities, its importance is given by the number of citations in articles on the Calculus Reform from various parts of the world, already mentioned in the caput of this paragraph.

About tabulating data and analyzing responses

The questionnaire applied (Appendix B – conceptual regarding derivatives) had 5 essay issues and 7 objective one; 6 of which had multiple items, totaling 30 questions to be answered.

We eliminated two questions, objective nº 1 item (d) and objective nº 3, item (e), due to typing errors noticed only during data tabulation; although we conducted a pilot test, this inconsistency was not observed. However, we understand that this did not invalidate the research, leaving us with 28 questions (11 essay and 17 objective). We eliminated the questionnaires of the students who did not give us interviews and analyzed the 21 remaining questionnaires by triangulating them with their respective 21 interviews, to maintain analytical consistency; we classified the levels of understanding evidenced by the answer to each question of each student according to the table below; then we counted the quantity of each code. It is worth noting that the levels are mutually exclusive; There were questions that only necessary an instrumental understanding to answer then, others that only had answers from relational understanding and some that both understandings could answer them.

Table 2

Codes for tabulating data (own research)

		understanding		
		Instrumental	Relational	Logical
Level	Evidence of failure	10	20	30
	Evidence of shortcoming	11	21	31
	Evidence of Success	12	22	32

Below (Table 3), we establish, according to the Calculus literature, the type of understanding necessary to answer each of the 28 questions and the way in which each of them was treated in the literature consulted.

Table 3

Understanding and type of topic (own research)

	Number and item	Type of understanding	How the content was treated in the books[14]
Essay questions	N° 1	Relational	Geometrically, algebraically and descriptive
	N° 2	Instrumental or relational	Geometrically, algebraically, numerically and and descriptive
	N° 3	Relational	Geometrically, algebraically and descriptive.
	N° 4	Instrumental	Numerically and algebraically
	N° 5	Instrumental and relational	Geometrically, numerically, algebraically and descriptive.
	N° 6 item (a)	Instrumental	Numerically and algebraically
	N° 6 item (b)	Instrumental	Numerically and algebraically
	N° 7 item (a)	Relational	Geometrically and descriptive
	N° 7 item (b)	Instrumental or relational	Geometrically and descriptive
	N° 7 item (c)	Instrumental or relational	Geometrically and descriptive
	N° 7 item (d)	Instrumental or relational	Geometrically and descriptive
Objective questions	N° 1 item (a)	Relational	Geometrically, algebraically and descriptive
	N° 1 item (b)	Instrumental	Geometrically, algebraically and descriptive
	N° 1 item (c)	Instrumental	Geometrically, algebraically and descriptive
	N° 1 item (d)	**Eliminated**	**Eliminated question**
	N° 2 item (a)	Instrumental	Numerically and algebraically
	N° 2 item (b)	Instrumental	Numerically and algebraically
	N° 2 item (c)	Instrumental	Numerically
	N° 2 item (d)	Relational	Geometrically
	N° 2 item (e)	Relational	Descriptive
	N° 3 item (a)	Relational	Geometrically
	N° 3 item (b)	Relational	Geometrically
	N° 3 item (c)	Relational	Geometrically
	N° 3 item (d)	Relational	Geometrically
	N° 3 item (e)	**Eliminated**	**Eliminated question**
	N° 4	Relational	Geometrically
	N° 5 item (a)	Instrumental or relational	Geometrically
	N° 5 item (b)	Instrumental or relational	Geometrically
	N° 5 item (c)	Relational	Geometrically
	N° 5 item (d)	Instrumental or relational	Geometrically

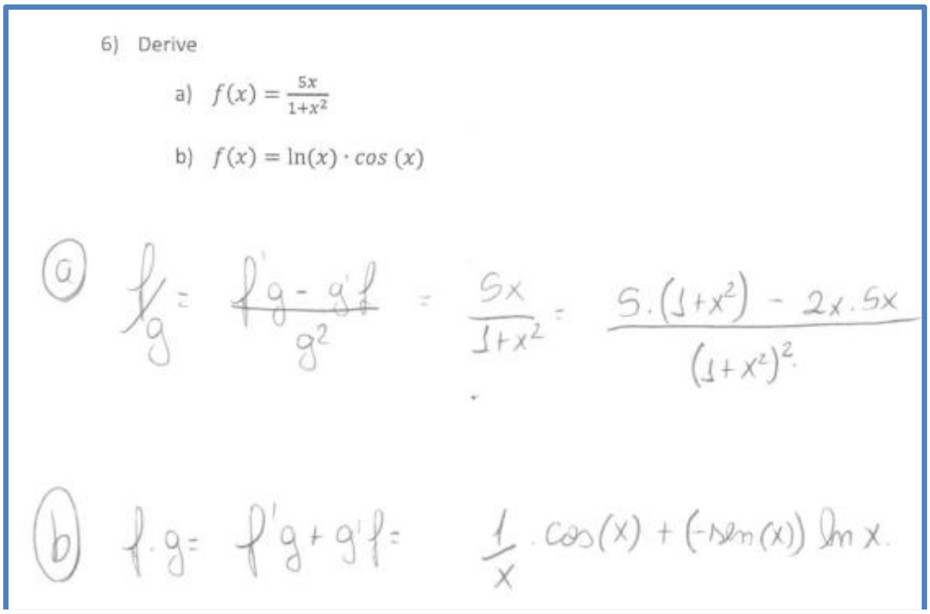
The interview was conducted after the questionnaire was completed, in a room reserved for the interviewer[15] and the interviewee, with a single question: "*What did you think about when answering this question?*" The student observed each question on the questionnaire that

[14] See footnote on page 347

[15]The interviews were conducted by three researchers linked to the research group Phenomenological Investigation in Mathematics Education – IFEM, at Unioeste.

he or she answered and told us what he or she thought about when answering each question. The answers were free and the interviewer began with the question mentioned above, said nothing else, did not interfere, other than repeating the same question for each question when necessary, and ended the interview with a “*thank you very much*”. We obtained 28 responses to each interview from each student, except for one, who answered only 27 (recording error).

Below (Figure 1, Figure 2 and Figure 3) is an example of a tabulation of the response to essay question nº 6 with two items. This question appears in the tabulation as two questions, identified as D6a and D6b.



6) Derive

a) $f(x) = \frac{5x}{1+x^2}$

b) $f(x) = \ln(x) \cdot cos\,(x)$

a $\frac{f}{g} = \frac{f'g - g'f}{g^2} = \frac{5x}{1+x^2} = \frac{5.(1+x^2) - 2x.5x}{(1+x^2)^2}$

b $f.g = f'g + g'f = \frac{1}{x}.\cos(x) + (-\operatorname{sen}(x))\ln x.$

Figure 1

Excerpt from a questionnaire – focus on instrumental understanding

00:02:23 Palestrante 1
E para a questão 6 o quê que você pensou?
00:02:27 Palestrante 2
Na questão 6, pelo menos no item (a).
00:02:30 Palestrante 2
Eu lembro das regrinhas, então eu sempre escrevo a elas para me nortear a regra.
00:02:36 Palestrante 2
E aí eu,
00:02:37 Palestrante 2
Fiz de acordo com a regrinha, e na (b) também, só que na (b), eu não estava com muita certeza da minha resposta.
00:02:50 Palestrante 1
E na sétima questão, o quê que você pensou para responder?

Figure 2

Excerpt from an interview transcript (student interviewed in Figure 1)

Questão	Tipo	Dissertativa												Diss	Diss
	Origem	Questionário						Entrevista							
	ordem	D6a	D6a	D6a	D6b	D6b	D6b	D6a	D6a	D6a	D6b	D6b	D6b	D6	D6
	Avaliação	instrumental	Relacional	Lógica	instrumental	Relacional	Lógica	instrumental	Relacional	Lógica	instrumental	Relacional	Lógica	Confiança	Revisou?
		12		31	12		31	12			12			1	RNA

Figure 3

Excerpt from the spreadsheet with its corresponding tabulations (student interviewed in Figure 1)

Looking first at the questionnaire, we consider that these responses are in accordance with the Calculus literature in force in the Brazilian curriculum. For us, items (a) and (b) are evidence of successful instrumental understanding of this content, which is why we assigned the code 12 to both items (instrumental column). There is no room for coding here regarding relational understanding (Table 3 – column 2), which is why we left the respective field blank, but we observed that the student did not make the connection between the symbolism and the notation and the mathematical idea according to the Calculus literature; for us, this response is evidence of shortcoming in logical understanding, which is why we assigned the score 31 to both items. Now, observing the transcript, we confirm the evidence of successful instrumental understanding of this content, but his statement does not provide us with any level of evidence of logical understanding, which is why we only filled in the "instrumental" columns of the interview columns. His statement, "*I wasn't very sure*", justifies confidence 1; this confidence may be related to the failure in logical understanding. The question was reviewed by the student (RNA) and was not changed, but it did not have answer suggestions in the objective part.

The penultimate column is the confidence placed in the answer (0 – I guessed the answer or did not answer; 1 – I am not sure if it is correct; 2 – I am sure it is correct). In the last column, the student must indicate to us whether the essay questions were reviewed after solving the objective questions (NR – I did not review; RNA – I reviewed and did not change the answer; RA – I reviewed and changed the answer). Some questions in part 1 (essay) have suggested answers in part 2 (objective); it was not said that there were suggestions there, the student should notice them. These two columns are intended to help us with grading doubts; for example, if a student gave an unclear answer to a question (there may be a suggested answer in the objective part) marked RA and even, so their confidence is 1, this for us is evidence of a shortcoming of understanding.

The maximum number of pieces of evidence for instrumental understanding (number of questions in which your answer is evidence for instrumental understanding) that a student can achieve is 16 (successes + shortcomings + failures) and 20 for relational understanding (successes + shortcomings + failures), adding up to a total of 16 + 20 = 36 pieces of evidence (Table 3 – column 2). Since 7 questions admit answers that can give us evidence of success/failure/failure in instrumental or relational understanding, or both (Table 3 – column 2), this tells us that the total number of pieces of evidence for success/failure/failure in instrumental understanding ranges from 9 (16 – 7) to 16, and the total number of pieces of evidence for success/failure/failure in relational understanding ranges from 13 (20 – 7) to 20. Combining the two understandings, this number ranges from 29 to 36. It is worth noting that this number, 36, was not reached. The number 29 does not contradict the number of questions, namely 28, since essay question no. 5 requires both types of understanding to answer it, therefore this question necessarily received two grades level (one instrumental and one relational). As for logical understanding, each question can receive a grade level; we have a maximum of 28 grades of this understanding for each student.

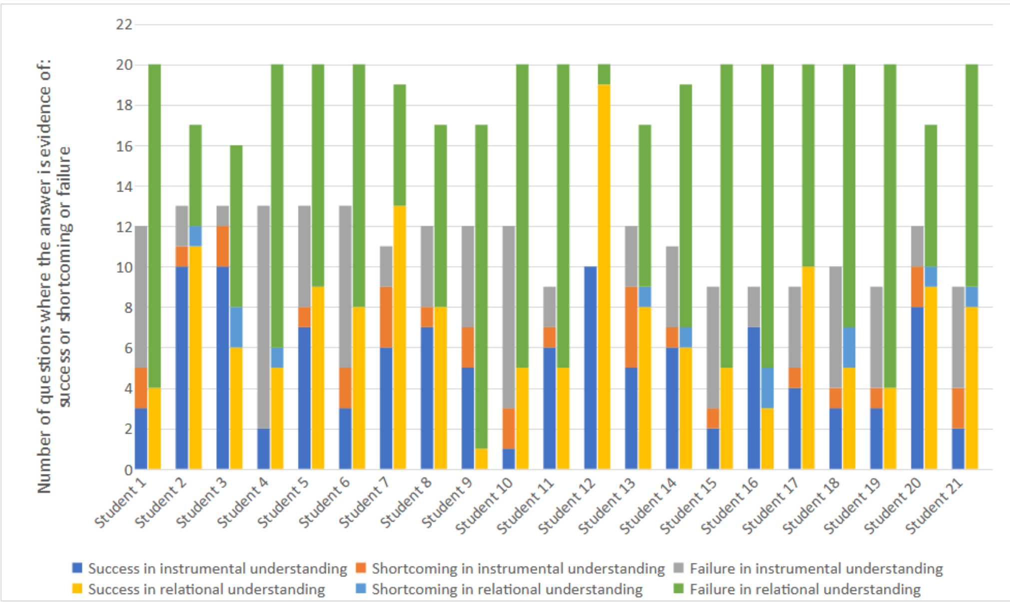


Figure 4

Overview of successes, shortcomings and failure (questionnaire)

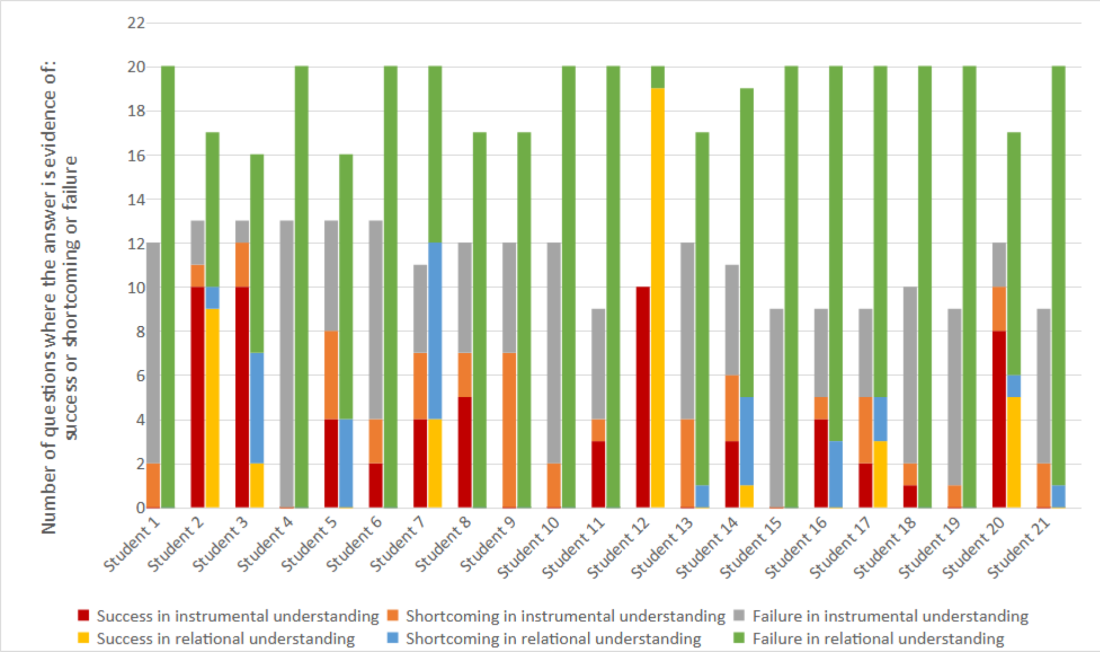


Figure 5

Overview of successes, shortcomings and failures (interview transcripts)

We consider that "understanding" means successful in instrumental and logical relational understandings, and as the DCN-M states that curricula must provide students with the *ability to express themselves in writing and orally with clarity and precision*, we can make the following connection: Evidence of success (in instrumental + relational + logical understandings) in the questionnaire confirmed (triangulated) by the interview is directly related to the ability to express oneself in writing and orally with clarity and precision.

Results

About analysis of sociology-economic data

The socioeconomic framework is relatively balanced in terms of gender, with 10 males and 11 females. No other genders were reported. This relative balance remains when we compare the genders with: marital status, age, residential area, housing, education and resides with (Table 4 and Table 5).

Table 4

Marital Status / Age / Residential Area (own research)

Gender	Marital Status		Age	Residential area	
quantity	Single	Married	Average / standard deviation	Urban	Rural

Male – 10	8	2	(26,7)[16] / 7	9	1
Female – 11	10	1	(21,8)[17] / 1	10	1

Table 5

Housing / Education / Lives with (own research)

Gender quantity	housing		Education		Who do they reside with?	
	owner	rented	Incomplete university education	Complete university education	Relatives	others
Male – 10	6	4	9	1	8	2
Female – 11	7	4	11	0	7	4

There is a small imbalance in the data: source of income and total family income (Table 6); this small variation in income, declared by the students, only allows us to state that the male gender has a higher income.

Table 6

Total family income (own research)

Income	Male	Female
Até 1320,00	1	3
from 1320,00 to 3960,00	3	3
from 3960,00 to 5280,00	3	3
from 5280,00 to 13200,00	2	2
Beyond 13200,00	1	0

The imbalance is greater in terms of color; the female gender is predominantly white (Table 7) compared to the male gender, for which there is a smaller imbalance between white and black/brown.

Table 7

[16] If we remove two students over 30, the average of the remainder is 22.4 with a standard deviation of 1.3.
[17] One female student completed the entire socioeconomic questionnaire except her age.

Source of income / Color (own research)

Gender quantity	Source of income		Color	
	Parents / scholarship	Employment / others	White	Black or brown
Male – 10	4	6	6	4
Female – 11	7	4	10	1

We calculated the failure rate the first time the student took the Calculus course (Table 8). There is also a balance in this rate, if we compare the class formats (hybrid/remote and in-person). It is worth noting that some students declared “in-person format” in 2021/2022, at the end of the pandemic.

Table 8

Situation when they took Calculus course for the first time

Format	**Total who attended**	**Total who failed**	**% who failed**
In-person	9	4	44
Hybrid/remote	12	5	42

These data situate the production of data and the context in which students are immersed in undergraduate courses in mathematics teacher training[18], allowing interpretations about the relationship between their level of understanding and their general socioeconomic conditions, however, this is not the focus of this article.

About evidence of instrumental and relational understanding

Of the 629 pieces of evidence (sum of all successes/failures/failures of all 21 students) only 6% of the evidence in the questionnaire appears at a higher level in the interview for the same question (verbalized better than written). The majority, 74.1%, of the evidence (success/failure/failure) in the questionnaire remained in the interview, and in 25.3% of the analyses the level dropped. This shows, through the questionnaire-interview triangulation, that the analysis of the interview confirms or lowers the level of evidence of understanding in the

[18]In Brazil *undergraduate course in mathematics teacher training* is called “Licenciatura em Matemática”

answers to the questions in the questionnaire. From now on we calculate all rates (except drops from the observed in the questionnaire to the heard in the interview) based on the numbers of evidence data obtained by the interview in relation to the maximum number of expected evidence, 16 for instrumental understanding and 20 for relational understanding.

About evidence of logical understanding

We had difficulty classifying the questions regarding the students' logical understanding, since they wrote and spoke little about mathematics in the questionnaire and in the interview, leaving us with almost no evidence of success, shortcomings or failure of their understanding. As a first result, we can say that our research did not manage to describe this understanding of the 21 students surveyed. Furthermore, it is possible to infer that students in general do not have fully developed mathematical communication skills in relation to ideas pertinent to calculus, which may be related to the lack of focus on understanding, which requires communicational aspects.

The subsequent analysis will address the other two types of understanding and their articulations.

About Analysis of Instrumental Understanding Data

As mentioned above, there are 16 questions where the student can provide evidence of their level of instrumental understanding, but in 9 of these, only instrumental understanding is required to answer them. For the other questions, one or the other understanding can answer them. Some students gave us 9 pieces of evidence of instrumental understanding (sum of successes, shortcomings and failures), while others gave us up to 13 pieces of evidence of this understanding (Figure 5).

We observed that there were 4 (four) students who, in addition to presenting us with at least 8 pieces of evidence of success in the instrumental understanding required to answer the respective questions, had a 0% drop[19] from the questionnaire to the interview in evidence of success in instrumental understanding, that is, all the evidence of success in the questionnaire for these four is reflected in the interview. On the other hand, we had 8 that dropped 100%, that is, none of the evidence of success in the questionnaire was confirmed in the interview. The rest

[19] Drop rate = 1 – (number of pieces of evidence of success in interview ÷ number of pieces of evidence of success in questionnaire) in percentage numbers.

presented less than 8 pieces of evidence of success; not all of them confirmed in the interview transcript (Figure 6).

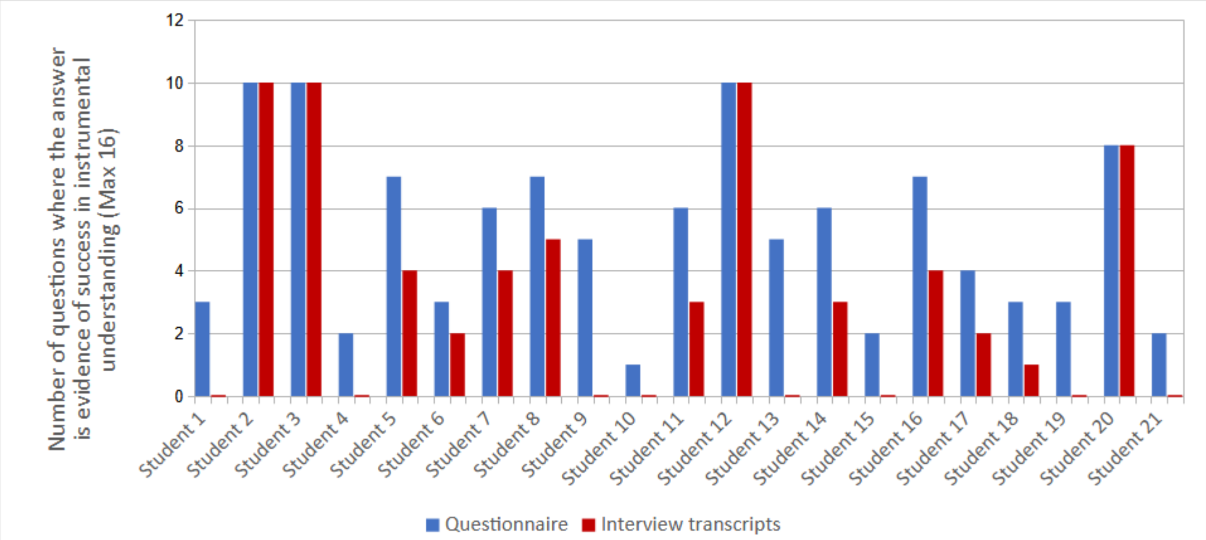


Figure 6

Success in instrumental understanding

The evidence of failure is well distributed across quartiles when we look at the interview transcripts. As stated above, the interview either confirms or lowers the level of understanding noted in the questionnaire.

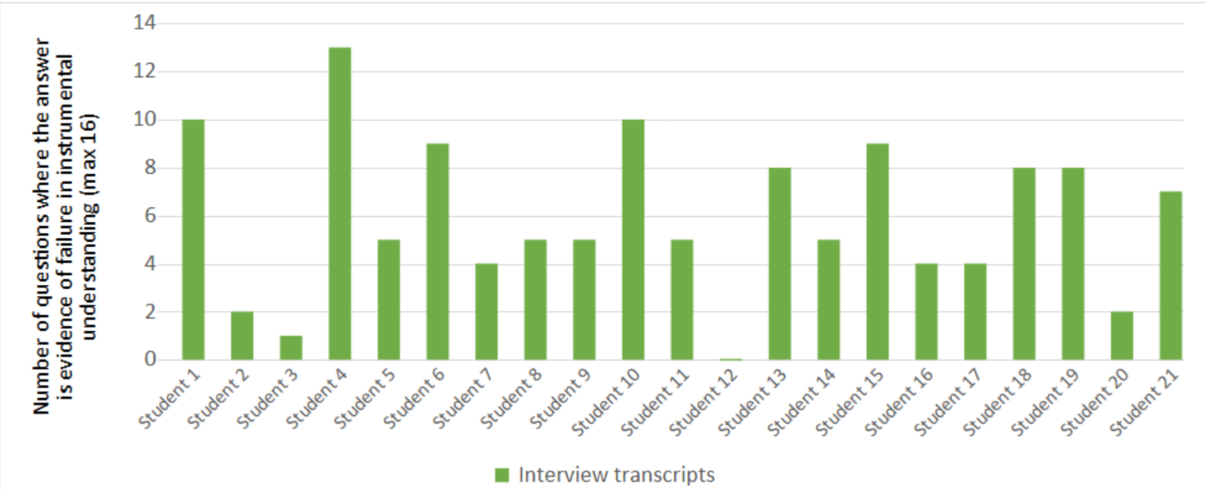


Figure 7

Failures in instrumental understanding (interview transcript)

43% of students (8 of 21) gave us 38% or more evidence of failure in instrumental understanding; 57% of students (13 of 21) gave us less than 50% pieces of evidence of failure in instrumental understanding. We can see in the tabulation that the 4 students with more than

50% pieces of evidence of success (Figure 7) are the students with less than 25% pieces of evidence of failure in instrumental understanding.

About Combined Data Analysis: Instrumental and Relational Understanding

As stated above (Table 3 – column 2), there are 20 questions where the student can provide evidence of their level of relational understanding, but in 12 of these, only relational understanding is required to answer them and 1 (one) requires both understandings. For the other 7 questions, one or the other understanding can answer them. We would have at most 20 pieces of evidence of relational understanding and 16 pieces of evidence of instrumental understanding.

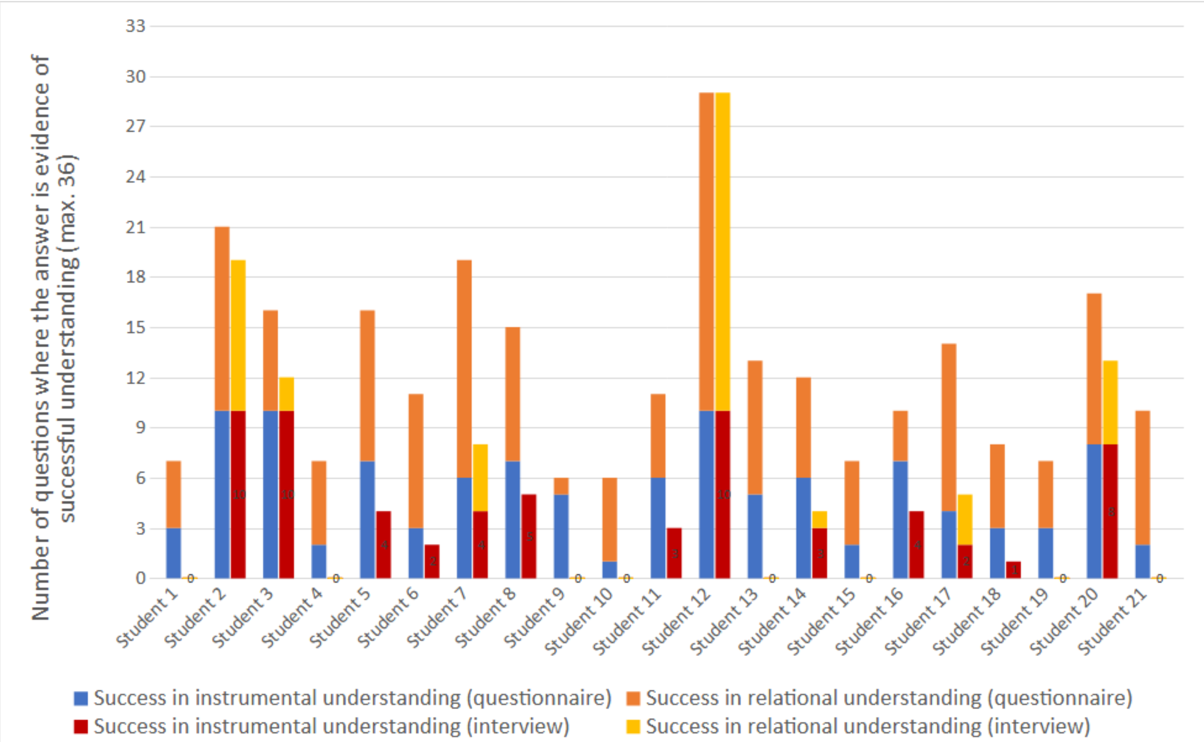


Figure 8

Success in instrumental and relational understanding (questionnaire and interview)

Only 2 students gave us evidence of more than 50% success (number of pieces evidence of successes in instrumental understanding + number of pieces of evidence of successes in relational understanding) in the questionnaires and confirmed by their interviews; another 2 students (students 3 and 20), who had achieved relative success in instrumental understanding, did not achieve as much success in the combined analysis.

When comparing the drop in the number of successes evidenced in the questionnaire to the interview, only the 4 students mentioned (the same ones from the instrumental analysis[20]) above had a drop of less than or equal to 25%; the other 17 students (Figure 8) had a drop of more than 50%, and 8 (the same 8 with a 100% drop in the instrumental analysis) had a 100% drop, that is, we were unable to confirm any evidence of success in the transcription of the interview of 8 of the 21 students. It is worth noting that the lack of success may be either shortcoming or failure in the combined understandings; we analyze these possibilities later.

When comparing the 17 students who had a drop of more than 50% in the success rate of the questionnaire toward interview, we found that 15 of them are the same ones who presented more than 50% evidence of failure in the combined understandings (relational and instrumental) in the interview (Figure 9).

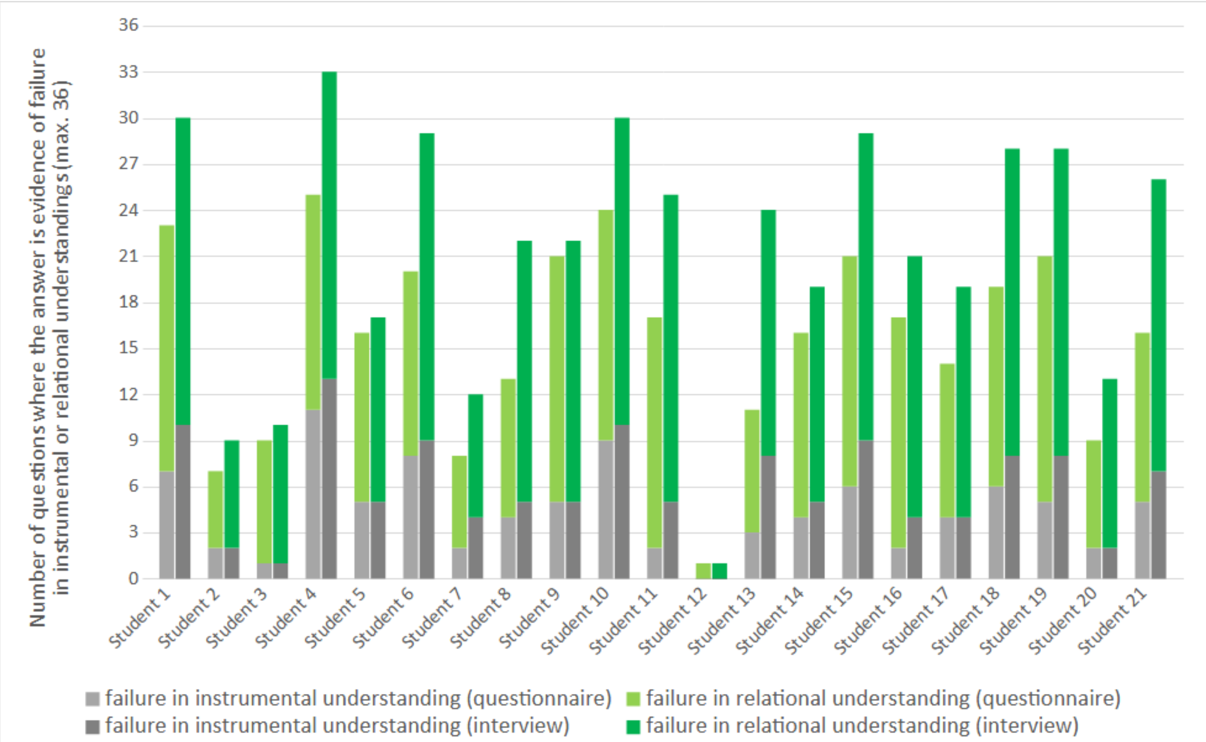


Figure 9

Failure in instrumental and relational understanding (questionnaire and interview)

Discussions

When we wonder about understanding of the concepts of derivative of a variable by undergraduate students in Mathematics (future teachers) at universities in western Paraná, we

[20] Checked in tabulation

advanced with the research in different directions, from socioeconomic aspects to indications of students' understanding of the topic.

The socioeconomic status does not make student life difficult, as 85% are single, 90% live in urban areas, and 71% live with relatives (parents and non-parents). The average age of 22.4 years[21] and the average enrollment in the course between 2018 and 2021 tell us that 18 students (85%) are young adults who took a short break from studying and entered university. There is a relative imbalance in income (table 6), with 47% reporting a family income of less than three minimum wages and 53% reporting a family income of more than three minimum wages, with males generally reporting a higher family income. These data do not showed a direct correlation between the socioeconomic status and the level of understanding expressed by successes/failures/failures. Therefore, it can be said that the level of understanding is related to the curricular model implemented, which does not emphasize the development of understanding, except among those students who already have some cognitive skill favored by various factors unknown to us.

Regarding conceptual aspects, we observed a correlation (Figure 8 and Figure 9): Students with more than a 50% drop in success did not show many errors but showed more than 50% failures in the combined analysis of understanding, that is, 15 out of 21 students showed incomprehension (failure in combined understanding) in more than 50% of the concepts addressed in the 28 questions. This tells us that students have a lot of difficulty verbalizing what they write, as two interviewees told us (Figure 10 and Figure 11).

00:02:31 Palestrante 1

E na sétima questão o quê que você pensou?

00:02:35 Palestrante 2

Hum. Ai eu não, não lembro, não lembro como fazer isso, não lembro. Então não respondi por isso.

Figure 10

Transcript of an excerpt from a student interview[22]

[21] If we remove one unreported age and two ages over 30, the average of the remainder (18 students) is 22.4 with a standard deviation of 1.3.

[22] “Palestrante 1” is the interviewer. “Palestrante 2” is the student. It’s a transcriber’s default.

00:00:00 Palestrante 1

Nós vamos começar pela questão dissertativa.

É ... o quê você pensou para responder a primeira?

00:00:07 Palestrante 2

Eu lembrei apenas do jeito que resolvi assim, tipo a definição certinha ali eu não lembrava, mas eu tipo, tentei esboçar, um jeito de fazer.

00:00:19 Palestrante 2

E tentar resolver.

Figure 11

Transcript of an excerpt from another student's interview

Thus, there appears to be a correlation between non-verbalization of writing and lack of understanding, since it is through relational understanding that instrumental and logical aspects are properly apprehended in an ordered and intelligible structure. The triangulated analysis of the interviews and questionnaires highlights the absence of relational understanding, since students were unable to express articulated understandings in a situation different from that in which they learned the concepts, as we can infer from Skemp (1987).

The four students who showed a drop of less than 25% (from questionnaire toward interview) in pieces of evidence of success on combined understanding are the same ones who showed a 0% drop in pieces of evidence of success from the questionnaire to the interview when we looked only at their instrumental understanding. This suggests a correlation between relational understanding, instrumental understanding and knowledge retention for lasting learning, because in the exclusive analysis of the data related to instrumental comprehension, we did not find any student with more than 50% pieces of evidence of success on relational understanding besides the four students mentioned above.

Trevisan and Mendes (2018), as mentioned above, highlight that, in general, students, when entering different undergraduate courses, present a study dynamic developed in Basic Education, prioritizing aspects related to memorization and mechanization of procedures, instead of understanding and attribution of meaning. As a vicious cycle, Barufi (1999), already warned us of the continuity of this practice in higher education, to minimize the failure in the construction of meaningful knowledge. However, our research points to a more worrying fact, that is, not even the instrumental aspects seem to be consolidated, asserting the need to invest in aspects that articulate relational and instrumental understanding.

Although our focus is not to evaluate teaching methodologies, teaching performance or curricula, nor to point out educational policies or socioeconomic conditions that lead undergraduate students to understand or not understand concepts, the results show strong evidence of non-lasting learning, since 76% (16 students) stated (according to our research) that they have already taken a subject with derivative content more than once, either repeating it due to failure or taking the subsequent one. For this reason, it is necessary to advance the debate on curricula and especially on the way they are implemented, in methodological and pedagogical terms. Furthermore, it is necessary to investigate and develop proposals and strategies that allow us to understand the background of students, both intellectually and in terms of their training in intellectual life.

Pozo (2002) had already warned that learning based on memorization and repetition is not very long-lasting and not very transferable. In the same way, however, specifically for mathematical learning, Skemp (1987) clarified that instrumental understanding alone is also not long-lasting.

In general, we can say that 17 students showed strong evidence of shortcoming and failure in the relational and instrumental understanding of the derivative concepts addressed in the questions.

What is missing for the 17 to be successful? For Skemp, Pozo and many others mentioned above, teaching must focus on deep understanding, which is effective, long-lasting and transferable. This raises the question: “Will a teacher be able to base his/her classes on understanding without having experienced classes like this in his/her training?”

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Limitations of this study

Although the topic is widely mentioned, it is in a naturalized way, since there are still few studies on the understanding of concepts in mathematics in Higher Education, which extends to Calculus. Even restricting the content to the “derivative”, many of its justifications and consequences were not evaluated. We did not evaluate the understanding of the chain rule and L'Hospital's method; we avoided numerical calculations of non-polynomial functions, since this would take time that would go beyond the post-doctoral study of the first author; we even avoided graphs of these functions; we did not talk about functions defined by several sentences; we did not touch on certain formal concepts, such as the mean value theorem, etc. What questions should we include in the questionnaire so that we could investigate our students' understanding of concepts of derivatives, their justifications and their consequences? Of course, we avoided questions with long answers, even so there was an immense volume of exercises in the Calculus books. We realized that more work is needed to improve the results of this research to help verify lasting learning, and thus assist teachers in restructuring curricula, teaching plans and classroom practices.

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Apêndice A

Questionário Socioeconômico

Universidade:____

1. Nome:____data: ____/ _____ / 2023

2. Gênero:
   a. ( ) masculino
   b. ( ) feminino
   c. ( ) outro: ____________________
   d. ( ) prefiro não declarar

3. Idade: ____________________________

4. Você se considera:
   a. ( ) Branco
   b. ( ) Preto
   c. ( ) Pardo
   d. ( ) Amarelo
   e. ( ) Indígena
   f. ( ) Prefiro não declarar

5. Estado civil:
   a. ( ) solteiro(a)
   b. ( ) casado(a)
   c. ( ) divorciado(a)
   d. ( ) viúvo(a)
   e. ( ) outro____________________

6. Tem quantos filhos(as)?
   a. ( ) nenhum
   b. ( ) 1
   c. ( ) 2 ou mais filhos

7. Escolaridade:
   a. ( ) Ensino Superior incompleto
   b. ( ) Ensino Superior completo
   c. Pós-graduação
      i. ( ) Especialização.
      ii. ( ) Mestrado.
      iii. ( ) Doutorado
      Área: ______________________

8. Você possui alguma deficiência?
   a. ( ) Sim
   b. ( ) Não.
   Em caso afirmativo, indique o tipo:
      i. ( ) Deficiência física
      ii. ( ) Def. visual
      iii. ( ) Def. mental
      iv. ( ) Defi. auditiva
      v. ( ) outra: _________________

9. Residente em:
   a. ( ) Área Urbana
   b. ( ) Área Rural.

10. Atualmente, você reside:
   a. ( ) com os pais
   b. ( ) com parentes (não pais)
   c. ( ) com amigos
   d. ( ) sozinho(a)

11. Número de moradores na sua residência:___________

12. Condições de moradia:
    a. ( ) Própria
    b. ( ) Alugada
    c. ( ) Cedida
    d. ( ) outro: _________________

13. Renda mensal:
    a. ( ) até R$ 1.320,00
    b. ( ) de R$ 1.320,00 até R$ 2.640,00
    c. ( ) de R$ 3.960,00 até R$ 5.280,00
    d. ( ) de R$ 5.280,00 a R$ 13.200,00
    e. ( ) mais de R$ 13.200,00

14. Principal fonte de renda:
    a. ( ) pais
    b. ( ) bolsa
    c. ( ) emprego
    d. ( ) outra: ____________________

15. Em que ano você entrou nesse curso de licenciatura em matemática? ______________________________

16. Em que semestre/ano (e formato) você cursou (primeira vez) a disciplina que tem o conteúdo de derivadas? Ano: ________
    a. ( ) Presencial
    b. ( ) Remota
    c. ( ) Híbrida

17. Você já foi reprovado na disciplina que tem o conteúdo de derivada?
    a. ( ) Sim
    b. ( ) Não

18. Você já cursou mais de uma vez a disciplina que tem o conteúdo de derivada?
    a. ( ) Sim
    b. ( ) Não

Apêndice B

Questionário (parte conceitual referente às derivadas)

Projeto de pesquisa

Conceitos de cálculo diferencial: compreensão de estudantes de licenciatura em Matemática de universidades do oeste do Paraná

Pesquisador: Jorge Fernandes de Lima Neto
Supervisor: Tiago Emanuel Klüber

Questionário – Questões dissertativas

Parte 1 de 2

Nome:____

Universidade:____

Data:____

Instruções: As respostas são livres. Você pode colocar ou o conceito ou as ideias intuitivas que você aprendeu durante as aulas sobre derivadas. Faça esboços se considera-los necessário. Não tenha vergonha, solte sua imaginação.

1) Qual é o seu entendimento sobre derivada de uma função?

Note: this comment in italics was not in the student questionnaire.
We intend to verify the students' relational understanding of the concept of derivative. What does the student know about the more general relations that lead to the definition of a derivative function? For example, the approximation of the tangent line by secant lines to a fixed point, and rethinking this fixed point as any point.

Quão confiante você está em relação a sua resposta? ____

0 – Chutei a resposta ou não respondi

1 – Não tenho certeza se está correta;

2 – Tenho certeza de que está correta.

☐ Revisei e **alterei** a resposta após responder a parte 2

☐ Revisei e **não alterei** a resposta após responder a parte 2

2) Se f é uma função derivável no número a, o que representa $f'(a)$?

(Você pode dar uma resposta ou algébrica, ou numérica, ou geométrica ou ainda intuitiva)

Note: this comment in italics was not in the student questionnaire.
We intend to verify students' relational or instrumental understanding of the concept of derivative at a point. The previous question is a counterpoint to this one. The idea here is to assess the understanding of the difference between a local concept and a global concept. The student can present general ideas, such as the rate of change at a point or the slope of the tangent line to the curve at a point, or the tangent of the angle that the tangent line to a point makes with the x-axis, or even the algebraic limit rule.

Quão confiante você está em relação a sua resposta? ____

0 – Chutei a resposta ou não respondi

1 – Não tenho certeza se está correta;

2 – Tenho certeza de que está correta.

☐ Revisei e **alterei** a resposta após responder a parte 2

☐ Revisei e **não alterei** a resposta após responder a parte 2

3) Se f é uma função duas vezes derivável em um intervalo numérico. O que suas derivadas f' e f'' dizem sobre a função f ?

(Você pode dar uma resposta ou algébrica, ou numérica, ou geométrica ou ainda intuitiva)

Note: this comment in italics was not in the student questionnaire.
Unlike the first two, here we intended to assess the student's relational understanding, since the question "What do [...] say about [...]" involves general ideas and suggests an answer without applying procedures or formulas. There was a possibility that the answer would be a reproduction of phrases from mathematical jargon; but the answer to Objective Question #3 gave us an indication of the type of understanding we are looking for here.

Quão confiante você está em relação a sua resposta? ____

0 – Chutei a resposta ou não respondi

1 – Não tenho certeza se está correta;

2 – Tenho certeza de que está correta.

☐ Revisei e **alterei** a resposta após responder a parte 2

☐ Revisei e **não alterei** a resposta após responder a parte 2

4) (STEWART, 2013, p. 138, nº 23) Se $f(x)=3x^2-x^3$, encontre $f'(1)$ e use-o para encontrar uma equação da reta tangente à curva $y=3x^2-x^3$ no ponto $(1,2)$.

Note: this comment in italics was not in the student questionnaire.
The purpose of this purposely closed question (since a sequence of procedures was required) was to assess instrumental understanding. Knowledge of the sequence of procedures and calculations was necessary to obtain the answer.

Quão confiante você está em relação a sua resposta? ____

0 – Chutei a resposta ou não respondi

1 – Não tenho certeza se está correta;

2 – Tenho certeza de que está correta.

☐ Revisei e **alterei** a resposta após responder a parte 2

☐ Revisei e **não alterei** a resposta após responder a parte 2

5) Se $f(x)=3x^2-x^3$, $x \in R$, encontre seus pontos críticos, seu ponto de inflexão e esboce seu gráfico.

Note: this comment in italics was not in the student questionnaire.
Here, we continue the intention of evaluating the student's instrumental understanding that began in the previous question. It is worth noting that it was necessary for the subject to have a relational understanding of the content so that at the end of the sequence of procedures and calculations, he could sketch the graph of a function by interpreting some data provided by its derivatives.

Quão confiante você está em relação a sua resposta? ____

0 – Chutei a resposta ou não respondi

1 – Não tenho certeza se está correta;

2 – Tenho certeza de que está correta.

☐ Revisei e **alterei** a resposta após responder a parte 2

☐ Revisei e **não alterei** a resposta após responder a parte 2

6) Derive

a) $f(x)=\frac{5x}{1+x^2}$

b) $f(x)=\ln(x)\cdot\cos(x)$

Note: this comment in italics was not in the student questionnaire.
Here we wanted to assess the instrumental understanding of the students. We only assessed whether the formula was memorized and applied according to the Calculus literature.

Quão confiante você está em relação a sua resposta? ____

0 – Chutei a resposta ou não respondi

1 – Não tenho certeza se está correta;

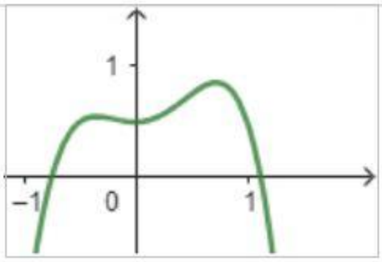
2 – Tenho certeza de que está correta.

☐ Revisei e **alterei** a resposta após responder a parte 2

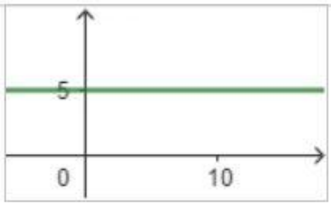
☐ Revisei e **não alterei** a resposta após responder a parte 2

7) Esboce o gráfico da função $f'(x)$ para as funções dadas. Dê razões para sua escolha.

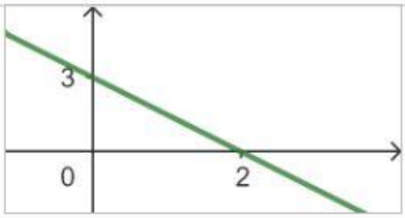
a)



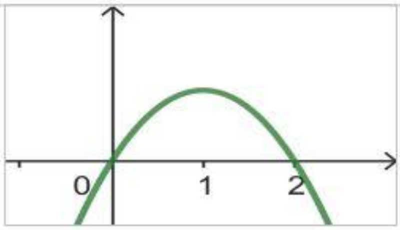
b)



c)



d)



Note: this comment in italics was not in the student questionnaire.
Our intention in this question was to assess only the students' relational understanding; since we left the values in the graphs, we received answers where the student used his instrumental understanding in items (b) (c) and (d) (constructed the equation, derived it and sketched the graph); we also received answers where the student used his relational understanding: identified the affine and quadratic functions or identified critical points (general concepts) and sketched the approximate graph of their derivatives. We assessed the student's instrumental and relational understanding.

Quão confiante você está em relação a sua resposta? ____

0 – Chutei a resposta ou não respondi

1 – Não tenho certeza se está correta;

2 – Tenho certeza de que está correta.

☐ Revisei e **alterei** a resposta após responder a parte 2

☐ Revisei e **não alterei** a resposta após responder a parte 2

Projeto de pesquisa

Conceitos de cálculo diferencial: compreensão de estudantes de licenciatura em Matemática de universidades do oeste do Paraná

Pesquisador: Jorge Fernandes de Lima Neto
Supervisor: Tiago Emanuel Klüber

Questionário – Questões objetivas

Parte 2 de 2

Nome: ____

Universidade: ____

Data: ____

1) Complete as lacunas com V, F ou D.

- V significa verdadeiro
- F significa falso
- D significa não sei ou tenho dúvida.

Admita que f é uma função derivável no número a,

a) ( ) A derivada $f'(a)$ é a taxa instantânea de variação de $y=f(x)$ em relação a x quando $x=a$,

b) ( ) A derivada de uma função f em um número a, denotada por $f'(a)$, é $f'(a)=\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ se o limite existir.

c) ( ) A derivada de uma função f em um número a, denotada por $f'(a)$, é $f'(a)=\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ se o limite existir.

d) ( ) A derivada da função f no número a é o número L, se para todo $\varepsilon>0$ existe um existe um correspondente número $N>0$ tal que: se $|x-a|<N$, então $\left|\frac{f(x)-f(a)}{x-a}-L\right|<\varepsilon$

Note: this comment in italics was not in the student questionnaire.
Here we intended to assess both the instrumental and relational understanding of the students, since the first item is a verbal definition talking about rate of change (general concept) and the other items are variations of the formal definition (rule) of derivative at a point. Triangulation with the interview gave us a lot of evidence of their level of understanding.

Quão confiante você está em relação a sua resposta? ____
0 – Chutei a resposta ou não respondi
1 – Não tenho certeza se está correta;
2 – Tenho certeza de que está correta.

2) Complete as lacunas com V, F ou D.

- V significa verdadeiro
- F significa falso
- D significa não sei ou tenho dúvida.

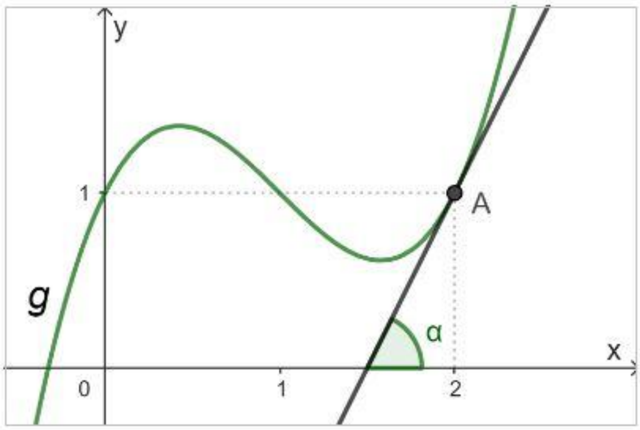
Considere que g é uma função real e $g(x)=x^3-3x^2+2x+1$.

a) ( ) $g'(2)=\lim_{x\to 2} g(x)$

b) ( ) $g'(2)=\lim_{h\to 0} \frac{g(2+h)-g(2)}{h}$

c) ( ) $g'(2)=3$

d) ( ) $g'(2)=tg\,\alpha$



e) ( ) A derivada da função g no número 2 é a inclinação da reta tangente ao gráfico de g no ponto $(2,1)$.

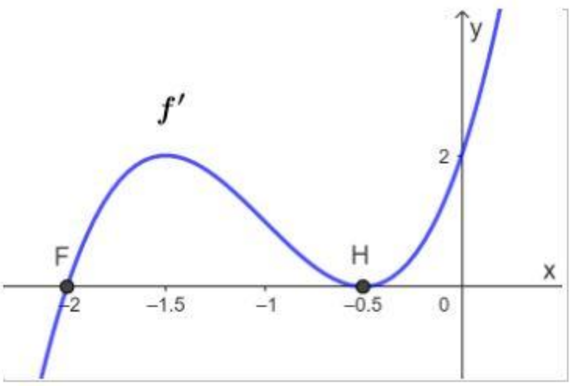
Note: this comment in italics was not in the student questionnaire.
Here we also intended to assess both the instrumental and relational understanding of the derivative at a point, but this time numerically and graphically. This question was also included here to help us assess the student's understanding of essay question 2. Triangulation with the interview gave us more evidence of their level of understanding.

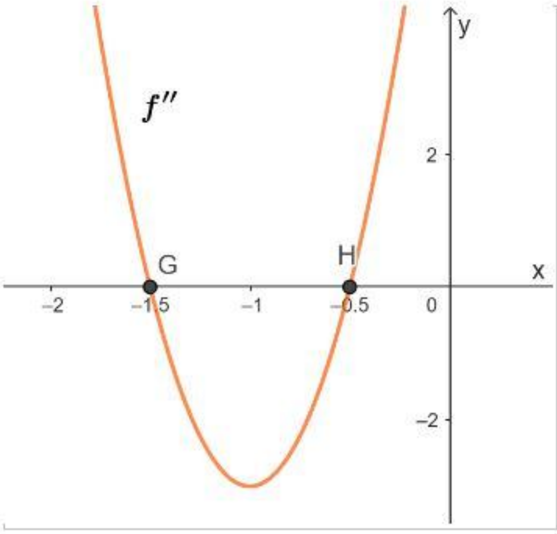
Quão confiante você está em relação a sua resposta? ____
0 – Chutei a resposta ou não respondi
1 – Não tenho certeza se está correta;
2 – Tenho certeza de que está correta.

3) Abaixo temos os esboços dos gráficos de f' e de f''. O que podemos dizer da função f?

Complete as lacunas com V, F ou D.

- V significa afirmação verdadeira
- F significa afirmação falsa
- D significa não sei ou tenho dúvida.





a) ( ) A função f tem dois máximos locais.

b) ( ) A função f tem um máximo e um mínimo local.

c) ( ) A função f tem somente um mínimo local.

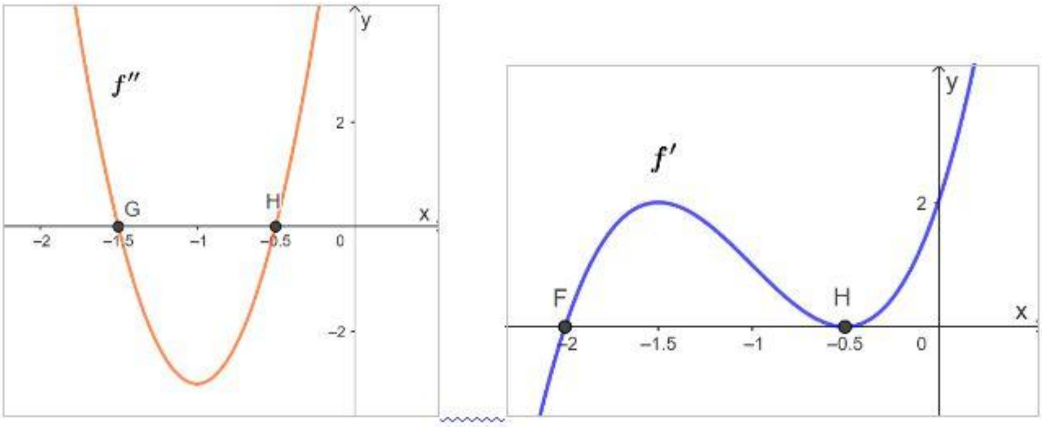
d) ( ) A função f tem dois pontos de inflexão.

e) ( ) A função f tem não tem máximo local.

Note: This comment in italics was not in the student questionnaire.
Here and next we wanted to assess the students' relational understanding. What do the derivatives tell us about their function?

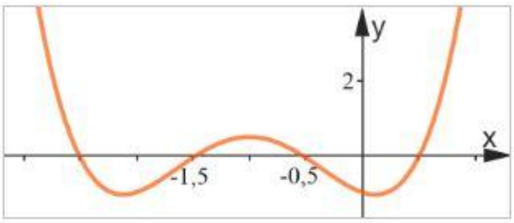
Quão confiante você está em relação a sua resposta? ____
0 – Chutei a resposta ou não respondi
1 – Não tenho certeza se está correta;
2 – Tenho certeza de que está correta.

4) Observe os esboços dos gráficos das derivadas primeira e segunda da função f,

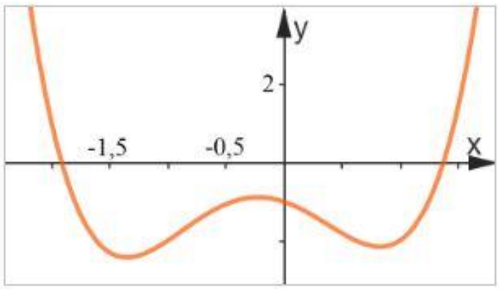


Marque o possível esboço gráfico da função f,

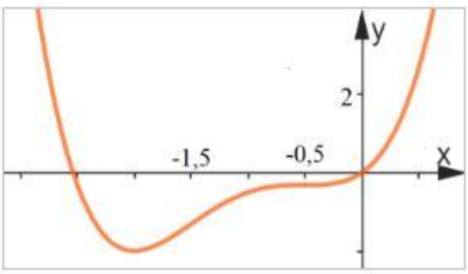
(   ) (a)



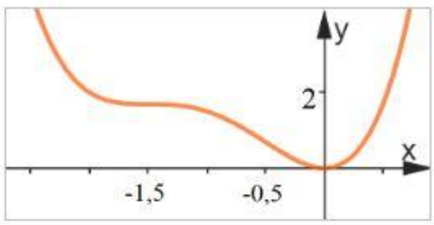
(   ) (b)



(   ) (c)



(   ) (d)



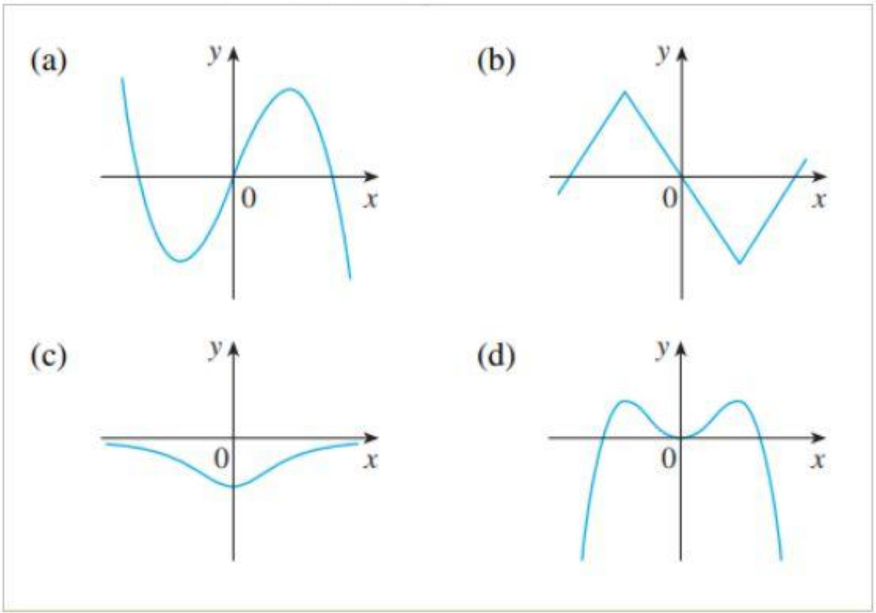
Quão confiante você está em relação a sua resposta? ____

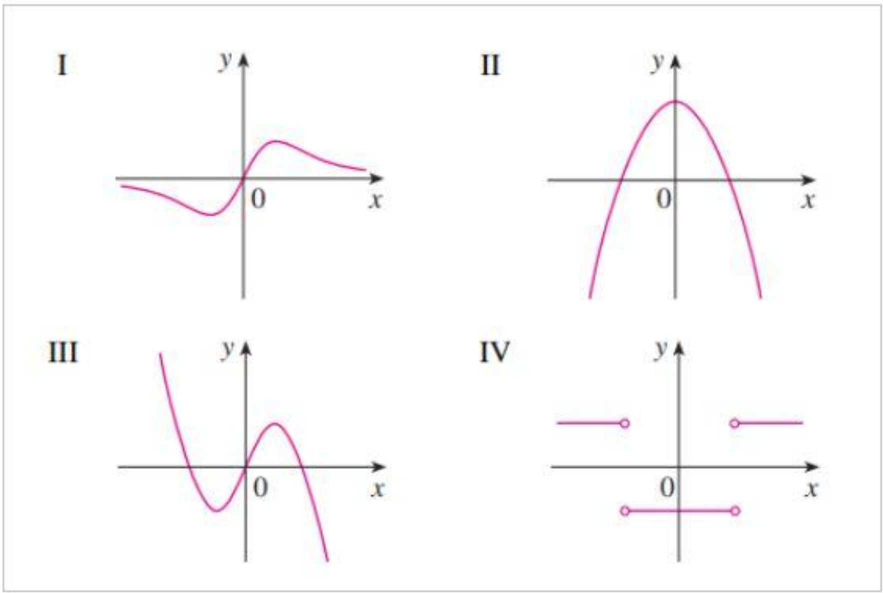
0 – Chutei a resposta ou não respondi

1 – Não tenho certeza se está correta;

2 – Tenho certeza de que está correta.

5) (Stewart, 2013, pág. 147, nº3) Associe o gráfico de cada função (a) – (d) com o gráfico da sua possível derivada em I – IV.





Note: this comment in italics was not in the student questionnaire
Here we wanted to evaluate again the relational (or instrumental) understanding of the concepts of calculus regarding the identification of the graphs of their derivatives. Your statements helped us in the evaluation of the relational understanding.

Quão confiante você está em relação a sua resposta? ____
0 – Chutei a resposta ou não respondi
1 – Não tenho certeza se está correta;
2 – Tenho certeza de que está correta.