EMP Educação Matemática Pesquisa
ISSN 1983-3156
Qualis A 1
http://dx.doi.org/10.23925/1983-3156.2024v26i3p142-174

Some considerations pertaining to the notion of an epistemological reference model (ERM) for calculus and analysis

Algunas consideraciones relativas a la noción de modelo epistemológico de referencia (MER) para el cálculo y el análisis

Algumas considerações sobre a noção de modelo epistemológico de referência (MER) para cálculo e análise

Quelques considérations relatives à la notion de modèle épistémologique de référence (MER) pour le calculus et l'analyse

Pierre Job[1]
ICHEC Brussels Management School, Belgique
Docteur en Sciences
http://orcid.org/0000-0003-0592-8783

Kevin Balhan[2]
Université de Liège (ULiège), Belgique
Docteur en Sciences
http://orcid.org/0009-0001-5025-0791

Abstract

A central feature of didactics is the questioning of knowledge. This questioning can be done in particular through an epistemological reference model (ERM). After explaining what we mean by this notion, we present an ERM of *calculus* and analysis. This MER then serves as a basis for discussing the epistemological vigilance function that a ERM should help to perform. More specifically, we show to what extent the theoretical framework adopted for the design of a ERM conditions the way in which the function of epistemological vigilance can be exercised, on the basis of the characteristics of this ERM.

Keyword: Epistemological reference model, Knowledge modelling, Epistemological vigilance, Fundamental situation, Praxeology.

Resumen

Una característica central de la didáctica es el cuestionamiento del conocimiento. Este cuestionamiento puede hacerse, en particular, a través de un modelo epistemológico de referencia (MER). Tras explicar lo que entendemos por esta noción, presentamos un MER de

[1] pierre.job@ichec.be.
[2] kevin.balhan@uliege.be.

cálculo y análisis. Este MER sirve a continuación de base para discutir la función de vigilancia epistemológica que un MER debería contribuir a realizar. Más concretamente, mostramos en qué medida el marco teórico adoptado para el diseño de un MER condiciona la forma en que puede ejercerse la función de vigilancia epistemológica, a partir de las características de este MER.

Palabras-clave: Modelo epistemológico de referencia, Modelización del saber, Vigilancia epistemológica, Situación fundamental, Praxeología.

Resumo

Uma característica central da didática é o questionamento do conhecimento. Esse questionamento pode ser feito especialmente por meio de um modelo de referência epistemológica (MER). Depois de explicar o que queremos dizer com essa noção, apresentamos um MER de cálculo e análise. Em seguida, esse MER serve de base para discutir a função de vigilância epistemológica que um MER deve ajudar a desempenhar. Mais especificamente, mostramos até que ponto a estrutura teórica adotada para o projeto de um MER condiciona a maneira pela qual a função de vigilância epistemológica pode ser exercida, com base nas características desse MER.

Palavras-chave: Modelo de referência epistemológico, Modelização do conhecimento, vigilância epistemológica, Situação fundamental, Praxeologia.

Résumé

Une caractéristique centrale de la didactique est le questionnement du savoir. Ce questionnement peut se faire notamment au travers d’un modèle épistémologique de référence (MER). Après avoir explicité ce que nous entendons par cette notion, nous présentons un MER du *calculus* et de l’analyse. Ce MER nous sert alors de point d’appui pour discuter la fonction de vigilance épistémologique qu’un MER doit contribuer à exercer. Plus spécifiquement, nous montrons dans quelle mesure le cadre théorique adopté pour la conception d’un MER conditionne la manière dont la fonction de vigilance épistémologique peut être exercée, sur base des caractéristiques de ce MER.

Mots clés : Modèle épistémologique de référence, Modélisation du savoir, Vigilance épistémologique, Situation fondamentale, Praxéologie.

Some considerations relating to the notion of an epistemological reference model (ERM)

A central assumption of French didactics, initiated in the 1970s by Brousseau (1998) and then by Chevallard (1991), is the need to question knowledge and to move it out of a state in which it is considered to be an unquestioned black box and, even more importantly, not open to questioning, as Bosch and Chevallard express it when they say of didactics that

> Its original singularity consists in taking as the primary object to be studied (and therefore to be questioned, modelled and problematised according to the rules of scientific activity), not the learner subject or the teacher subject, but the mathematical knowledge that they are supposed to study together, as well as the mathematical activity that their common study project will lead them to carry out. (Bosch and Chevallard, 1999, p. 79)

This assumption that questioning is necessary sets didactics apart from other disciplines, including pedagogy and (cognitive) psychology. In pedagogy, Develay asserts that

> [...] didactics assumes that the specificity of content is a determining factor in the appropriation of knowledge. Whereas pedagogy focuses on the relationship between the teacher and the pupil and between the pupils themselves (Develay, 1998, p. 266).

Matheron (2009) goes in the same direction with regard to psychology, pointing out that "Psychology does not take into account the distinction to be made between different types of knowledge [...], seen as given and unquestionable [...]" (p. 38). However, this distinction between didactics and psychology is not simply a way of demarcating different approaches, of marking one's territory so to speak, but raises fundamental questions about the value and validity of approaches that do not take (sufficient) account of the specificities of knowledge. Indeed, Matheron (2009) points out that failure to question knowledge also leads to a denial of its "different specific modes of learning, teaching and study" (p. 38).

This poses all the more of a question given that, from the outset of didactics, Brousseau imported from Bachelard's work (1934) the notion of epistemological obstacle, which essentially expresses the idea that the acquisition of knowledge can only be achieved by rejecting prior knowledge, prior knowledge that constitutes the meaning of the targeted knowledge. The validity of this obstacle-clearing hypothesis is well supported by several decades of research in mathematics didactics (Job & Schneider, 2014), even if, since then, didactics research has considered other avenues to explore (Artigue, 1990a). The reader may wish to consult Tricot and Sweller (2016) for a discussion of possible reasons for the above-mentioned 'specific knowledge blindness' in some parts of the educational sciences. In this context, it is hardly surprising that epistemology plays an important role in didactics if we consider that this discipline, part of philosophy

> is the critical study of the postulates, conclusions and methods of a particular science, considered from the point of view of its evolution, in order to determine its logical origin, value and scientific and philosophical significance[3].

The multiplicity of publications devoted to the links between epistemology and didactics also testifies to the strong link between these disciplines. A recent example is Bächtold, Durand-Guerrier and Munier (2018), who devote a book to the question of the links between didactics and epistemology. Epistemology as a discipline is not a monolithic block made up, once and for all, of indubitable historical facts. Epistemologists, and in particular science epistemologists, have developed epistemological theories that sometimes coincide and sometimes conflict, such as Kuhn (1996), Popper (1973), Lakatos (1994) and Feyerabend (1979), to name but a few. So there is no single way of questioning knowledge. This observation of plurality leads us to the notion of the epistemological reference model (ERM). Generally speaking, we define an ERM as a didactic construction, a model, which expresses in one form or another the epistemological characteristics of knowledge that appear central to the researcher, enabling him to control the meaning and coherence of the practices in which knowledge is used. This is the function of epistemological vigilance (Chevallard, 1991). As a model, a ERM of knowledge is not necessarily unique. This immediately raises the following questions. If there is plurality, how should a ERM be designed? On what basis? How can a MER be put to the test? How can epistemological vigilance be exercised through this potential plurality?

We obviously do not claim to be able to address all these issues in the space of this article. Our contribution will be much more modest and will be structured as follows. In the first part, we set out the theoretical framework on which our work is based. This framework is essentially made up of an articulation between the theory of didactic situations (TDS) and the anthropological theory of the didactic (ATD). We explain how the notion of ERM is conceived within this framework. In the second part, based on this theoretical framework, we outline a ERM for calculus and analysis that we have been using for several decades now in our research laboratories, the LADIMATH and the LADICHEC. In the third and final part, we use this ERM for calculus and analysis to sketch out some answers to the question of how to exercise epistemological vigilance. Finally, the distinction between calculus and analysis will be

[3] From the online dictionary of the Centre National de Ressources Textuelles et Lexicales (CNRTL): https://www.cnrtl.fr/definition/%C3%A9pist%C3%A9mologie

clarified in the second part of this article and will be an important feature of our ERM. For the moment, let us just say that analysis is the period of differential and integral calculus initiated by Cauchy when he first formalised the notion of limit in ε-δ, and calculus the period preceding Cauchy.

Part One. Theoretical framework: The concept of ERM in the light of TDS and ATD

In this section, we present the theoretical framework that serves as a sub-base for thinking about the notion of epistemological reference model (ERM). This theoretical framework is essentially based on an articulation between the theory of situations (TDS) initiated by Guy Brousseau (1998) and the anthropological theory of the didactic (ATD) initiated by Yves Chevallard (1992), based on the theory of didactic transposition (Chevallard, 1991). The articulation of distinct theoretical frameworks is not without question, as Perrin-Glorian (1999) has shown, and there is a considerable risk of lapsing into a misplaced syncretism.

However, the articulation we propose to set out will be duly motivated, precisely in relation to the questioning of knowledge and the question of ERM. We will show how this articulation on the question of knowledge is, so to speak, inevitable. Given the use we wish to make of our theoretical framework, we shall confine ourselves to presenting the elements of TDS and ATD that are directly useful to us, and refer the reader to the references given above for further developments, as well as to the more specific references that will appear throughout the rest of the presentation.

The contribution of TSD to the issue of MER: The notion of fundamental situation

In TDS, an ERM of knowledge is constructed from the notion of a fundamental situation. To a first approximation, a fundamental situation of a knowledge is a set of problems to which the knowledge provides an "optimal" solution. Let's refine this notion in successive layers by taking a closer look at the notion of problem and the qualifier "optimal", the meaning of which can vary according to the knowledge under consideration. This plurality of meanings of the qualifier "optimal" will be reflected in the distinction that will be made in the course of the presentation between fundamental situations in the *strict sense* and fundamental situations in the *broad sense*.

The notion of problem in TSD

In TDS, the notion of problem refers to something more precise than its usual understanding, where a problem is, according to the definition given by the Larousse online, "a point about which one wonders, a question which lends itself to discussion, which is the subject of arguments, of various theories". Firstly, as researchers, teachers and learners, we have probably all encountered and/or used problems (in the common sense) whose resolution does not really require the use of the knowledge that the teacher is going to use in front of the learners to carry out the resolution. This type of disconnection is commonplace in the introductory problems proposed in school textbooks under the heading "activity", which is not without its problems when learners solve these problems by means other than that envisaged by the teacher, since the latter can be undermined by the legitimate demands of learners who wonder why their "divergent" solutions have been discarded.

For a detailed example, see Schneider (2008, p. 41). By contrast, in TDS, only those problems that cannot be solved without the knowledge in question are considered to be real problems. A problem is therefore not a problem in itself, but is linked to knowledge and forms a pair. The function of problems in TDS is therefore to characterise knowledge by the fact that it cannot be dispensed with in order to solve them. This indispensability is the first meaning of the adjective "optimal". Imaginatively speaking, problems act as "equations" and knowledge is the solution to this system of "equations". From this point of view, the meaning of knowledge lies in its function as an "optimal" tool for solving a set of problems. In TDS, the meaning of knowledge is therefore inseparable from the problems that characterise it and from the history of its constitution as an 'optimal' tool for solving these problems.

Mathematical knowledge can carry several meanings, which cannot be reduced to one another, although they are articulated. This is particularly true of knowledge that has a FUGS character, i.e. formalising, unifying, generalising and simplifying (Robinet, 1984; Dorier, 1990). Examples include the notion of limit, to which we will return later, or the notions of linear algebra (Dorier, 1990). This multiplicity of meanings implies that a given piece of knowledge is likely to be modelled by several fundamental situations that take on the different meanings of the knowledge. For this type of knowledge, an ERM can therefore be made up of several fundamental situations. This multiplicity of meanings is an initial gateway to the notion of the institutional relativity of knowledge, which will be developed further in the context of the articulation of TDS with ATD.

Secondly, problems in TDS are also different from ordinary problems in terms of the notion of epistemological obstacle mentioned in the introduction. Following Bachelard, Brousseau considers that

> error and failure do not have the simplified role that we sometimes want them to play. Errors are not simply the result of ignorance, uncertainty or chance, as is believed in empiricist or behaviourist theories of learning, but the result of prior knowledge, which was useful and successful, but which now proves to be false or simply unsuitable. Errors of this type are not erratic and unpredictable; they are obstacles. In both the teacher's and the pupil's workings, error is constitutive of the meaning of the knowledge acquired. (Brousseau, 1983, p. 171)

The function of the problems that make up a fundamental situation may therefore also be, in addition to the indispensability/"optimality" characteristic mentioned above, to challenge the pupils' prior knowledge that is an obstacle to the acquisition of the targeted knowledge, and the overcoming of which is constitutive of that knowledge. We refer the reader to the following references for an in-depth discussion of the notion of epistemological obstacle (Artigue, 1990a; Schneider, 2008). Following this initial approach to the notion of fundamental situation and the contrast established between problem in the TDS sense and problem in the usual sense, it seems important to emphasise that a fundamental situation is not a problem-situation, the latter more often than not constituting an undue transposition of the former. A fundamental situation is therefore above all a modelling of knowledge and not a teaching project:

> [...] It is sometimes tempting to consider that, in the theory of situations, the notion of fundamental situation serves, above all, to describe and create teaching situations [...]. We then forget that this notion constitutes - also and above all - the key instrument proposed by this theory to characterise mathematical knowledge. (Bosch and Chevallard, 1999, pp. 80-81)

This being the case, didacticians are keen to have a positive impact on the education system and also put forward teaching proposals based on the notion of a fundamental situation. This dialectic (which is not an assimilation) between knowledge modelling and teaching project is discussed in particular in Perrin-Glorian (2019). See also Henrotay et al (2011), Krysinska and Schneider (2010), Lebeau and Schneider (2009), Ngan Nguyen, Rosseel and Schneider (2009), Rosseel and Schneider (2011), Schneider et al (2016) for examples of teaching proposals.

Let's continue with the qualifier "optimal". We will speak of a fundamental situation in the strict sense (Schneider, 2008) when the knowledge provides an "optimal" solution to the problems of the situation in an "absolute" way. For example, rationals considered as linear operators provide an "absolute" answer to the problem of enlarging jigsaw puzzles (Brousseau, 1998) insofar as any other procedure including additive procedure is invalidated in the action itself: the pieces of the jigsaw puzzle that are enlarged by another procedure cannot be assembled.

Fundamental situation in the strict and broad sense

However, the notion of a fundamental situation in the strict sense is too restrictive to allow all mathematical knowledge to be modelled. The notion of limit provides an example that highlights the limitations of this modelling of knowledge. Indeed, the deductive architecture of the calculus constituted by analysis can be based on the notion of limit, but, following the work of Robinson (1966), it can also be based deductively on the notion of infinitesimal. We cannot therefore consider that the notion of limit provides an "absolute" optimal solution to the problem of deductively founding the calculus. There are at least two possible foundations, and the notion of limit is only one of them. This is what Bloch (1999) expresses when she notes the

> non-necessity of the validation system of classical analysis: historically, mathematicians have hesitated for a long time, as we know, between validations of the "classical" type (inequalities, majorations) and validations by indivisibles, before settling on a theory. (Bloch, 1999, p. 188).

If the notion of fundamental situation in the strict sense is undermined, the idea of characterising knowledge by a set of problems is nonetheless possible, as long as we agree to relax the demand that knowledge constitute an "optimal" answer in an "absolute" sense in favour of a more "relative" vision. The example of the notion of limit suggests that this notion provides a solution deemed "optimal" to the problem of the deductive foundation of the calculus, by the institution "(standard) analysis", whereas the notion of infinitesimal provides a solution deemed "optimal" by the institution "non-standard analysis". We will therefore define the notion of fundamental situation in the broad sense (Schneider, 2008) of a piece of knowledge S in an institution I as a set of problems for which I considers that S provides an 'optimal' solution to these problems. In the broadest sense, knowledge, its meaning and its scope are relative to an institution, which decrees within it what knowledge is or is not acceptable and functional for its needs.

The contribution of ATD to the question of ERM: The notions of institution and praxeology

The notions of institution and institutional relativity

The idea of institutional relativity was already present in Brousseau's work, particularly when he highlighted the fact that the meaning of knowledge can be "correct in relation to the history of this concept, in relation to the social context, in relation to the scientific community" (Brousseau, 1998). What is lacking, however, is an understanding of the notion of institution that would enable us to encompass not only the usual institutions but also those mentioned above, such as '(standard) analysis' and 'non-standard analysis'.

This is where the anthropological theory of the didactic (ATD) comes in, one of the characteristics of which, underlined by the qualifier "anthropological", is precisely to study the relationship between institutions and knowledge. Let's look at the notion of institution in ATD. According to Chevallard (1992, p. 86) "everything is an object", including people and institutions. "From the point of view of the "semantics" of the theory, anything can be an object" (p. 86) and "An object exists as soon as a person X or an institution I recognises this object as existing (for it)" (p. 86). Consequently, "an institution can be just about anything" (p. 88). In particular, an institution exists from the moment one or more people agree to refer to it. In this sense, "standard analysis" and "non-standard analysis" are institutions because of the people who practise each form of analysis.

We see in the plasticity of the basic notions of ATD and specifically that of institution not a desire for theoretical casualness but the very consequence of the inclusion of this theory in the field of anthropology. The broad definition of the notion of institution is, in a certain sense, the expression of a methodology, an invitation to adopt the appropriate institutional level in order to make sense of the 'real' and flush out the phenomena that make it intelligible. This, at least, is how we interpret what Chevallard (1999, p. 221) has to say:

> The crucial point here, the implications of which we will gradually discover, is that the ATD situates mathematical activity, and therefore the activity of studying mathematics, within the whole range of human activities and social institutions. Yet this epistemological bias leads those who subscribe to it to cross in all directions - or even to ignore - many of the institutional boundaries within which it is customary to stand, because, ordinarily, we respect the division of the social world that the established institutions, and the current culture which disseminates their messages to satiety, present to us as self-evident, almost natural, and ultimately obligatory. According to this vulgate of the 'culturally correct', to speak validly of the didactics of mathematics, for example, presupposes that we are talking about certain distinctive objects - mathematics, first of all, and then, jointly and severally, students, teachers, textbooks, etc. - to the exclusion of others. - to the exclusion of almost any other type of object, and in particular of all

> those that are too quickly believed to be scientifically irrelevant because they appear to be culturally alien to the objects held to be emblematic of the issues involved in the didactics of mathematics. (Chevallard, 1999, p. 221)

Is it not also one of the functions and virtues of the scale of levels of didactic co-determination to draw our attention to the interest, and even the necessity, of looking beyond institutions in the usual sense in order to climb to the higher levels of this scale? We refer the reader to Chevallard (2002) for more details on this scale and to Job and Schneider (2010) for other examples of institutions related to the analysis which illustrate both the advantageous plasticity of the notion of institution, the use of the scale of levels of co-determination and, more finely, the didactic modelling work that can be involved in choosing relevant institutions, if not creating such institutions.

The notion of praxeology

The institutional relativity of knowledge underlines the different relationships that institutions can have to knowledge, which is expressed by different practices involving this knowledge. ATD provides a tool for analysing these practices with the notion of praxeology, which constitutes an anthropological model for didactic purposes. Briefly, a praxeology is a quadruplet made up of, in order, a task to be performed, a technique for performing that task, a technology which justifies that the technique does indeed enable the task in question to be performed, and a theory which provides a higher level of justification for the technique, by playing a technological role in relation to the technology then envisaged as the technique of the task which consists in justifying the pair (task, technique). The pair (task, technique) is called the "practical block" and corresponds to know-how, and the pair (technology, theory) is called the "logos block" and corresponds to knowledge. Let us make a few comments on the notion of praxeology, confining ourselves to the elements that we consider to be directly useful for our thinking on ERM. Further details on the notion of praxeology can be found in Bosch & Chevallard (1999), Chevallard (1992) and Schneider (2008).

The notion of praxeology makes it possible to model activities that would readily be labelled as "mathematical", such as the following. The task is to solve a quadratic equation, the technique is to apply the discriminant's formula, the technology is that which shows how the discriminant can determine possible solutions from a second-degree formatting likely to lead to factorization, the theory is the theory of equations. This is just one example, and the technique envisaged is not the only one possible, nor indeed is the technology considered. See, for example, Bosch and Chevallard (1999) for other possible choices.

A major use of the notion of praxeology for us is to enable us to reformulate the notion of fundamental situation in the broad sense. We will say that a fundamental situation in the broad sense of a knowledge S in an institution I is a set of praxeologies for which the knowledge S constitutes a technique judged 'optimal' by I for the tasks of these praxeologies. We will therefore describe these praxeologies as fundamental, in reference to the relationship with the notion of fundamental situation. The notion of fundamental situation in the broad sense of the term, considered in the light of (fundamental) praxeologies, thus provides the theoretical framework within which we consider the notion of ERM. The ERM that we consider below will be constructed on the basis of fundamental praxeologies. Other models of knowledge are obviously possible. We do not claim that fundamental praxeologies constitute the alpha and omega of knowledge modelling. On the other hand, it seems important to us to put on the table as clearly as possible the point of view adopted to approach the study of knowledge. The relevance of such an approach will be discussed in the third part of the article in relation to the notion of ERM in general.

We should now point out that, from the point of view of fundamental praxeologies, knowledge appears as a technique for accomplishing a certain number of (fundamental) tasks. At the same time, we have emphasised that the logos block corresponds to knowledge, which the example of equations of the second degree given above tends to confirm. So where is knowledge located in ATD? The answer to this question is provided by Bosch and Chevallard (1999), who point out that the "technique/technology/theory distinction is functional" (p. 86). Thus, in ATD, depending on the institutional and praxeological contexts considered, what is technical can also be envisaged as a technology or a theory, but also as a task. It is the needs encountered that will determine the relevant functions to be adopted in a given institution. Formulated differently, the notion of praxeology is a polymorphous tool and it is up to the didactician to determine which praxeological functions will be used to best highlight the specificities of the knowledge being studied.

In particular, the task function offers the potential for iterative modelling. In particular, this means that through tasks, a praxeology can model the type of practice that gives rise to other praxeologies. This iterative character of the praxeological model seems essential to us and will be used in what follows to formulate our ERM of calculus and analysis, via the introduction of two types of praxeologies (I and II). See also Bourgade, Cirade and Durringer (2023) for further developments on the question of the localisation of knowledge in ATD.

TSD as a safeguard in relation to the notion of economy of thought

The central use of the notion of praxeology in our theoretical framework might lead one to think that the support provided by TDS was merely an intermediary that can now be forgotten. But this is not the case. TDS continues to play a central role as a safeguard against certain uses that may be made of ATD and that we consider to be deviant. This is obviously not to criticise ATD, a theory to which we are fully committed, but rather some of its uses. There is a great temptation, when we have the praxeological formalism of the ATD at our disposal, to choose as the ERM a set of praxeologies whose legitimacy would be proven by the simple fact of having formulated the ERM in praxeological language, as if having recourse to the tools of the ATD were in itself a guarantee of the epistemological validity of the model adopted.

This type of deviance seems to us inevitable as soon as we have a sufficiently formalised theory where, like mathematics, we can make the formalism run on empty and give the illusion that significant mathematics has been achieved. The link (in reverse) with TDS is as follows. The fact that the optimality of knowledge is relative to an institution should not lead us to think that any praxeology is acceptable for modelling knowledge. The technique used must have a certain efficiency to accomplish the task in question. We will take this effectiveness as a postulate based on different references. At a relatively general level, socio-constructivist epistemology (Fourez, Englebert-Lecomte & Mathy, 1997) asserts that scientific theories:

> are creations of the human mind, adopted temporarily for their effectiveness in carrying out a given project or interpreting phenomena. But the same concepts are rejected or modified when this effectiveness is challenged. So it's not a question of believing in them, but of testing their limits. (Schneider, 2011, p. 177)

This notion of efficiency is well attested in other disciplines, such as physics, as Mach (1987) defends by speaking of an "economy of thought", an expression which we have adopted by extending its use to mathematics. In fact, the history and epistemology of mathematics can also attest to the fact that mathematical knowledge, and in particular mathematical models, are constructed to effectively achieve the goals assigned to them. For example, at a macroscopic level, the entire undertaking of the Bourbaki group and the creation of the structures that revolutionised 20th century mathematics can be seen as the very expression of this desire for economy of thought:

> Structures' are tools for the mathematician; once he has discerned, between the elements he is studying, relations that satisfy the axioms of a structure of a known type, he immediately has at his disposal the whole arsenal of general theorems relating to structures of this type, whereas previously he had to painstakingly forge for himself

> means of attack whose power depended on his personal talent, and which were often encumbered with unnecessarily restrictive hypotheses stemming from the particularities of the problem being studied. (Bourbaki, 1948, p. 42)

We refer the reader to Job and Schneider (in press) for more details on this notion of economy of thought.

Second part. Draft of a ERM for calculus and analysis

In this second part, we present a ERM of calculus and analysis that makes use of the theoretical framework explained in the previous part. For lack of space, we can only give a broad outline and refer the reader to Job (2011), Job & Schneider (in press) and Schneider (2008, 2011) for more details.

By calculus, we mean the period of differential and integral calculus (Boyer, 1949) which precedes the genesis of analysis initiated by Cauchy and which extends roughly from antiquity with Archimedes to the beginning of the 19th century. As a counterpoint, we designate by the term analysis the period that is initiated with Cauchy who founded for the first time differential and integral calculus on the notion of algebraised limit, i.e. expressed in ε-δ. Each of these periods is considered to be an institution in the sense of ATD (Job, 2011; Job & Schneider, 2010).

Essentially, our MER is made up of two fundamental praxeologies (in the sense mentioned above), which respectively model calculus and analysis. The modelling adopted is particular in that these two praxeologies constitute models of the mathematical activity of which these two periods are the result. From this point of view, calculus and analysis are seen as products of these two processes. We will therefore call these praxeologies the "Constitution of the calculus" praxeology and the "Constitution of analysis" praxeology respectively, to emphasise their character as processes.

These two praxeologies are themselves instances of two types of praxeology, I and II, which also model two aspects of mathematical activity that have played (and still play) a central role in the development of mathematics. Broadly speaking, the first type of praxeology models the activity of 'Determining magnitudes' and the second type models the activity of 'Designing a deductive architecture'.

The distinction between calculus and analysis adopted is motivated on two levels. The first is historical and epistemological legitimacy. The transition between the two periods constitutes a major epistemological break in terms of both the aims and objectives and the modes of validation adopted, which modify the meaning and scope of the knowledge involved in these two periods (Job, 2011). A second, more specifically didactic is linked to the didactic phenomena that the model makes it possible to highlight, precisely in connection with the notion of epistemological obstacle mentioned above. This level is addressed in the third part of the article, where we also discuss, in the light of the didactic analyses that this choice allows, the reasons why we have chosen not to provide a 'direct' model of calculus and analysis, but rather of the processes that lead to them.

Type I praxeologies and the "constitution of the calculus" praxeology

Part 1. Type I praxeologies

The type of task of a type I praxeology consists in determining preconstructed objects in the sense of Chevallard (1991). An object is preconstructed when its existence is not questioned, because it is considered to be self-evident, not subject to discussion. The status of preconstructed object can result from different circumstances, which may reinforce each other. The first consideration relates to the significance of the senses. The object exists because our senses allow us to perceive it. A second consideration is institutional. In certain institutions, the existence of an object may be culturally established to the point of excluding any doubt. For example, $\sqrt{2}$ has been manipulated for generations by secondary school pupils. In French-speaking Belgium, no one is able to prove its existence in the sense of deductive mathematics by referring to one or other construction of the real numbers (Boniface, 2002).

Despite this, what pupil would agree to question this existence, such is its pervasiveness in ordinary classroom practice, apart from having been pushed into such questioning through an encounter organised specifically for this purpose, an encounter based, for example, on a fundamental situation? In an anthropological sense, $\sqrt{2}$ exists de facto. By their very nature, preconstructs can be the subject of definitions, but their existence is not subordinate to these definitions. The definition of a preconstructed object is more often a matter of description than of the desire to create deductive support (Job, 2011), as is the case in the deductive mathematics typical of Type II analysis and praxeologies (see below).

Based in particular on certain intuitions (Fischbein, 2010) relating to these preconstructs, we develop one or more techniques for determining them. Essentially, the validation of these techniques cannot be based exclusively on a deductive theory, because within such a theory, preconstructed objects have no legitimacy, only deductively defined objects do. The validation of a technique must therefore also be based on a level of rationality (Rouy, 2007) other than the strictly deductive level. This other level of rationality constitutes a special case of technology in the sense of the ATD (Bosch & Chevallard, 1999), which can be described as "pragmatic": the technique is justified because it produces results in line with those obtained using another technique that has already been validated elsewhere. This type of validation has been ubiquitous throughout the history of mathematics, particularly throughout the period when the calculus was being formed, before the advent of analysis, and seems to be unavoidable at this first praxeological level.

Part 2. The praxeology of the "constitution of calculus

Within the calculus, four major types of tasks can be pinpointed: optimising quantities, determining tangents to curves, determining instantaneous velocities, and determining areas of surfaces and volumes of solids. These tasks concern the preconstructed objects tangent to a curve, instantaneous velocity, area and volume which, in this period, are not defined by means of limits in the sense of analysis, or any other strictly deductive definition. Thus the tangent at a point on a curve is not defined algebraically by an equation of the form $y = ax + b$ with a certain derivative as its slope, but as an object of synthetic geometry which has only one point in common with the curve. The area under a curve is not defined as an integral but determined by the geometric objects that delimit its surface, such as the segment of a parabola whose area, according to Archimedes, is determined by the delimitation of the segment of the parabola by a line joining two of its points (Edwards, 1994). The existence of these preconstructs is therefore not in question; the conviction of their existence is based on what the Greeks established in previous centuries, combined with a certain evidence, conveyed by visual perceptions.

The techniques used to determine these preconstructs are multiple. These include the calculation of limits, which is still in its embryonic stage and has therefore not yet been expressed in ε-δ, or the more efficient calculation of derivatives and primitives, but also a whole series of variants based on the notion of infinitesimal. It is not possible to detail all these techniques in the space of this article. Suffice it to say a few words about the calculation of limits at the embryonic stage, to give an idea of what it is all about and to contrast it with the modern version of the notion of limit. We refer the reader to Boyer (1949), Edwards (1994) and Grabiner (2005) for more details on these different techniques.

This embryonic calculation of limits consists of deleting terms from an algebraic expression in order to determine the object under consideration. Fermat (1896) used this type of calculation when he tried to solve the optimisation problem of dividing the segment AC into E, so that $AE \times EC$ was maximum (Figure 1.).

Figure 1.

Illustration of Fermat's sharing problem.

To find this maximum, Fermat developed a technique he called the method of adequality. It is based on the idea that the variations of a quantity are "small" in the "vicinity" of a maximum. Essentially, it works as follows. Let a and b be the lengths of segments AE and AC respectively. The product of the lengths to be maximised is $ba - a^2$. Let e be an increase in the length of segment AE. If e is "small", $ba - a^2 \approx ba - a^2 + be - 2ae - e^2$. After algebraic simplifications, we obtain $be \approx 2ae + e^2$ and after division by e, we are left with $b \approx 2a + e$. Fermat then concludes by cancelling e in this last expression, without any compensation game, to obtain $b = 2a$. We therefore need to divide the segment into two equal parts to maximise the area studied.

This technique is similar to a modern calculation of the derivative of a function based on its definition as the limit of a function with an average rate of change, i.e. $\lim_{e \to 0} \frac{f(a+e)-f(a)}{e}$, with $f(a) = ba - a^2$ and $f(a+e) = ba - a^2 + be - 2ae - e^2$. In fact, at a purely manipulative level, in this limit calculation, we cancel out e in the expression obtained once all the algebraic simplifications have been completed and the division by e has been carried out.

Beyond the similarities in form, which allow us to describe Fermat's method as an embryonic calculation of limits, there is a major difference between these two conceptions of the limit. For Fermat, the status of e is problematic because, depending on the stage of calculation, it designates a non-zero quantity at the moment of division by e and then a zero quantity in the final stage. Fermat's contemporaries, including Descartes, Mersenne and Roberval (Clapié & Spiesser, 1991), were not fooled by this and were quick to criticise the logical flaws in his adequality method. In the present sense of the notion of limit, this ambiguity is not present. If formally, calculating the derivative amounts, in the particular case discussed, to equating e to zero, the procedure is perfectly legitimate and justified by the ε-δ definition of the notion of limit. Fermat, however, continues to make use of his method of adequality on the basis of pragmatic justifications. On the one hand, he argues that his method has an almost unlimited scope and allows him to solve hitherto unknown problems:

> Thanks to this method, we have found the centres of gravity of figures terminated by straight and curved lines, as well as those of solids and many other things that we can deal with elsewhere, if we have the time. (Fermat, 1896, p. 123)

What's more, his method gives results in line with others previously obtained, such as determining the tangent at a point of a parabola, a result known since antiquity, or the optimisation problem considered above.

Newton later developed this technique to solve problems of instantaneous velocity, and Leibniz used it to determine tangents to curves. Despite these advances, their technological discourses, rooted in kinematics and geometry respectively, remained pragmatic in nature. In fact, for the founders of differential and integral calculus, like Fermat, the status of the infinitesimals involved remains open to question because of their dual status: sometimes non-zero, sometimes zero. Leibniz, for example, considered the tangent to be either a straight line with only one point in common with a curve, or a straight line with a portion in common with the curve between two "infinitely close" points. It wasn't until the work of Cauchy that this pitfall was overcome and differential and integral calculus developed a stronger theoretical foundation.

This transition to Cauchy is typical of the link between type I and type II praxeologies. At the end of type I praxeologies, the preconstructs will be defined on the basis of the techniques that determine them in order to give rise to deductive reasoning, which leads us to type II praxeologies that model mathematical activity in its strictly deductive component.

Type II praxeologies and the "constitution of analysis" praxeology

Part 1. Type II praxeologies

The type of task of a praxeology of type II consists in designing a deductive architecture of a more or less large portion of mathematics or of an extra-mathematical domain, whether the latter belongs to the sensible world or to other already constituted disciplines (physics, biology, economics, linguistics...). A type II praxeology is therefore a model of a particular type of mathematical activity.

A deductive architecture is a set of definitions and propositions that verify the following overall characteristics. We will confine ourselves to a few broad outlines, as our aim is not to provide an in-depth theory of deductive architectures, but to outline enough of their elements to provide a basis for the rest of the presentation. We refer to Patras (2001) for a study of this notion.

There are two kinds of propositions. Either they are demonstrated deductively; or they are accepted without demonstration with the status of axioms. Demonstrating deductively means that the deductions made constitute instances of the rules of inference of mathematical logic which are applied to definitions previously introduced (in the sense given below) or to propositions previously demonstrated deductively or having the status of axiom. The inference rules adopted are typically those of the "classical" logic of predicates, but other types of logic are also possible, such as intuitionistic logic (Largeault, 1992) linked to the constructive mathematics initiated by Brouwer.

A proposition accepted without demonstration is called an axiom. If an axiom is not demonstrated in the same way as a proposition, this status nevertheless entails a particular task. It involves proving the coherence of the axioms of the deductive architecture under consideration. Demonstrating the coherence of a set of axioms consists in exhibiting a set of objects whose existence is proven and which satisfy these axioms. The question of coherence takes us well beyond the scope of this article and the interested reader can turn to the correction and completeness theorems by consulting Mendelson (2015) for more details on this question.

Let's move on to definitions. In a deductive architecture, the purpose of a definition, which is not simply to introduce notation or terminology, is to characterise a type of object in a deductive way. Characterising a type of object deductively means that only this definition and the properties demonstrated on the basis of this definition can be used in (deductive) practices involving this type of object. In particular, a deductive definition must be accompanied by a proposition that demonstrates the existence of at least one object that satisfies it. If we define the notion of filter (Bourbaki, 2007) in topology, we still need to prove the existence of filters.

The "design a deductive architecture" type of task can take various forms, depending on the more or less local/global nature of the architecture to be designed, and on whether it has to be created from scratch or constitutes a more or less significant reworking of a previous architecture. In particular, at a local level, it may involve defining (deductively this time, and no longer merely descriptively) objects which have hitherto been preconstructed, in the sense referred to above in type I praxeologies, in order to control their properties and use through deductive reasoning alone.

But it can also involve demonstrating a proposition, giving a simpler demonstration, generalising it and therefore potentially reformulating it in a significant way to extend its field of applicability, weakening unnecessary hypotheses, etc. These different forms are obviously not mutually exclusive and usually interact with each other. The demonstration of a proposition may, for example, require the deductive definition of new objects which did not previously exist as preconstructs, at the same time as the reformulation of its statement, both at the level of the hypotheses and the thesis.

At a more global level, we can start thinking about the structural aspects of an architecture by trying to create an axiomatic that is as simple as possible, where the axioms are not redundant. This involves proving the independence of these axioms. At the same time, we can reflect on the axioms to be chosen to allow the main propositions of the architecture to be demonstrated with minimum effort, and on the form to be given to these results so that they can also be used as instrumental tools. This focus on deductive aspects may have a cost, and may further reduce the possibility of a connection with the sensible world and with what some would call intuition, at least in its ordinary sense (Fischbein, 2010).

One possible technique for accomplishing (locally) such a task of deductive shaping is the dialectic of (attempted) proofs and refutations highlighted by Lakatos (1984). To initiate such a dialectic, the techniques used in type I praxeologies to determine quantities can serve as inspiration for formulating deductive definitions.

This technique is obviously not algorithmic in a strict sense. Generally speaking, there is no "canonical" technique for carrying out the "design a deductive architecture" type of task. The non-algorithmic nature of this technique is partly explained by the fact that this type of task can generally be accomplished in different ways. In general, there is no single deductive shaping of a given domain. This is shown by the example of the notion of limit used above to motivate the introduction of fundamental situations in the broad sense. On a larger scale, the history of mathematics is made up of successive deductive recasts. This is the well-known question of foundations, with the emergence of set theory and modern logic, and more recently the emergence of category theory as a possible new foundation for mathematics (Bell, 1981).

Let us now turn to the "logos" block of type II praxeologies. The most obvious aspect of validation is probably the following. The technique used to build a deductive architecture will be validated insofar as the corresponding task has been accomplished. Given the hyper-focus on the deductive, the task has been accomplished if, in particular, the propositions of the architecture are well demonstrated deductively, the axioms are coherent, and the definitions refer to non-empty classes of objects. However, this aspect does not allow us to close the validation question. Indeed, as already emphasised, apart from the Lakatosian dialectic mentioned above, which offers some broad outlines, there is no algorithmic technique for designing a deductive architecture, and the resulting architecture is not necessarily unique.

Producing a deductive architecture is an open problem where the techniques and responses adopted may vary according to the institutions under consideration and according to validation criteria that are not limited to strictly deductive considerations. A certain subjectivity infiltrates even what might initially be considered a paragon of deductive rationality. As a result, depending on the degree of maturation of an architecture and of competing architectures, an architecture may be adopted simply because it is the only one in existence at a given period, despite its flaws, or because an architecture is well established in terms of a certain tradition that refuses to be overturned by another.

Aesthetic criteria (Feyerabend, 1979) may also come into play: choices of demonstrations and/or arrangements judged more elegant than others. Aesthetic considerations are not disassociated from criteria that might more readily be linked to deductive rationality, showing the intertwining of different levels of argument. Thus, one architecture may be preferred to another because it leads to a smaller number of more general basic propositions from which the other results of the architecture can be derived. More than one mathematician might consider such an architecture more beautiful, precisely because of the perception of its greater simplicity.

Our aim is not to enumerate all the arguments that may be involved in the 'logos' block of a type II praxeology, but rather to raise awareness of the plurality of levels of rationality involved, and hence of the anthropological dimension at the very heart of the deductive, and of the need, in the constitution of a MER falling within a type II praxeology, to take full account of this anthropological dimension, which governs institutional practices linked to knowledge, including transpositive practices, just as much as the deductive (Chevallard, 1991).

The irruption, so to speak, of non-strictly deductive considerations within type II praxeologies seems to us all the more important to emphasise, since the resulting deductive architecture will typically be put to use as a theory in the "logos" block of various praxeologies, from a strictly deductive perspective. This different view of deductive architecture in the process of constitution and constituted deductive architecture is an integral part of our ERM and will be put to use in the third part of the article.

Part 2. Praxeology as the constitution of analysis

The "constitution of analysis" praxeology which models the "analysis" period is an instance of type II praxeologies which corresponds to the particular task "Constitute a deductive architecture of the calculus". The articulation mentioned above with the praxeologies of type I and specifically the praxeology "constitution of the calculus" is clearly present. The techniques for determining the preconstructs of this praxeology are used to create deductive models: areas are defined as definite integrals, velocities and tangents using derivatives. Many historians regard Cauchy as a central figure in the development of analysis:

> Émile Borel, who had collaborated on the publication of Cauchy's works, wrote [...] that it was Hermite who had taught him "to know and admire Cauchy", and this admiration continued to grow as he got to know "better the man who was truly the creator of modern analysis ». (Dugac, 2003, 93)

This authorship is largely due to the central role that Cauchy gave to the notion of limit in his treatise on analysis (Cauchy, 1821). This notion was already present before him, but he seems to have been the first person we know of, and whose work had an impact, to have given a deductive definition of the notion of limit and to have really used it to define the notions of derivative, convergence of a series, etc., but also to prove propositions such as the inequality of the mean, using the now archetypal ε-δ technique.

Given this centrality of the notion of limit, it can be argued that this notion constitutes a key element of the technique enabling a deductive shaping of the calculus within the "constitution of standard analysis" institution. Moreover, this definition of the limit is part of a Lakatosian dialectic, as argued in Job (2011), based on the work of Grabiner (2005) and Lakatos (1984). In a few words, Cauchy seems to have been inspired, in constructing his definition of limit, by an algebraic lemma present in the work of Ampère, who also worked on differential and integral calculus, as well as a proof scheme present in Lagrange's attempt to estimate the error committed by truncating the Taylor series of a function to a given order. In this sense, we have just provided a fundamental praxeology of the notion of limit for the institution of (standard) analysis (Job & Schneider, 2010).

Third part. Generic function of a ERM: Exercising epistemological vigilance

The generic function of a ERM is to enable the exercise of a certain epistemological vigilance. What does it involve? How is it exercised? Generally speaking, exercising epistemological vigilance means passing judgement on practices involving knowledge by means of an ERM of that knowledge. More specifically, to what extent are these practices in line with the ERM? In other words, are they legitimate from the epistemological point of view highlighted by the MER? At first sight, these questions may seem trivial, even self-evident. We will show in what follows that this is not the case, that this exercise is central and makes it possible, in particular, to render intelligible practices that would not otherwise make sense. These practices can be situated at different levels. They include the practices of teachers, learners, curricula and reference frameworks, textbooks and researchers. Let's use our ERM of calculus and analysis to explore this dimension of epistemological vigilance.

Localisation of knowledge in ATD

Let's ask ourselves the question of the praxeological location and therefore the modelling of knowledge in ATD. One obvious location, if we follow the theory set out in the first part of the article, is the "technological-theoretical" block, since this block is called the "logos block". However, as we pointed out earlier, the "technical/technological/theoretical distinction is functional" (Bosch & Chevallard, 1999, p. 86) and what is technical in a given institutional and praxeological context may be task, technology or theory in another context. We can therefore locate knowledge, elsewhere than in the technological-theoretical block, as a technique. This possibility of localisation at the level of technique is what the praxeological types introduced above testify to, and specifically the second one, since the notion of limit appears there as a technique for accomplishing the task of giving a deductive structure to the calculus.

The functional character of praxeologies can also be seen clearly in our ERM, through the distinction made between type I and type II praxeologies, which constitute models of particular facets of mathematical activity and therefore of processes, and the resulting praxeologies. In the "constitution of analysis" praxeology, "limit" knowledge has the status of a technique and subsequently acquires the status of a theory in the (standard) "analysis" praxeology. This possibility, offered by the ATD, of considering knowledge according to different functions, is not a mere formal potentiality, a kind of theoretical gimmick, but constitutes a truly strategic place for exercising epistemological vigilance, as we shall now illustrate.

Neglect, misunderstandings and lack of articulation around type I and II praxeologies and pragmatic and deductive levels of rationality.

In Job (2011) we designed a didactic engineering based on our ERM of calculus and analysis which organises an encounter between secondary school students enrolled in a section which prepares them for higher education with a strong mathematical content (engineers, mathematicians, physicists) and the Lakatosian character of the notion of limit, i.e. type II praxeologies. This didactic engineering (Artigue, 1990b, 2002) is thought of as a research device of a phenomenotechnical nature (Schneider & Job, 2014) and not as a teaching project.

It shows that the students have an epistemological relationship based on empirical positivism, which is essentially the idea (Fourez, Englebert-Lecomte and Mathy, 1997) that scientific concepts are 'exact' reflections of the world that can be discovered by observing 'objective facts', independent of the observer's interpretations and observation devices.

This relationship constitutes an obstacle to the acquisition of the Lakatosian character of the notion of limit, sufficiently resistant and "invalidating" over time (see below) to qualify it as epistemological. This relationship manifests itself in particular in the fact that the students see the notion of limit as a preconstruct (in the sense mentioned above). To all intents and purposes, they consider that "everyone can see what the notion of a limit is all about". Fundamentally, for them, even if it can be used in a demonstration, a definition of this notion only describes a "reality" that is supposedly shared. The experimentalists' objections to their proposed definitions, based on counter-examples that invalidate them as a tool of proof in demonstrations of properties, are therefore not really legitimate in their eyes and are more a matter of "nitpicking" and a certain amount of bad faith.

This empirical positivist relationship is not restricted to the notion of limit; it is also present at the level of the main concepts of calculus and analysis, such as derivatives and integrals, and its demonstration is also based on our ERM of calculus and analysis (Balhan & Schneider, 2022; Gantois, 2012; Gantois & Schneider, 2012; Job & Schneider, 2014; Rouy, 2007; Schneider, 1988, 1991, 1992). It manifests itself in a number of ways, including the following: "curvilinear" areas and volumes cannot be determined exactly from "rectilinear" areas and volumes; an instantaneous speed cannot be determined exactly from average speeds; a tangent is a prime object in relation to its slope and is therefore not defined by the notion of derivative, which merely expresses a calculative property. This posture of empirical positivism is not limited to calculus and analysis, as shown by the work of our LADIMATH and LADICHEC research laboratories, which demonstrate its pervasiveness and resistance in various fields of mathematics, earning it the title of generic epistemological obstacle (Schneider, 2008; Schneider, 2013).

Over and above the obstacle dimension, what this research highlights is the need and necessity for pupils, for various reasons, to draw on mathematical concepts that are still in a preconstructed state, in order to enable them to experience the limits of this preconstructed status and, consequently, the need for a deductive review. This presupposes that they can be experienced over a relatively long period of time. On the other hand, it also means enabling them to make sense of the techniques for determining quantities, such as limits, derivatives and integrals, on the basis of arguments that they are in a position to accept. "Able to receive" refers to arguments that answer pupils' legitimate questions about the ability of techniques to achieve what they claim to achieve. This need is obviously linked to the notion of the epistemological obstacle mentioned above.

For example, it is a question of offering students access to arguments which lead not only to the acceptance of the existence of an instantaneous velocity but also to its method of determination by means of an embryonic form of limit calculation based on average velocities. We refer to the same references as above on this aspect, emphasising in particular the fruitful possibility of using a kinematic context to bring out an embryonic form of the calculus of limits and derivatives and as a gateway to the fundamental theorem of analysis. It is in this context, where the pupils are questioning themselves, that the level of pragmatic rationality and the praxeologies of type I take on their full meaning, by taking advantage of the anchoring to sensitive experience, and to common knowledge acquired in other disciplines, to weave networks of meanings that make sense for these pupils, through the development of pragmatic arguments, which admittedly cannot be reduced to canonical deductive mathematics, but which nonetheless have perfect epistemological legitimacy, as the historical development of calculus and analysis attests.

It should be stressed that, without this stage of working on the preconstructs, in conjunction with the level of pragmatic rationality, we can hardly expect a 'direct' entry into 'hard' deductive mathematics to be successful. In addition and at the same time, such an approach prevents students from working through the obstacle of empirical positivism and does violence to their culture and experience of the sensible world in a way that makes mathematics abstruse for many. Aside from bureaucratic compliance with curricula and standards, what has been gained by sticking to the strictly deductive? Not much, if we follow Rouy (2007), who shows that for secondary school teachers, the deductive level of rationality of type II praxeologies is emblematic of mathematics and mathematical activity, to the extent that the pragmatic level of rationality is masked, or even discredited, because it is deemed not to be 'rigorous' (Rouy, 2007). Given the difficulties, if not the impossibility, of bringing 'hard' deductive mathematics to life at secondary level, in the absence of a perception of another level of rationality, teachers are more often than not reduced to relying on 'hole-in-the-wall' praxeologies in which the 'technological-deductive' block is formally present but essentially absent. This praxeological vacuum is then essentially filled by ostensive practices, some of which are assumed and others disguised (Salin, 1999).

This raises the question of the value of bringing the pragmatic level of rationality to life outside the school context. In the training of future mathematicians and secondary school teachers, working on this level of rationality seems to us to be essential. The epistemological obstacles encountered by pupils do not suddenly disappear when they become students and enter university. Once again, immersing them from the outset in strictly deductive mathematics, as is customary in many countries, focuses them on another form of rationality, but without offering them the opportunity to confront these obstacles. In fact, decades of experience in teacher training in our research laboratories show that these obstacles are just as present among future teachers (Schneider & Job, 2016) as they are among in-service teachers themselves (Job, 2011). More specifically, experiments carried out at the University of Liège in Belgium (Job, 2023) show that the posture of empirical positivism, just as much as among secondary school students, constitutes an obstacle to a properly Lakatosian understanding of the notion of limit by future teachers, and that this posture resists 'hard' and systematic confrontations organised on the basis of our MER of calculus and analysis.

In addition, the lessons prepared by these future teachers on the notion of limits are organised in such a way as to reinforce the posture of empirical positivism in the students who follow them, thus creating a vicious circle that could be described as an "empiricist loop". This state of affairs does not seem to be unique to French-speaking Belgium. The available data are still being studied at this stage, but experiments conducted in September 2023 in various Brazilian states as part of the GECEMS research group suggest similar conclusions.

This inability of students and future teachers to enter into a Lakatosian dialectic can be clarified by our ERM as an inability to distinguish between the "constitution of analysis" praxeology and the resulting "analysis" (standard) praxeology, i.e. to distinguish between process and resultant of this process. Formulated differently, the very idea that the definition of a limit can be constructed 'from scratch' for the purposes of proof, rather than simply presented and 'justified' on the basis of ostensive practices, seems to be absent for more than one teacher. Experiments conducted in Belgium and Brazil that are still being studied indicate that, in essence, (future) teachers will reject $\forall \varepsilon > 0\ \exists n \geq 1 : |a_n - a| < \varepsilon$ as a definition of the convergence of a sequence in favour of $\forall \varepsilon > 0\ \exists n \geq 1\ \forall m \geq n : |a_m - a| < \varepsilon$ on the grounds that the former does not verify the same properties as the latter. In so doing they make the assumption that the second is indeed the one and only possible definition of the convergence of a sequence but without having constructed this definition in the least, so much so that it appears to be the one and only possibility of defining convergence.

This episode is probably the tree that hides the forest. As far as analysis is concerned, the adjectives "deductive", "formalised" and "rigorous", which are readily attached to it, might lead one to think that they are the subject of a consensus to the point of perfect transparency. This is not the case, but teachers are not necessarily aware of it. Experiments carried out with future teachers show that they are reluctant to consider proofs inspired by Cauchy (1821) as 'rigorous' on the grounds that they do not include quantifiers and do not explicitly rely on predicate logic, even though Cauchy is considered to be one of the founding fathers of analysis.

This experience seems to us to be indicative of a phenomenon - to transpose an expression used by Chevallard in relation to ostensives (Bosch & Chevallard, 1999) - of a reduction in the epistemological thickness of the 'deductive', the 'formal' and the 'rigorous', understood here not simply as qualifiers, but as referring to as many institutions in the sense of ATD (Job & Schneider, 2010). This reduction effectively consists in reducing the constitution of a deductive architecture to the use of predicate logic. This reduction seems untenable if we consider that in Cauchy's time predicate logic did not exist. There is therefore not one and the same rigour, one and the same 'analysis' but rather an evolving history of these institutions (Job, 2011) where Cauchy's major contribution to analysis was, as the title of his work (Cauchy,

1821) indicates, to algebraise the calculus by providing precisely the $\varepsilon - \delta$ formulation. This story continues to this day, notably with Weierstrass, whose contribution was to base the calculus on a clear notion of number, and Cantor and others who opened up the possibility of basing the calculus and, more broadly, the entire mathematical edifice on set theory, predicate logic and, more recently, category theory.

This epistemological reduction is not unique to learners and teachers. Researchers themselves are not immune and the notion of ERM therefore appears all the more fundamental for the researcher himself and the control of his own practices and the legitimacy of the research results announced. Consider, by way of example, the work of Przenioslo (2005) and Swinyard (2011) analysed in more detail in Job and Schneider (2014). Both work on similar principles. They propose that students, starting from an initial definition of a limit, construct new ones that are adapted to ever larger sets of functions, on the basis of examples and counter-examples. From a distance, this type of task may resemble a Lakatosian dialectic. But only from a distance, because the successive definitions envisaged are not evaluated by their ability to provide deductive proofs, or to determine quantities.

They are assessed on the basis of their ability to describe the perceptible characteristics of the functions of a reference set. These definitions therefore do not fit into either a type I or a type II praxeology. Their epistemological credibility is therefore open to question. On closer examination, this type of task is a sophisticated form of ostension (Brousseau, 1998) in which the students have to construct the definition that the teacher has in mind, rather than the definition or definitions that meet the needs of a legitimate mathematical problem. We will conclude by saying that getting students to produce a definition of a limit is not in itself evidence of the acquisition of this notion, which would be tantamount to confusing the result of a process, the definition of a limit, with the process itself, in this case a type II praxeology, the process which gives meaning to the result.

Conclusion

The ERM of calculus and analysis presented in the previous section has proved its worth for several decades now, as shown by the publications it has generated. However, it is not without its questions. It was initially designed to study analysis at the end of secondary school and the beginning of higher education, and was therefore not developed with the study of subsequent developments in analysis in mind, which are definitely not limited to spaces of functions of real variables. However, experience in the field shows that, after the first year of university, students may find themselves equally at a loss when faced with more sophisticated versions of real analysis which use different types of increasingly general spaces, from metric spaces to topological spaces, including Banach spaces, topological vector spaces, convergence spaces and many others which form the bedrock of modern analysis.

The notion of limit itself is not restricted to the real context and takes various forms in the above-mentioned spaces, applying to ever more sophisticated objects such as directed systems and filters, and even beyond the usual field of analysis to functors in category theory. So how can our ERM evolve to take on these different incarnations and pursue didactic analysis? This is the question we need to address in our next investigations.

We have illustrated how the function of epistemological vigilance can be exercised using our ERM of calculus and analysis. These examples are not insignificant. They allow us to emphasise that this possibility of exercising epistemological vigilance is closely linked to the nature of the ERM adopted, a nature which is itself conditioned by the theoretical framework in which this ERM is conceived. Taking account of the institutional relativity of knowledge has allowed us to distinguish between two levels of rationality within type I and type II praxeologies. This distinction has made it possible to locate some of the problems associated with calculus and analysis in the lack of articulation and awareness of these levels of rationality and their specificities. The functional character of the praxeological modelling of ATD has enabled us to locate knowledge at different praxeological levels and, in particular, to consider analysis as much as a theory intervening in different praxeologies as the result of a modelling activity with type II praxeologies.

Following on from the previous characteristic, this distinction, coupled with the notion of fundamental situation inherited from TDS, and therefore of knowledge, considered as the "optimal" solution of a set of problems, makes it possible to restore a certain epistemological depth to the main notions of analysis (limit, derivative, etc.) and to consider them as the expression of a modelling process involving a Lakatosian dialectic. This epistemological revival then serves as a tool for pointing out the epistemological obstacle of empirical positivism, as well as the incoherence of some of the ordinary practices relating to calculus and analysis. By extension, highlighting these characteristics raises the question of what an MER in general can do without these. In particular, what would be the scientific value of ERM or theoretical frameworks without such characteristics?

If epistemological questioning is posed as central in didactics, as an extension of the previous question, it seems inevitable to us to also question the links between the ontological theories (implicit and explicit) on which researchers base themselves and the nature of the ERM they design. In Job (2011), we raise the question of the link between teachers' ontological postures, and specifically Platonism, and the way in which they approach questions of teaching and learning. This type of questioning seems to us to be just as relevant to be developed at researcher level, but has been less addressed to our knowledge. One contribution along these lines is the article by Radford, Miranda and Vergel (2023), which examines knowledge in the context of objectification theory. What ontological position(s) does our ERM bear possible traces of ? Does our ERM make it possible to question these and other ontological positions? These are just some of the questions we need to address.

References

Artigue, M. (1990 a). Épistémologie et didactique. *Recherches En Didactique Des Mathématiques*, 10(2.3), 241-286.

Artigue, M. (1990 b). Ingénierie didactique. *Recherches En Didactique Des Mathématiques*, 9(3), 281-307.

Artigue, M. (2002). Ingénierie didactique : quel rôle dans la recherche didactique aujourd’hui ? Dans A. Terrisse (Dir.), Les Dossiers des Sciences de l’Éducation N°8, *Didactique des disciplines scientifiques et technologiques : concepts et méthodes* (pp. 59-72). Presses Universitaires du Mirail.

Bachelard, G. (1934). *La formation de l'esprit scientifique. Contribution à une psychanalyse de la connaissance objective*. Vrin.

Bächtold, M., Durand-Guerrier, V., & Munier, V. (Eds.) (2018). Epistémologie & didactique : Synthèses et études de cas en mathématiques et en sciences expérimentales. Presses universitaires de Franche-Comté.

Balhan, K., & Schneider, M. (2022). L’apprentissage de l’analyse à l’épreuve du théorème fondamental. *Recherches En Didactique Des Mathématiques*, *42*(2), 241–283.

https://revue-rdm.com/2022/lapprentissage-de-lanalyse-a-lepreuve-du-theoreme-fondamental/

Bell, J.L. (1981). Category Theory and the Foundations of Mathematics. *The British Journal for the Philosophy of Science*, 32(4), 349-358.

Bloch, I. (1999). L'articulation du travail mathématique du professeur et de l'élève dans l'enseignement de l'analyse en première scientifique. *Recherches en Didactique des Mathématiques*, 19(2), 135-194.

Boniface, J. (2002). *Les constructions des nombres réels dans le mouvement d'arithmétisation de l'analyse*. Ellipses.

Bosch, M., & Chevallard, Y. (1999). La sensibilité de l'activité mathématique aux ostensifs. Objet d'étude et problématique. *Recherches en Didactique des Mathématiques*, 19(1), 77–124. https://revue-rdm.com/1999/la-sensibilite-de-l-activite/

Bourbaki, N. (1948). L'architecture des mathématiques. In F. Le Lionnais (Ed.), *Les grands courants de la pensée mathématique* (pp. 35-47). Actes Sud.

Bourbaki, N. (2007). *Topologie générale*. Springer.

Bourgade, J.-P., Cirade, G., & Durringer, C. (2023). Le « savoir » comme fonction. *Caminhos da Educação Matemática em Revista*, 13(4), 9-41.

Boyer, C. (1949). *The History of the Calculus and Its Conceptual Development*. Dover.

Brousseau, G. (1983). Les obstacles épistémologiques et les problèmes en mathématiques. *Recherches En Didactique Des Mathématiques*, 4(2), 165–198.

Brousseau, G. (1998). *Théorie des situations didactiques*. La pensée sauvage.

Cauchy, A.-L. (1821). *Analyse algébrique*. Presses de l'école polytechnique.

Chevallard, Y. (1991). *La transposition didactique* (2e ed.). La Pensée Sauvage.

Chevallard, Y. (1992). Concepts fondamentaux de la didactique : perspectives apportées par une approche anthropologique. *Recherches En Didactique Des Mathématiques*, 12(1), 73-111.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches En Didactique Des Mathématiques*, 19(2), 221–266.

Chevallard, Y. (2002). Organiser l'étude. 3. Écologie & régulation. Dans J.-L. Dorier, M. Artaud, M. Artigue, R. Berthelot, & R. Floris (Eds.), *Actes de la XIe école d'été de didactique des mathématiques* (pp. 41-56). La Pensée Sauvage.

Clapié, M., & Spiesser, M. (1991). *Des problèmes d'extrema chez Fermat à la notion de dérivée*. IREM de Toulouse.

Develay, M. (1998). Didactique et pédagogie. Dans Jean-Claude Ruano-Borbalan, *Éduquer et Former*. Éditions Sciences humaines.

Dorier, J. (1990). Analyse historique de l'émergence des concepts élémentaires d'algèbre linéaire. *Cahier de DIDIREM*, 7.

Doukhan, C. (2022). Comment l'articulation entre théorie de l'activité et théorie anthropologique éclaire la transition secondaire-supérieur : le cas des probabilités conditionnelles. *Annales de Didactique et de Sciences Cognitives*, 27, 133-167.

Dugac, P. (2003). *Histoire de l'Analyse*. Vuibert.

Edwards, C. H. (1994). *The Historical Development of the Calculus*. Springer.

Fermat, P. de (1896). *Œuvres de Fermat, Tome III*. Gauthier-Villar.

Feyerabend, P. (1979). *Contre la méthode, Esquisse d'une théorie anarchiste de la connaissance*. Le Seuil.

Fischbein, E. (2010). *Intuition in Science and Mathematics*. Springer.

Fourez, G., Englebert-Lecomte, V., & Mathy, P. (1997). Nos savoirs sur les savoirs. De Boeck Université.

Gantois, J.-Y. (2012). Un milieu graphico-cinématique pour apprendre les dérivées. Potentialités et limites [Thèse de doctorat, Université de Liège].

Gantois, J.-Y., & Schneider, M. (2012). Une forme embryonnaire du concept de dérivée induite par un milieu graphico-cinématique dans une praxéologie 'modélisation'. *Recherches en Didactique des Mathématiques*, 32(1), 57-99.

Grabiner, J. (2005). *The Origins of Cauchy's Rigorous Calculus*. Dover.

Henrotay, P., Krysinska, M., Rosseel, H., & Schneider, M. (2011). *Des fonctions taillées sur mesure*. Presses universitaires de Liège.

Job, P. (2011). Étude du rapport à la notion de définition comme obstacle à l'acquisition du caractère lakatosien de la notion de limite par la méthodologie des situations fondamentales/adidactiques [Thèse de doctorat, Université de Liège].

Job, P. (2023, 12 septembre). *Didactique et notion de limite. Vers un modèle épistémologique de référence (MER) partagé* ? [Conférence] Séminaire de didactique des mathématiques, Pontifícia Universidade Católica de São Paulo (PUC-SP).

Job, P., & Schneider, M. (2010). Une Situation Fondamentale pour le Concept de Limite ? Question de Langage de Culture ? Comment la TAD Permet-elle de Problématiser cette Question ? Dans A. Bronner, M. Larguier, M. Artaud, M. Bosch, Y. Chevallard & C. Ladage (dir.), *Actes du 2ième congrès international sur la théorie anthropologique du didactique (TAD) : Diffuser les mathématiques (et les autres savoirs) comme outils de connaissance et d'action* (pp. 615-632). Uzès (France) du 31 octobre au 3 novembre 2007 : IUFM de l'académie de Montpellier.

Job, P., & Schneider, M. (2014). Empirical positivism, an epistemological obstacle in the learning of *calculus*. *ZDM Mathematics Education*, 46, 635–646.

Job, P., & Schneider, M. (sous presse). Quel enseignement pour préparer les apprenants à la modélisation mathématique ? In Commission Inter-IREM Didactique (org.), *Rencontres autour de la compétence "Modéliser"*. IREM de Poitiers.

Krysinska, M., & Schneider, M. (2010). *Emergence de modèles fonctionnels*. Presses universitaires de Liège.

Kuhn, Thomas S. (1996). *The Structure of Scientific Revolutions* (3rd ed.). University of Chicago Press.

Lakatos, I. (1984). *Preuves et refutations. Essai sur la logique de la découverte mathématique*. Hermann.

Lakatos, I. (1994). *Histoire et méthodologie des sciences : programme de recherche et reconstruction rationnelle.* Presses Universitaires de France.

Largeault, J. (1992). *L'Intuitionisme*. Presses universitaires de France.

Lebeau, C., & Schneider, M. (2009). *Vers une modélisation algébrique des points, droites et plans*. Presses universitaires de Liège.

Mach, E. (1987). *La Mécanique. Exposé historique et critique de son développement* (Bertrand, É., Trad.). Sceaux : J. Gabay (œuvre originale publiée en 1883).

Matheron, Y. (2009). *Mémoire et étude des mathématiques : une approche didactique à caractère anthropologique*. Paideia.

Mendelson, E. (2015). *Introduction to Mathematical Logic* (6th ed.). Routledge.

Ngan Nguyen, G., Rosseel, H., & Schneider, M. (2017). *Une approche heuristique d'une géométrie calculatoire*. Presses universitaires de Liège.

Patras, F. (2001). *La pensée mathématique contemporaine.* Presses Universitaires de France.

Perrin-Glorian, M.-J. (1999). Problèmes d'articulation de cadres théoriques : l'exemple du concept de milieu. Recherches En Didactique Des Mathématiques, 19(3), 279–322. https://revue-rdm.com/1999/problemes-d-articulation-de-cadres/

Perrin-Glorian, M.-J. (2019). L'ingénierie didactique entre recherche et ressource pour l'enseignement et la formation des maitres. *Caminhos da Educação Matemática em Revista*, 9(1), 45-82.

Popper, K. (1973). *Logique de la découverte scientifique*. Payot.

Przenioslo, M. (2005). Introducing the concept of convergence of a sequence in secondary school. *Educational Studies in Mathematics, 60*(1), 71–93.

Radford, L., Miranda, I., & Vergel, R. (2023). Savoir mathématique et action didactique dans la théorie de l'objectivation. *Caminhos da Educação Matemática em Revista*, 13(4), XX – XXX.

Robinet, J. (1984). *Ingénierie didactique de l'élémentaire au supérieur* [Thèse de doctorat, Université Paris VII].

Robinson, A. (1966). *Non-standard analysis*. Princeton University Press.

Rosseel, H., & Schneider, M. (2009). *Des grandeurs inaccessibles à la géométrie du triangle*. Presses universitaires de Liège.

Rosseel, H., & Schneider, M. (2011). *Ces nombres qu'on dit imaginaires sont-ils vraiment des nombres* ? Presses universitaires de Liège.

Rouy, E. (2007). Formation initiale des professeurs du secondaire supérieur et changement de posture vis-à-vis de la rationalité [Thèse de doctorat, Université de Liège].

Salin, M.-H. (1999). Pratiques ostensives des enseignants. Dans G. Lemoyne & F. Conne , *Le cognitif en didactique des mathématiques* (pp. 327-352). Les Presses de l'Université de Montréal.

Schneider, M. (1988). Des objets mentaux 'aires' et 'volumes' au calcul des primitives [Thèse de doctorat, Université catholique de Louvain-la-Neuve].

Schneider, M. (1991). Un obstacle épistémologique soulevé par des 'découpages infinis' des surfaces et des solides. *Recherches en Didactique des Mathématiques*, 11(2.3), 241-294.

Schneider, M. (1992). À propos de l'apprentissage du taux de variation instantanée. *Educational Studies in Mathematics*, 23, 317-350.

Schneider, M. (2008). *Traité de didactique des mathématiques. La didactique par des exemples et contre-exemples*. Les Éditions de l'Université de Liège.

Schneider, M. (2011). Ingénieries didactiques et situations fondamentales. Quel niveau praxéologique? In C. Margolinas, M. Abboud-Blanchard, L. Bueno-Ravel, N. Douek, A. Fluckiger, & P. Gibel (Eds.), *En amont et en aval des ingénieries didactiques : actes de la XVe école d'été de Didactique des Mathématiques* (pp.175-206). La Pensée Sauvage.

Schneider, M. (2013). Un obstacle épistémologique comme trait d'union des travaux d'un laboratoire de didactique des mathématiques. In C. Sylvie, & M. Haspekian (Eds.), *Actes du Séminaire National de Didactique des mathématiques 2012* (pp. 215-228). IREM de Paris.

Schneider, M., Balhan, K., Gerard, I., Henrotay, P. (2016). *Du calcul infinitésimal à l'analyse mathématique*. Presses universitaires de Liège.

Schneider, M., & Job, P. (2016). Ingénieries entre recherche et formation, *Éducation et didactique*, 10(2), 91-112. https://journals.openedition.org/educationdidactique/2508

Swinyard, C. (2011). Reinventing the formal definition of limit: the case of Amy and Mike. *Journal of Mathematical Behavior, 30*, 93–114.

Venturini, P. (2012). Action, activité, « agir » conjoints en didactique : discussion théorique. *Éducation et didactique*, 6(1), 127-136.