EMP Educação Matemática Pesquisa
ISSN 1983-3156
Qualis A1
http://dx.doi.org/10.23925/1983-3156.2024v26i3p422-441

Between intuition and formalization of calculus: applications of the derivative illustrated in comics

Entre a intuição e a formalização do cálculo: aplicações da derivada ilustradas em histórias em quadrinhos

Entre la intuición y la formalización del cálculo: aplicaciones de la derivada ilustradas en cómics

Entre l'intuition et la formalisation du calcul : applications de la dérivée illustrées en bandes dessinées

Fabiana Alves dos Santos[1]
Universidade Federal do Oeste Bahia (UFOB)
Doutorado em Matemática
https://orcid.org/0000-0002-5271-9231

Fabio Nunes da Silva[2]
Universidade Federal do Oeste Bahia (UFOB)
Doutorado em Matemática
https://orcid.org/0000-0003-3761-3348

Lauriclecio Figueiredo Lopes[3]
Universidade Federal do Oeste Bahia (UFOB)
Mestrado em Matemática
https://orcid.org/0000-0002-3356-5644

Priscila Santos Ramos[4]
Universidade Federal do Oeste Bahia (UFOB)
Doutorado em Matemática Aplicada
https://orcid.org/0000-0003-3394-764X

Abstract

In this article, the results of a didactic experiment are presented whose objective was to analyze the potential of Comics in promoting learning in Higher Education, especially in the Differential Calculus I component, during the remote period resulting from the COVID health emergency - 19. The study was based on notions of intuition and formalization, in addition to assumptions from the Anthropological Theory of Didactics, particularly mathematical praxeology and praxeological analysis. A praxeological analysis of the comics produced was carried out,

[1] fabiana.santos@ufob.edu.br
[2] fabionuness@ufob.edu.br
[3] lauriclecio@ufob.edu.br
[4] priscilasr@ufob.edu.br

identifying the predominant epistemological model in this experimentation. The application of derivative concepts to problems related to real situations resulted in an improvement in the student's relationship with knowledge, contributing to the construction of meaning in learning. Aspects inherent to the process that occurs between intuition and the formalization of a concept were highlighted. A paradigm shift was observed in the teaching of Calculus by focusing on the proposition of creative situations. The implications of this research for the training of Higher Education teachers are also highlighted.

Keywords: Teaching calculus, Comic, Epistemological reference model, Intuition, Formalization.

Resumo

Neste artigo, são apresentados os resultados de uma experimentação didática cujo objetivo foi analisar as potencialidades da História em Quadrinhos (HQ) na promoção da aprendizagem no Ensino Superior, especialmente no componente de Cálculo Diferencial I, durante o período remoto decorrente da emergência sanitária do COVID-19. O estudo foi embasado em noções de intuição e formalização, além de pressupostos da Teoria Antropológica do Didático, particularmente a praxeologia matemática e a análise praxeológica. Realizou-se uma análise praxeológica das HQs produzidas, identificando o modelo epistemológico predominante nesta experimentação. A aplicação dos conceitos de derivada a problemas relacionados a situações reais resultou em uma melhoria na relação do estudante com o conhecimento, contribuindo para uma construção de sentido na aprendizagem. Foram evidenciados aspectos inerentes ao processo que ocorre entre a intuição e a formalização de um conceito. Observou-se uma mudança de paradigma no ensino do Cálculo ao enfocar a proposição de situações criativas. Destacam-se também as implicações desta pesquisa para a formação do professor do Ensino Superior.

Palavras-chave: Ensino de cálculo, História em quadrinhos, Modelo epistemológico de referência, Intuição, Formalização.

Resumen

En este artículo se presentan los resultados de un experimento didáctico cuyo objetivo fue analizar el potencial de las historietas en la promoción del aprendizaje en la Educación Superior, especialmente en el componente de Cálculo Diferencial I, durante el período remoto derivado de la emergencia sanitaria COVID-19. El estudio se basó en nociones de intuición y formalización, además de supuestos de la Teoría Antropológica de la Didáctica, particularmente

la praxeología matemática y el análisis praxeológico. Se realizó un análisis praxeológico de las historietas producidas, identificando el modelo epistemológico predominante en esta experimentación. La aplicación de conceptos derivados a problemas relacionados con situaciones reales resultó en una mejora en la relación del estudiante con el conocimiento, contribuyendo a la construcción de significado en el aprendizaje. Se destacaron aspectos inherentes al proceso que se da entre la intuición y la formalización de un concepto. Se observó un cambio de paradigma en la enseñanza del Cálculo al centrarse en la proposición de situaciones creativas. También se destacan las implicaciones de esta investigación para la formación de docentes de Educación Superior.

***Palabras clave*:** Enseñanza del cálculo, Cómic, Modelo epistemológica de referencia, Intuición, Formalización.

Résumé

Cet article présente les résultats d'une expérience didactique dont l'objectif était d'analyser le potentiel des bandes dessinées pour favoriser l'apprentissage dans l'enseignement supérieur, en particulier en calcul différentiel I, pendant la période d'éloignement résultant de l'urgence sanitaire COVID-19. L'étude s'appuie sur les notions d'intuition et de formalisation, ainsi que sur les hypothèses de la théorie anthropologique de la didactique, en particulier la praxéologie mathématique et l'analyse praxéologique. Une analyse praxéologique des bandes dessinées produites a été réalisée, identifiant le modèle épistémologique prédominant dans cette expérience. L'application des concepts dérivés à des problèmes liés à des situations réelles a permis d'améliorer le rapport de l'élève à la connaissance, contribuant ainsi à la construction du sens dans l'apprentissage. Les aspects inhérents au processus qui se déroule entre l'intuition et la formalisation d'un concept ont été mis en évidence. Un changement de paradigme a été observé dans l'enseignement du calcul en se concentrant sur la proposition de situations créatives. Les implications de cette recherche pour la formation des enseignants de l'enseignement supérieur sont également soulignées.

***Mots-clés* :** Enseignement du calcul, Bande dessinée, Modèle épistémologique de référence, Intuition, Formalisation.

Between Intuition and Formalization of Calculus: Applications of Derivatives in Comics

This article aims to describe and analyze the potential of Comics in promoting learning in Higher Education, specifically within the context of the Differential Calculus I component, which constitutes the flowchart of courses in the area of Exact Sciences: Sanitary and Environmental Engineering, Degree in Mathematics, Interdisciplinary Bachelor of Science and Technology and Bachelor of Physics. Comics were utilized with a dual purpose: as an active methodology and a teaching resource, during the remote period, due to the COVID-19 health emergency, when online classes were planned and conducted remotely.

Initially, it is necessary to highlight that Calculus is one of the components with the highest failure rates (Pinto, 2008; Pereira, 2009), among the curricular components that make up the curricular matrix of courses in the area of Exact Sciences. This scenario has encouraged research on the teaching and learning process of the main mathematical notions used in this component. Among the dimensions of the problems that the teaching of Calculus in Higher Education in Brazil faces are difficulties related to mathematical concepts, which can sometimes be difficulties of a historical-epistemological nature (Silva, 2011). This may indicate that the reference epistemological model – REM, which supports the teaching of Calculus, is not well defined in institutional practices, or that the choices for disseminating knowledge imply the aforementioned difficulties, including the explanation of the transferred REM itself. In addition, there may still be difficulties arising from the abstract nature of the concepts at stake.

Recognition of the challenges faced in teaching Calculus motivated the development of an experience in the Differential Calculus I component, whose syllabus covers the topics of Limit of Functions and Limit of Sequences, as well as Derivatives and their applications. These topics are an integral part of the curricula of the Sanitary and Environmental Engineering, Degree in Mathematics, Interdisciplinary Bachelor of Science and Technology, and Bachelor of Physics courses. The initiative to create comics, which will be analyzed in this context, was developed with the aim of exploring the concepts of limits, derivatives and their possible applications with students of these courses, during the pandemic period, when classes were taught remotely.

The aforementioned analysis will consider a discussion on the role of intuition and formalization in the teaching and learning of Differential Calculus, based on two epistemological reference models that can direct mathematical practices at this level of education, namely: the pragmatic and the rationalist. The rationalist model follows the criteria

of mathematical formalization, while the pragmatic model, without disregarding these criteria, is more focused on the application of knowledge.

We infer that the students constructed the comics, in a way, following one of these two epistemological models, which occurred intuitively. In the results and discussion section, we compare the productions of these students with such models, highlighting what is manifested in accordance with the mathematical praxeologies presented in each comic. Evidence of intellectual intuition will be observed, which in such records denotes a need for formalization, highlighted in the descriptions of techniques for solving mathematical tasks, which the authors of this article extracted from the students' productions. Thus, we aim to conduct this analysis based on the study of mathematical praxeologies (Chevallard, 1999), contained in the mathematical records of comic books, to then discuss the epistemological model that guided the mathematical practices present in this material.

The context of the experiment was composed of a Mathematics teacher who works in the aforementioned undergraduate courses and students of these courses. The curricular component in question had as its basic bibliographic reference the book *Um Curso de Cálculo*, written by Hamilton Luiz Guidorizzi. The authors of this manuscript are researchers in the area of Pure and Applied Mathematics and in the area of Mathematics Education, working in the aforementioned undergraduate courses in components of the Core of specific knowledge, and in postgraduate studies, in the Professional Master's Degree in Mathematics in a National Network in said university. They currently carry out studies on teaching and learning Calculus, some are part of research groups on teaching Calculus. Following the text, the following will be presented: study on the role of intuition in Calculus and on the process of formalization in Calculus, the context of experimentation, praxeological analyzes of comics, didactic consequences of using these tasks for teaching and learning the calculus of derivatives and, finally, some considerations that signal possible generalizations resulting from the analyzes and point out open questions for the continuation of the study.

The role of intuition in Calculus

Mathematical intuition is a topic of interest to many researchers from various fields, such as Mathematics, Mathematics Education and Mathematical Philosophy (Marinho, 2019; Javaroni & Reis, 2005; Correia, 2010). Another topic that has provoked the interest of many researchers is the teaching of Calculus (Abreu & Silva, 2011; Alves, 2016; Trevisan & Tavares, 2017). This section aims to conceptualize sensitive intuition and intellectual intuition, as well

as to provide a reflection on the role of intuition in the process of teaching and learning Calculus.

It is important to emphasize that although other texts have contributed to this work, the text by Meneghetti (2009) serves as the fundamental basis for the ideas presented in this section. According to the author, sensation refers to the effect of an object on our capacity for representation, to the extent that we are affected by it. The intuition that arises from our connection with the object through sensation is called "sensitive intuition". On the other hand, intellectual intuition provides us with direct and immediate knowledge of the principles of reason, of the necessary relationships between beings or ideas, and of the truth of an idea or a being. In the context of teaching and learning in mathematics, for the student to achieve new knowledge he uses intellectual intuition, based on previous knowledge and personal and cultural perceptions, to establish new conjectures until the formalization of concepts.

To illustrate, imagine a scenario where at a crafts fair a merchant sold three lamps and two personalized bags, totaling R$260.00 in sales. But what would be the individual prices of a lamp and a bag? In this problem situation, it is possible to observe sensitive intuition when students try to determine the prices of the lamp and the bag by suggesting solutions. For instance, they might suggest that each lamp costs R$20.00, totaling R$60.00, and each bag costs R$100.00, totaling R$200.00, which adds up to a total sales of R$260.00. After some questions, students can certainly realize that there are other possibilities to obtain the total revenue, leading them to consider other hypotheses for the problem.

Intellectual intuition becomes evident when students assign variables to the objects "lamp" and "bag", think of the problem as a two-variable solution, discuss the types of solutions and seek a generalization for the problem.

It is common and desirable that intuition is encouraged in mathematics teaching. Authors of Calculus books (Guidorizzi, 2013; Stewart, 2021; Flemming, 2010) begin their explanations based on intuitive ideas of the concept that will be taught, gradually instigating the reader's knowledge until they are led to the formalization of the concept. To illustrate, we list below the intuitive ideas about the concept of derivatives presented in volume 1 of the book *Um Curso de Cálculo*, whose author is Hamilton Luiz Guidorizzi.

i. Establishes a motivation: To determine the angular coefficient of the tangent line T to the graph of the function f at the point $(p, f(p))$ with a point p in the domain of the function f;

ii. Consider a secant line S_x that passes through the points $(p, f(p))$ and $(x, f(x))$, whose angular coefficient is $\frac{f(x)-f(p)}{x-p}$;

iii. Presents a graph to illustrate the situation that was described;

iv. It states that when *x* tends to p, the angular coefficient of the straight line S_x tends to $f'(p)$, where $f'(p) = \lim_{x \to p} \frac{f(x)-f(p)}{x-p}$;

v. It states that as *x* approaches p, the straight line S_x tends to the position of the straight line T of equation $y - f(p) = f'(p)(x - p)$;

vi. Presents a graph to illustrate the situation described;

vii. Establishes another motivation: movement of a particle linked to an oriented straight line.

The ideas presented above demonstrate that the intuition expressed relies on an understanding of concepts such as functions of a real variable, equations of straight lines, tangent lines at points on function graphs, secant lines, and limits. Consequently, it can be inferred that the author employs intellectual intuition.

In the practice of teaching Calculus, it is possible to observe that the process of intuition plays an important role in the student's education. In this process, students are motivated by teacher to reflect, connect previous knowledge, formulate hypotheses, and discover new meanings, in addition to being challenged to formulate new problems.

In the following section, we discuss formalization in Calculus, but we previously note that this is a mathematical formalization and is not the formalization studied in the field of Philosophy of Mathematics.

Formalization process in Calculus

Differential Calculus is a mathematical tool used to study the behavior of a certain class of functions and is applied to solve problems in mathematics itself and in various fields of science, such as Physics, Biology, Engineering, among others. Historically, scientists Isaac Newton (1642-1727) and Gottfried Wilhelm Leibniz (1646-1716) are credited with developing Differential Calculus. Today there are several bibliographic references that deal with the topic (Guidorizzi, 2013; Stewart, 2021; Flemming, 2010) and each one presents an approach to the concept. In this section, we discuss the process of mathematical formalization in the teaching of Calculus in the sense of exploring the formal definition, axioms and ideas for demonstration.

Mathematical formalization allows relating the intuitive approach and/or problem situation to the theoretical field of the concept to be studied, where classic results, consequences

and applications are addressed. According to Snapper (1984), in the article "As três crises da matemática" (In English, "The three crises of mathematics"), formalizing an axiomatic theory means choosing an appropriate first-order language for the theory. The language of first-order vocabulary consists of five items: i) A list of enumerable quantities of variables, ii) Symbols for common language connectives, iii) The equals sign, iv) The two quantifiers, to represent "whole" and "exists" and v) Symbols for undefined terms (the parameters)

Next we discuss the formalization of the concept of derivatives presented in the book *Um Curso de Cálculo*, by Hamilton Luiz Guidorizzi. The relationship between intuition and formalization can be noted when, based on the ideas presented previously, the author introduces the formal definition of the concept of derivative, stated below.

Definition: Let f be a function and p be a point in its domain. The limit

$$\lim_{x \to p} \frac{f(x) - f(p)}{x - p}$$

when it exists and is finite, it is called the *derivative of* f *in* p and is indicated by $f'(p)$ (read: f line of p). Like this,

$$f'(p) = \lim_{x \to p} \frac{f(x) - f(p)}{x - p}$$

if f admits a derivative at p, then we say that f is *derivable* or *differentiable at* p.

From the definition above, it can be seen that the formal definition of derivative is presented as a limit, a concept that must be previously understood by the student. Then, the author presents a notation for the derivative and, finally, defines a differentiable function at the point. We can also notice the mathematical formalism when abstract objects, such as the function f and the point f in the domain of f are considered, and the definition of the derivative of f at the point $p, f'(p)$ is stablished. From the discussion of the intuitive ideas listed above, the author uses the angular coefficient of the tangent line at the point $(p, f(p))$ to introduce the concept of derivative. The conclusion was obtained from the variation of the secant line that contains the points $(x, f(x))$ and $(p, f(p)), x, p$ in the domain of f.

Thus, the analysis of intuitive ideas and the formalism of the definition shows how much knowledge is necessary to understand the definition of derivative, the first definition presented in the chapter of the aforementioned book. Moreover, this analysis allows us to affirm that mathematical formalism is based on the use of notations and symbols and this structure points to ideas studied in mathematical logic. These theoretical assumptions about intuition and formalization supported the analyzes presented in the subsequent section.

Analysis of comics produced by students pointing out the transition from intuition to formalization

In this section, the comics created by undergraduate students enrolled in the Differential Calculus I curricular component who participating in the experiment are described and analyzed. The analyses consider the role of intuition, formalization, and notions of the Anthropological Theory of Didactics – ATD (Chevallard, 1999), which employs Mathematical Praxeology and Praxeological Analysis.

Mathematical Praxeology (MP) is a theoretical-practical model used to represent the mathematical activity of individuals (teachers or students) within an institution (university, course, classroom, etc.). It acknowledges mathematical activity as a human endeavor mediated by mathematical objects.

A praxeology is represented as the quadruple [T, **τ**, Θ, Θ], where T is the type of task, **τ** is the associated technique, Θ is the technological discourse justifying the technique, and Θ is the associated theory (Chevallard, 1998). In this article, we refer to Mathematical Praxeology as PM, underlying the Praxeological Analysis adopted herein. This analysis focuses on understanding a person's mathematical activity, essentially modeling the mathematical activity produced by that person.

Praxeological Analysis highlights the elements of a praxeology. Drawing on Matheron (2000), we identify the tasks, which may be standard or non-standard, the techniques for solving these tasks, and the characteristics of the records used in these techniques. Additionally, we explore the technological-theoretical discourse integrating these tasks, aiming to discern an increasing complexity of MP.

Therefore, in this section, praxeological analyses of tasks extracted from the comics created by the participants were conducted, serving to highlight the records and, consequently, the characteristics of the mathematical practice produced in the curricular component under study.

The Praxeological Analysis, from this point onwards PA, consists of highlighting a mathematical action, based on a task, at least one standard technique for solving said task, and the theoretical discourse that justifies it. This model is based on Matheron's PA proposal (2000). Based on this information, we proceed with inferences regarding how intuition manifests itself, its relationship with mathematical formalization in these productions, and the epistemological model compatible with what was recorded in the comics.

It is worth noting that the tasks that make up the PA are adjustments to the mathematical actions indicated in the comics, carried out by the researchers, translated into language

compatible with the ATD, which is part of the theoretical-methodological framework of this study. Likewise, it should be noted that in the figures and transcription of the techniques presented in each Comics, some mistakes or typing errors made by the undergraduates were maintained, with corrected/adjusted expressions being presented in brackets, which will facilitate understanding of the PA discussion. In order to organize the text, we chose to present the comics with the aforementioned analyzes and show a synthesis of common aspects at the end of this section.

Context of the Comics

Comic1: the context of this story is an accident in offshore oil extraction, causing an oil spill, as can be seen in the Figure 1, below.

Figure 1.

Accident during offshore oil extraction (Students1 production, 2022)

The story continues with information regarding the attempt to solve the leak problem, as shown in Figure 2.

Figure 2.

Attempt to solve the problem (Students' production, 2022)

In the section of the comic, presented in Figure 3, there are elements to highlight a task that begins the praxeological analysis of the students' production.

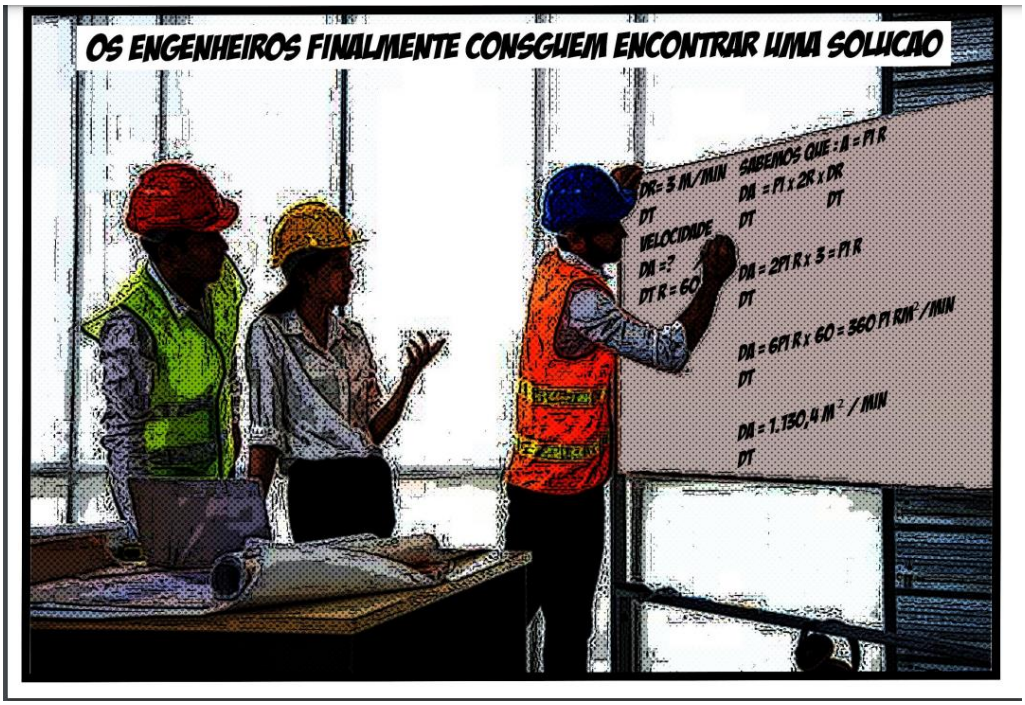

Figure 3.

Presentation of the solution (Student' production, 2022)

[T] Task 1: Determine the rate of variation of the area as a function of time, at the instant when the radius is 60 meters, knowing that the variation of the radius as a function of time is equal to 3m/min.

[**τ**] Technique: Consists of transcribing the resolution presented by the students in Figure 3.

$$\frac{dR}{dt} = 3m/min$$

$$velocity: \frac{dA}{dt} = ?$$

$$R = \ 60m$$

$$A = \pi R \ [A = \pi R^2]$$

$$\frac{dA}{dt} = 2\pi R \times 3$$

$$\frac{dA}{dt} = 6\pi R \times 60 = 360\pi R = 1.130{,}4 \ m^2/min$$

$$[\frac{dA}{dt} = 6\pi \times 60 = 360\pi = 1.130{,}4m^2/min]$$

[Θ, Θ] Speech that justifies the techniques: The derivative of a function represents the instantaneous rate of change of the dependent variable in relation to the independent variable. In this case, it is necessary to determine the variation of the area function in relation to time, $\frac{dA}{dt}$. However, the area is expressed in terms to the radius, and the radius is expressed in terms to time. The task presents the rate of change of the radius expressed in terms to time, and the equation below was obtained using the chain rule.

$$\frac{dA}{dt} = \frac{dA}{dR}\frac{dR}{dt}$$

Comic2: *The Avengers* face the challenge of closing a rectangular portal that could engulf the city. See Figure 4.

Figure 4.

Context of the comic "The Avengers: The Derivative of Power" (Students' production, 2022)

[T] Task 2: Determine how long it will take for the area of the rectangle to reach 132 square meters, knowing that the rectangle has a base of 10 cm, a height of 12 cm, both growing at 1 cm per second.

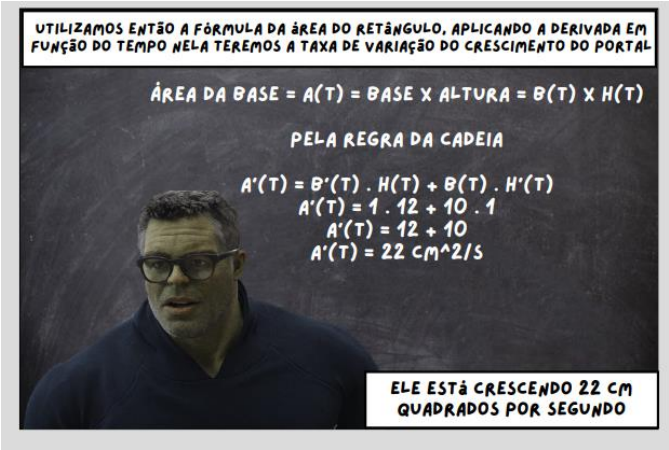

Figure 5.

Developing the strategy for a solution (Students' production, 2022)

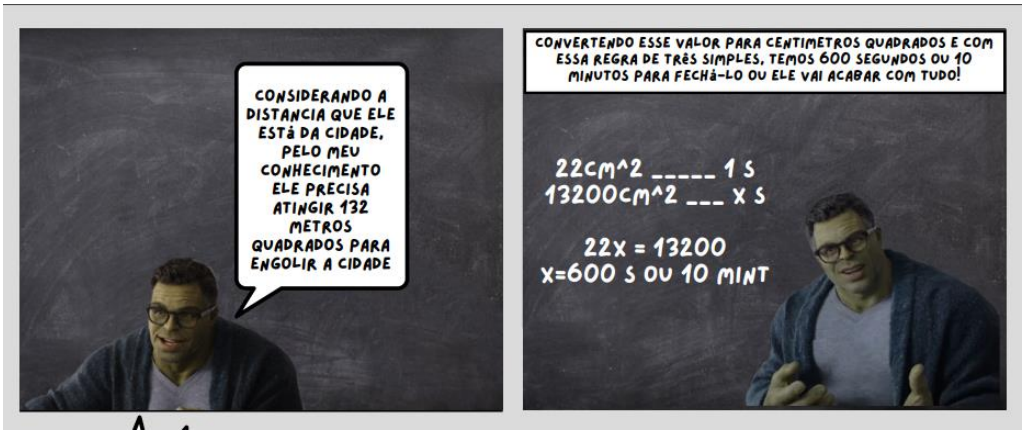

Figure 6.

Presentation of the solution (Students' production, 2022)

[**τ**] Technique:

$$A(t) = b(t)h(t)$$

$$A'(t) = b'(t)h(t) + b(t)h'(t)$$

$$A'(t_0) = 1 \times 12 + 10 \times 1 = 22cm^2/s$$

$$\frac{22}{13200} = \frac{1}{x} \left[\frac{22}{1320000} = \frac{1}{x}\right]$$

$$x = 600s \; or \; 10 \; min \; [x = 60000s = 1000min = 16h \; and \; 40 \; min]$$

[Θ, Θ] Speech justifying the technique: It is necessary to determine the growth velocity of the area of a rectangle, whose sides grow depending on the time. For this purpose, the Product Rule, presented by the students in the technique, was used. Then, a simple rule of three and measurement conversion are applied to determine how long it takes for the rectangle to reach

the given area. In the technique presented in the aforementioned Comic, there was a mistake in converting square meters to square centimeters.

Comic3: Teacher Girafales demonstrates the practical applications of derivatives in Physics.

Figure 7.

Context of the comic "Turma do Chaves em... Applications of the derivative in Physics" (Students' production, 2022)

The comic presents the definition of velocity and acceleration as the first and second derivatives of the position function, along with two activities. The first activity involves determining the velocity and acceleration of a particle at a specific instant, while the second activity involves finding the acceleration of a particle two seconds after the start of its movement. Here we only analyze the first activity.

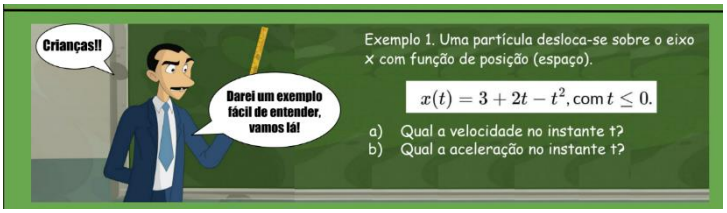

Figure 8.

Activities proposed in Comic3 (Student production, 2022)

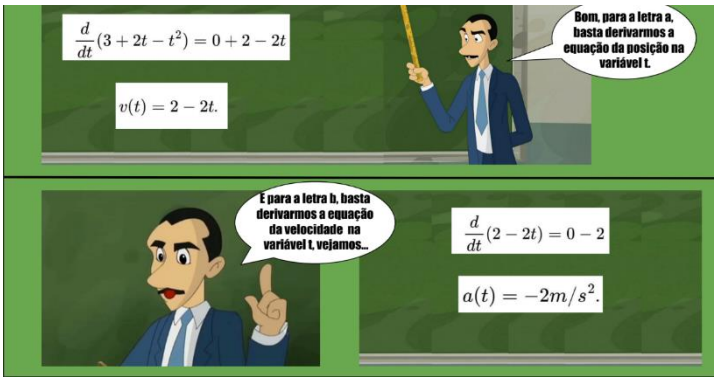

Figure 9.

Resolution of activity 1 of Comic3 (Students'production, 2022)

[T] Task: Determine the velocity and acceleration of a particle whose function is position $x(t) = 3 + 2t - t^2, t \geq 0$.

[**τ**] Technique:

$$x(t) = 3 + 2t - t^2, t \leq 0\ [t \geq 0]$$

$$a)\ \frac{d}{dt}(3 + 2t - t^2) = 0 + 2 - 2t = 2 - 2t$$

$$b)\ \frac{d}{dt}(2 - 2t) = 0 - 2 = -2$$

[Θ, Θ] Speech that justifies the technique: Considering that the concepts of instantaneous acceleration and velocity were used, the given position function is a polynomial function and the rule for deriving polynomial functions is used. Note a typing error when determining non-positive values for variable t that, in this context, represents time, so it is not negative.

Discussion of PA and didactic consequences of work with mathematical praxeology in comics

It is possible to notice from the analysis of the praxeologies of the three Comics presented in the previous section, especially from the analysis of their techniques, that mathematical practice is based on intellectual intuition and is consigned to a place that can represent a border between rational and pragmatic epistemological models. We understand that praxeologies are based on the reference epistemological model – REM rational, as the techniques are guided by justifications based on mathematical formalism. On the other hand, the rational discourse justifying the techniques did not emerge in the context of the comics; it remained in PA as the research process in the work of the authors of this article.

This fact leads to the inference that the idea of producing comics already signals a pragmatic bias, given that their authors, the undergraduates, seek application contexts as a backdrop for the stories. Moreover, it's important to note that pragmatic REM should not only manifest itself when referring to extra-mathematical applications, but also when thinking about applications with direct tasks that evoke the technological-theoretical discourse of the knowledge being taught or studied.

What reinforces the thesis that the pragmatic REM is manifested, more than the rationalist one, is the fact that the manipulation of algebraic symbols, present in the comics, do not give clues to the conceptual understanding of some characteristics related to formal Calculus, such as the analysis of points critical, or the proposition of other conceptual relationships beyond those presented in the theoretical moment of classes and based on the basic bibliography.

Furthermore, the techniques presented in the comics can be classified as standard and are mainly based on the basic bibliography, mentioned above. In this way, the praxeologies recorded in the Calculus book authored by Guidorizzi (2013) constitute the repertoire of standard techniques in the context of experimenting with working with comics. Even though this recommendation was not made explicit, it can be inferred about the manifestation of effects of the Teaching Contract (Brousseau, 2008), which here is characterized by the submission of the authors of comic books to believe that the techniques should follow the model of the textbook adopted as a basic reference, a common way of thinking among students who show the belief that one should follow the main book adopted by the teacher.

As stated in the Formalization of the Calculation section, the PA allowed us to infer that abstract objects are considered, namely the function f and a point p in the domain of that function, which implied the Calculation of the derivative of f at point p. More than that, it was observed that such praxeologies are firmly based on mathematical notations and symbols which, in turn, also signal the visualization of ideas from the field of mathematical logic.

Returning to the discussion about mathematical intuition, the data produced do not allow us to infer that sensitive intuition occurred because probably the student authors of the comics thought about the situation to apply a certain "rule" or certain "knowledge" already consolidated, which highlights intuition intellectual. Inferring about the aforementioned sensitive intuition, if it actually occurred, would require repeating the experience in a group in which, before explaining the mathematical theory, they faced the challenge of trying to solve the problem through different paths. This might bring the students' praxeologies closer of

situations that denote the praxeologies of mathematicians in the constitution of notions of limits and derivatives of functions.

Didactic consequences of using these tasks for teaching and learning Derivatives Calculus

It is possible to approach the didactic consequences of the development of the type of praxeology presented in comics from two perspectives, described below.

In the first, the use of comics in Higher Education, and in this case, Vergueiro (2004) points out some reasons for the pedagogical use of comics as a teaching tool, such as the efficiency of teaching when words and images are combined, the development of the habit of reading mathematical texts (emphasis added), the development of imagination and mathematical reasoning in the case of the experiment presented here. Despite the advantages, there are still few studies that discuss such contributions in Higher Education (Oliveira, 2010; Felix, 2016). One of the motivations for research on the work in the Differential Calculus component is the unsatisfactory performance of students. And considering what was pointed out by Vergueiro (2004), the consequences of the use of comics in the dissemination of knowledge of "derivatives of a function" are especially the development of imagination, formal mathematical reasoning and appropriation of appropriate language based on the reframing of reading. Furthermore, it can also be said that the praxeologies present in the comics are a work of resignification and overcoming some problems related to the first contact of students as beginners in the study of abstract mathematical theories.

In the second, concerning the didactic consequences of using the three tasks presented, in addition to understanding the Calculus application processes (which may allow an inference that the REM assigned to the teaching of Calculus in the reported context is pragmatic), such tasks represent a foundation for the study of integration and differential equations in the courses of Sanitary and Environmental Engineering and Physics, and Analysis, Differential Calculus in several variables and Integral Calculus, with tasks such as "to calculate the integral of f(x)", which is to be carried out, in a certain sense, the inverse operation of the derivation, that is, "to determine a function F(x) whose derivative is F'(x)=f(x), for Mathematics and Interdisciplinary Bachelor of Science and Technology courses.

In the case of the study on Differential Calculus in several variables, it is possible that, at least with regard to elements of a cognitive order, tasks that emerge from situations such as those shown in the comics directly imply the transition from Calculus in one variable to several variables, as suggested by Alves (2011).

In any case, work with the concept and applications of derivatives in comics is much closer to work based on a pragmatic REM than on a rationalist one, at least in the reported context.

General considerations

The proposal to work with comics provided undergraduate students with the opportunity to apply the concepts studied in Calculus to different problems related to real-life situations. This reveals significant implications for both teaching and learning. For teaching, it shows that Calculus classes can move away from the traditional model of exposition followed by solving exercises, approaching, instead, more active methodologies. For learning, it indicates that students better internalize knowledge when they perceive some practical meaning behind what they are studying, as was demanded in the construction of the comic.

One aspect to emphasize is the necessary observation by the teacher of the time required for the exercise of intuition, whether it be sensory or intellectual. This intuition should not be suppressed by excessive explanations and haste in proposing exercises. Furthermore, the process that occurs between intuition and formalization requires time, learning time. This learning process can be examined through praxeological analyses that highlight the tasks, the technique associated with each task, the technological discourse that justifies the technique, and the associated theory. The praxeological analysis, when referring to the knowledge about the undergraduate's mathematical practice, showed us the modeling of the participants' mathematical activity.

In the analyzed proposal, the pragmatic epistemological reference model predominated, as it explored the application contexts of the studied concepts. We reaffirm that among the didactic consequences of the development of the praxeology presented in comics (HQs), in Higher Education, the following can be listed: the efficiency of teaching by allowing the use of images associated with words; the exercise of reading mathematical texts; and the use of imagination and mathematical reasoning. Furthermore, it is worth highlighting the possibility of mitigating unsatisfactory performance of students in Calculus studies by improving the student's relationship with knowledge, specifically derivatives of a function, as it allows for the reframing and overcoming of problems that arise, as we stated, when students have their first encounter with abstract mathematical theories.

With these aspects observed, one can visualize a paradigm shift in the teaching of Calculus based on activities with comics. These activities focus on proposing creative situations

for applying the notions of derivatives, as one of the paths to the most significant development of the process of teaching and learning.

Additionally, the implication of this practice for the training of Higher Education teachers is noteworthy. The planning, development, recording, reflection, analysis, and sharing of results with a group of researchers, as well as the dissemination of experimentation through publications, all contribute to the departure of mathematics teachers from an individualistic approach focused solely on teaching. This demonstrates the possibility of viewing classroom practices as objects of study and emphasizes the importance of collectively thinking about teaching. It promotes the adoption of an epistemological model that genuinely fosters learning.

References

Abreu, O. H. D., & Silva, F. D. (2011). Uma discussão sobre o papel das definições formais no ensino e aprendizagem de limites e continuidade em Cálculo I. *Educação Matemática Pesquisa, 13*(3), 439-459.

Alves, F. R. V. (2011). *Aplicações da sequência Fedathi na promoção do raciocínio intuitivo no Cálculo a várias variáveis* [Doctoral thesis, Universidade Federal do Ceará]. Repository of UFC. https://repositorio.ufc.br/ handle/riufc/ 3166

Alves, F. R. V. (2016). Categorias intuitivas para o ensino do Cálculo: Descrição e implicações para o ensino. *Revista Brasileira de Ensino de Ciência e Tecnologia, 9*(3), 1-21. https://periodicos.utfpr.edu.br/rbect/article/view/1538/pdf

Brousseau, G. (2008). *Introdução ao estudo das situações didáticas: Conceitos e métodos de ensino*. Ática.

Chevallard, Y. (1998). Analyse des pratiques enseignantes et didactique des mathématiques: L'approche anthropologique. In R. Noirfalise (Éd.), *Actes de l'Université d'été Analyse des pratiques enseignantes et didactique des mathématiques*, La Rochelle, 4-11 juillet 1998 (pp. 91-120). IREM de Clermont-Ferrand. http://yves.chevallard.free.fr/spip/spip/article.php3?id_article=27.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en Théorie Antropologique du Didactique. *Recherches en Didactique des Mathématiques, 19*(2), 221-226. http://yves.chevallard.free.fr/spip/spip/IMG/pdf/Analyse_des_pratiques_enseignantes.pdf.

Correia, V. A. (2010). Intuição matemática em Jean Cavaillès. In *Estudios de Lógica, Lenguaje y Epistemología* (pp. 109-118). Universidad de Sevilla.

Felix, G. (2016). Produção de histórias em quadrinhos para a resolução de problemas matemáticos: O relato de uma experiência na iniciação à docência. *Annals of Encontro Nacional de Educação Matemática*. https://repositorio.utfpr.edu.br/jspui/bitstream/1/4034/1/LD_PPGMAT_M_Fonteque%2C%20Viviane%20Bergamini_2019.pdf

Flemming, D., & Gonçalves, M. (2010). *Cálculo A* (6th ed.). Pearson Universidades.

Guidorizzi, H. L. (2013). *Um curso de Cálculo* (5th ed., Vol. 1). LTC.

Javaroni, S. L. (2005). *A tensão entre rigor e intuição no ensino de Cálculo e análise: A visão de professores-pesquisadores e autores de livros didáticos* [Review of Doctoral thesis, by Reis, F. da S.]. *Bolema – Boletim de Educação Matemática, 18*(24), 125-132.

Marinho, E. E. S. (2019). *Intuição matemática* [Master's thesis, Universidade Estadual da Paraíba]. Repository of UEPB. https://sca.profmat-sbm.org.br/profmat_tcc.php?id1=4968&id2=170190115

Matheron, Y. (2000). Analyser les praxéologies: Quelques exemples d'organisations mathématiques. *lREM d' Aix-Marseille, 54*, 51-78.

Meneghetti, R. (2009). O Intuitivo e o Lógico no Conhecimento Matemático: Análise de uma proposta pedagógica em relação a abordagens filosóficas atuais e ao contexto educacional da matemática. *Bolema – Boletim de Educação Matemática, 22* (22), 161-188.

Oliveira, L. (2010). História em quadrinhos e matemática, essa conexão é possível? *Annals of Encontro Nacional de Educação Matemática*. https://conferencia.ciaem-redumate.org/index.php/xvciaem/xv/paper/viewFile/266/361

Pereira, V. (2009). *Cálculo no ensino médio: Uma proposta para o problema da Variabilidade.* [Master's thesis, Universidade Federal do Rio de Janeiro]. Repository of UFRJ. https://pemat.im.ufrj.br/images/Documentos/Disserta%C3%A7%C3%B5es/2009/MSc_13_Vinicius_Mendes_Couto_Pereira.pdf

Pinto, G. (2008). *Compreensão gráfica da derivada de uma função real em um curso de Cálculo semipresencial.* [Master's thesis, Universidade Federal do Rio de Janeiro]. Repository of UFRJ. https://pemat.im.ufrj.br/index.php/en/producao-cientifica/dissertacoes/2008/71-compreensao-grafica-da-derivada-de-uma-funcao-real-em-um-curso-de-calculo-semi-presencial

Reis, F. da S. (2001). *A tensão entre rigor e intuição no ensino de cálculo e análise: A visão de professores-pesquisadores e autores de livros didáticos* [Doctoral thesis, Universidade Estadual de Campinas]. Repository of Unicamp. https://repositorio.unicamp.br/Acervo/Detalhe/206743

Sá, C. C., & Rocha, J. (2012). *Treze viagens pelo mundo da matemática* (2nd ed.). SBM.

Silva, B. A. (2011). Diferentes dimensões do ensino e aprendizagem do Cálculo. *Educação Matemática Pesquisa, 13*(3), 393-413.

Snapper, E. (1984). As três crises da Matemática: O logicismo, o intuicionismo e o formalismo. *Humanidades, 11*(8), 85-93.

Trevisan, A. L., & Tavares, M. (2017). Integral antes de derivada? Derivada antes de integral? Limite, no final? Uma proposta para organizar um curso de Cálculo. *Educação Matemática Pesquisa, 19*(3), 353-373.

Vergueiro, W., & Rama, A. (2004). *Como usar as histórias em quadrinhos na sala de aula.* Contexto.