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Proposal for an expanded reference epistemological model for mathematics teaching

Propuesta de un modelo epistemológico de referencia ampliado para la enseñanza de las matemáticas

Proposition d'un modèle épistémologique à référence élargie pour l'enseignement des mathématiques

Proposta de um modelo epistemológico de referência expandido para o ensino da matemática

Teodora Pinheiro Figueroa[1]
Universidade Tecnológica Federal do Paraná (UTFPR)
https://orcid.org/0000-0001-8680-5202

Saddo Ag Almouloud[2]
Universidade Federal do Pará (UFPA)
https://orcid.org/0000-0002-8391-705

Abstract

This theoretical-methodological article aims to propose an expanded epistemological reference model for teaching mathematical concepts, based on the anthropological theory of didactic, more specifically, on the following theoretical constructs: expanded didactic triangle, archeschool, archedidactic transposition, vanishing point of personal and institutional relationships. The expanded epistemological reference model takes into account the fact that knowledge derives from praxeologies of various cultures, from various know-how in various institutions, and serves as a reference for the study of mathematical knowledge from the perspective of expanded didactic triangle, so that discussions can take place regarding the expansion of mathematical praxeologies and didactic praxeologies to establish a culture in those who teach and in those who learn in teacher training courses. As an example, a model of a expanded epistemological reference model related to differential and integral calculus is presented and the constructed model was applied to the fundamental theorem of calculus, proposing suggestions for possible mathematical praxeologies and didactic praxeologies, aligned with the vanishing point of calculus, to the study of problems that involve patterns of movement and change. We infer that the concept of vanishing point highlights the importance of the narrowing of different personal and institutional relationships with mathematical knowledge. This narrowing of relationships allows for increased discussions that make

[1] teodora.pinheiro@gmail.com
[2] saddoag@gmail.com

institutions spaces for debate, taking as a reference the epistemological reference model. In this perspective, the expanded epistemological reference model expands the spaces for these discussions, based on the expanded didactic triangle, which has as its principle the paradigm of questioning the world in educational institutions, which occurs through personal and institutional relationships with the object of knowledge.

Keywords: Expanded didactic triangle, Archeschool, archedidactic transposition, Fundamental theorem of calculus.

Resumen

Este artículo teórico-metodológico tiene como objetivo proponer un modelo epistemológico ampliado de referencia para la enseñanza de conceptos matemáticos, basado en la teoría antropológica del didactismo, más específicamente, en los siguientes constructos teóricos: triángulo didáctico ampliado, arqueescola, transposición archidáctica, punto de fuga de las relaciones personales e institucionales. El modelo epistemológico ampliado de referencia tiene en cuenta que los conocimientos derivan de praxeologías de diversas culturas, de diversos saberes en diferentes instituciones, y sirve como referencia para el estudio del conocimiento matemático desde la perspectiva del triángulo didáctico ampliado, de modo que puedan tener lugar discusiones sobre la expansión. de praxeologías matemáticas y praxeologías didácticas para establecer una cultura en quienes enseñan y quienes aprenden en los cursos de formación docente. A modo de ejemplo, se presenta un modelo de un modelo epistemológico ampliado de referencia relacionado con el cálculo diferencial e integral y el modelo construido se aplicó al teorema fundamental del cálculo, proponiendo sugerencias de posibles praxeologías matemáticas y praxeologías didácticas, alineadas con el punto de fuga del cálculo. El estudio de problemas que involucran patrones de movimiento y cambio. Inferimos que el concepto de punto de fuga resalta la importancia de los vínculos más estrechos entre diferentes relaciones personales e institucionales con el conocimiento matemático. Esta estrecha relación nos permite incrementar discusiones que conviertan a las instituciones en espacios de debate, tomando como referencia los modelos epistemológicos de referencia. Desde esta perspectiva, el modelo epistemológico ampliado de referencia amplía los espacios para estas discusiones, a partir del triángulo didáctico ampliado, que tiene como principio el paradigma del cuestionamiento del mundo en las instituciones educativas, que se da a través de relaciones personales e institucionales con el objeto de conocimiento.

Palabras clave: Triángulo didáctico ampliado, Arqueoescuela, Transposición arquidáctica, Teorema fundamental del cálculo.

Résumé

Cet article théorico-méthodologique vise à proposer un modèle épistémologique de référence élargi pour l'enseignement des concepts mathématiques, basé sur la théorie anthropologique du didactisme, plus spécifiquement, sur les constructions théoriques suivantes : triangle didactique élargi, archéo-école, transposition archididactique, point de fuite des relations personnelles et institutionnelles. Le modèle épistémologique de référence élargi prend en compte le fait que les connaissances dérivent de praxéologies de différentes cultures, de divers savoir-faire dans différentes institutions, et sert de référence pour l'étude des connaissances mathématiques dans la perspective de triangle didactique élargi, afin que des discussions puissent avoir lieu sur l'expansion de praxéologies mathématiques et de praxéologies didactiques pour établir une culture chez ceux qui enseignent et ceux qui apprennent dans les cours de formation des enseignants. À titre d'exemple, un modèle épistémologique de référence élargi lié au calcul différentiel et intégral est présenté et le modèle construit a été appliqué au théorème fondamental du calcul, proposant des suggestions de praxéologies mathématiques et praxéologies didactiques possibles, alignées sur le point de fuite du calcul, à l'étude de problèmes impliquant des modèles de mouvement et de changement. Nous en déduisons que le concept de point de fuite met en évidence l'importance des liens plus étroits entre les différentes relations personnelles et institutionnelles avec la connaissance mathématique. Cette relation étroite permet de multiplier les discussions qui font des institutions des espaces de débat, en utilisant comme référence les modèles épistémologiques de référence. Dans cette perspective, le modèle épistémologique de référence élargi élargit les espaces de ces discussions, basées sur le TDE, qui a pour principe le paradigme du questionnement sur le monde dans les institutions éducatives, qui se déroule à travers des relations personnelles et institutionnelles avec l'objet de connaissance.

Mots-clés : Triangle didactique élargi, Archéo-école, Transposition archidactique, Théorème fondamental du calcul.

Resumo

Este artigo de cunho teórico-metodológico, tem por objetivo propor um modelo epistemológico de referência expandido para o ensino de conceitos matemáticos, apoiando-se na teoria antropológica do didático, mais especificamente, nos seguintes constructos teóricos: triângulo didático expandido, arqueescola, transposição arquididática, ponto de fuga das relações pessoais e institucionais. O modelo epistemológico de referência expandido leva em consideração o fato de que o saber decorre de praxeologias de várias culturas, de vários saberes-fazer em diversas instituições, e serve como referência para o estudo de saberes matemáticos na perspectiva do triângulo didático expandido, para que ocorram discussões a respeito da expansão das praxeologias matemáticas e das praxeologias didáticas para instituir uma cultura naquele que ensina e naquele que aprende em cursos de formação de professores. Como exemplo, apresenta-se um modelo de um modelo epistemológico de referência expandido relacionado ao cálculo diferencial e integral e aplicou-se o modelo construído ao teorema fundamental do cálculo, propondo sugestões de possíveis praxeologias matemáticas e praxeologias didáticas, alinhadas ao ponto de fuga do cálculo, ao estudo de problemas que envolvem padrões de movimento e de mudança. Inferimos que o conceito de ponto de fuga evidencia a importância do estreitamento entre as diferentes relações pessoais e institucionais ao saber matemático. Este estreitamente das relações permite incrementar discussões que tornam as instituições espaços de debates, tendo como referência os modelos epistemológicos de referência. Nesta perspectiva, o modelo epistemológico de referência expandido amplia os espaços destas discussões, a partir do triangulo didático expandido, que tem como princípio o paradigma de questionamento do mundo nas instituições de ensino, o qual se dá por meios das relações pessoais e institucionais com o objeto de saber.

Palavras-chave: Triângulo didático expandido, Arqueescola, Transposição arquididática, Teorema fundamental do cálculo.

Proposal for an expanded reference epistemological model for mathematics teaching s

Research in didactics of mathematics focuses on contributions to issues related to phenomena occurring at the heart of teaching and learning processes, directly or indirectly intervening in the relationships between the student, the teacher, and the knowledge at stake. These contributions can be evidenced by the theories of the didactics of mathematics, which present theoretical models to study and investigate phenomena that occur in the universe of teaching and learning processes of mathematical knowledge.

Theoretical models are essential for the development of the field of research in didactics of mathematics.

Brousseau (1972) reports on one of the first theoretical models presented at a conference on learning structures. This model was considered a starting point for developing the theory of didactical situations (TDS), which models didactical situations and interactions between the learner, knowledge, and the *milieu*.

Gascón (2024) asserts that, for the TDS, models are not the psychocognitive mechanisms underlying the learning process but rather the situations considered as models of bodies of mathematical knowledge. Therefore, the research questions:

> How do students learn mathematics? What difficulties do they encounter? Through which mechanisms or cognitive processes do they acquire mathematical concepts (or how do they construct them)? What are the most appropriate methods for teaching those concepts? etc. (Gascon, 2024, p.3)

change in nature, to:

> What are the conditions that a situation must satisfy to materialize the specific knowledge it models? What are the foreseeable effects of this operation on the subjects involved and their production? Which game should be played to ensure knowledge is required? What relevant information or feedback should subjects receive from the *milieu* to guide their choices and achieve one type of knowledge rather than another? (Gascon, 2024, p.3)

This change in the nature of the questions signals the occurrence of new types of didactic problems, which focus on situations and require a more refined look within the scope of the epistemology of knowledge.

Given this need and considering the didactic transposition processes that are necessary for didactic situations to occur and for questions related to didactic problems to be answered, we find in Chevallard's anthropological theory of the didactic (ATD) (1992) the possibility of modeling knowledge/knowing at stake in human activity, based on the construct of praxeology

$\wp$, formed by the praxis block *[T, τ]*, know-how, and the logos block *[θ, Θ]* discourse on praxis, where *T* refers to the task type containing at least one task t, τ the technique or manner of performing a task of the type *T*, *θ* a technology, i.e., a discourse on technique, and *Θ* is the theory that constitutes technology.

The construct of praxeology opens many possibilities for the analysis and investigation of aspects related to how a given object of knowledge is interpreted (in the praxis and logos blocks) and consequently reveals its reason for being in the institutions that produce, develop, use, and disseminate this knowledge.

This construct is the result of an evolution in theoretical models in the field of didactics of mathematics research, resulting from questions arising from didactic problems, which emerge from the need to establish increasingly close personal and institutional relationships regarding the epistemology and genesis of mathematical knowledge.

Thus, Chevallard (2019) asserts that the first questions the ATD leads us to raise are: What is this knowledge that you call *ℎ* and claim to teach? Where does *ℎ* come from? How is *ℎ*σ legitimized - epistemologically speaking? Is *ℎ*σ viable in the long run? Or will it have to be reprocessed or even deleted?

The author puts forward the importance of this questioning for anyone teaching someone to learn something. There is a concern about fulfilling the curriculum proposal and a gap regarding "n" questions that could be asked by the various "why" and "what for" regarding *ℎ*. We can say that regarding the knowledge acquired, the personal relationship with the objects of knowledge is the result of the subjections to the institutions through which the subject has passed and that there is a risk that these subjections are reflected in the subjects as a guarantee that this knowledge is sufficient.

Given this scenario, this research raises the question of whether the evolution of theoretical models has converged toward the need for expanded reference epistemological models (EREMs), which present characteristics that enable discussions about mathematical and didactic praxeologies to answer the questions formulated by Chevallard (2019), and more specific questions regarding didactic organizations (Dos), such as: How should the praxeological equipment of those who teach someone to learn something be? These issues directly impact the didactic transposition processes that occur in mathematics teacher education courses.

This theoretical-methodological article proposes an expanded reference epistemological model (EREM) based on the anthropological theory of the didactic, specifically the expanded didactic triangle (EDT), the archeschool, the archedidactic transposition, and the vanishing point of personal and institutional relationships.

Below we will explain key elements that are part of the development of the EREM.

Vanishing Point of Personal and Institutional Relationships

The vanishing point (Figure 1) is the object of knowledge of the anthropological systems on which SDT is based, as it is at the heart of human relations. These human relations in SDT and, in general, in Mathematics Didactics, are based around an object of knowledge. This object is the core of the discussions that occur in Didactic Transposition processes.

The process of External Didactic Transposition involves discussions between the members of the noosphere, which can be represented by people who occupy different positions in different types of institutions, such as teachers, educators, book authors, editors, and politicians. In these discussions regarding the process of Didactic Transposition of wise knowledge, the knowledge to be taught present in curricula and textbooks becomes an object that exists for each of the institutions that make up the noosphere, establishing what can be called the institutional relationship $R_I (p, o)$ revealed by each representative.

The process of Internal Didactic Transposition is carried out by the teacher in a process of Transposition of the knowledge to be taught to the knowledge to be taught. This process is impacted by this teacher's personal relationship with knowledge, which we denote by $R (x, o)$, as well as by the institutional relationship $R_I (p, o)$ of this teacher with the knowledge in question.

In this case, it can be said that an object of knowledge ***o*** is the Vanishing Point of the relations $R (x, o)$ and $R_I (p, o)$, since o is the reference of both relations. Everything that x knows about ***o*** determines the level of this relationship among the “n” perspectives of ***o***, which represent the different reasons for ***o***'s existence in universes of different types of communities in which this knowledge lives.

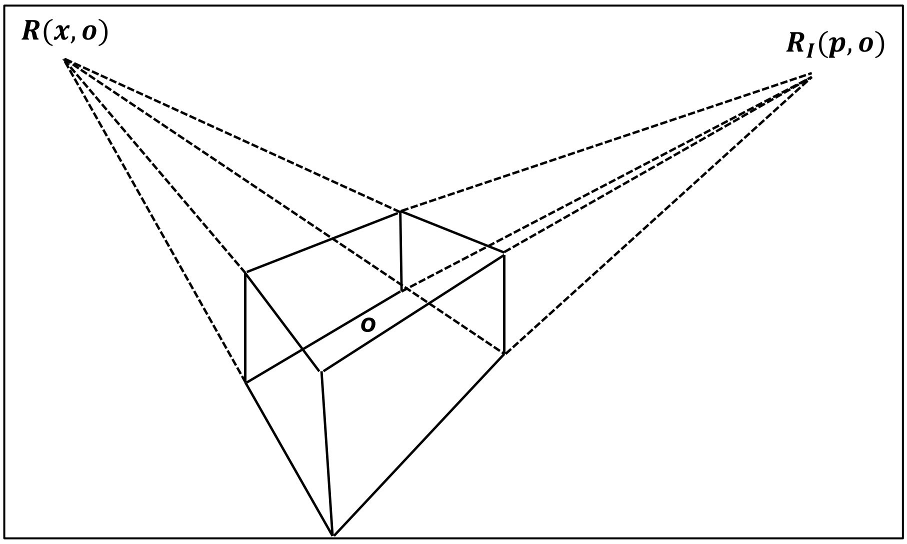

Figure 1.

Vanishing point (our own production)

We can say that mathematical knowledge is the result of the know-how of diverse multicultural universes of the Babylonian, Egyptian, Chinese, Indian, and Islamic peoples and that it continues to evolve in a globalized world. The same mathematical object presents different types of reinterpretations among people, such as the various ways of solving the second-degree equation (Taub, 2022).

Wozniak, Bosch, and Gascon[3] claim that knowledge is the product of human constructions, and its place and function differ depending on places, societies, and time. This fact makes explicit the existence of different perspectives of ***o*** (object of knowledge) and how much the praxeological equipment of those who appropriate these different perspectives, or reasons for being of ***o***, can be expanded.

Given this, the following question is: Which perspectives of ***o*** live in educational institutions —or more specifically, in teacher education institutions? The answer to this question depends on how external and internal didactic transposition processes have occurred.

We assert that when we have access to the different perspectives of ***o***, we have access to the different types of know-how, praxis block, in which you can open discussions about the logos block.

Strømskag and Chevallard (2024) have reported in their research that, in terms of the ATD, students have demonstrated a limited capacity for the logos component of their praxeology related to concavity and inflection points. This deficiency hampers their ability to adequately explain and justify or refute the techniques they employ. They assert that the possible

[3] https://ardm.eu/qui-sommes-nous-who-are-we-quienes-somos/yves-chevallard/

discrepancy in mathematical knowledge due to how the didactic transposition processes occur results from the different roles the logos block plays in one institution or another.

Thus, the institutional transposition[4] has violated the logos, causing ruptures in the justifications of the techniques carried out and identification of the validity of other know-how or the validation of other techniques. This could reveal other aspects of the object of knowledge and, therefore, other reasons for being.

This situation led us to reflect on the importance of proposing theoretical reference models as contributions to discussions on the existence of other mathematical know-how in teacher education courses to narrow the level of engagement between $\boldsymbol{R(x,o)}$ and $\boldsymbol{R_I(p, o)}$ and thus put to the test the logos that live in the institutions from the perspective of mathematical and didactic praxeologies.

This discussion culminated in the need to consider the constructs *archeschool* (Stromskag; Chevallard, 2024) and *archedidactic transposition* (Artaud; Bourgade, 2022), which are explained in the next section.

Archeschool and archedidactic transposition

Figure 2 presents a reinterpretation and expansion of the didactic triangle proposed by Brousseau (1986) to model the phenomena related to the interactions between those who teach, those who learn, and knowledge within school activity. In this case, the rereading and expansion are characterized by what we denote as the expanded didactic triangle (EDT), which envisions the possibility of analyzing the didactic phenomena that can occur in any relationship between those who teach and those who learn in the different spaces where teaching and learning occur.

In the EDT structure (Figure 2), the steps refer to the “n” institutions of the spaces where something is learned and taught. The connections represented by the colored half-lines from the base of Figure 2 relate who teaches and who learns and reveal that through these interactions, knowledge is acquired through the praxeologies existing in the “n” institutions. But what happens if the praxeologies existing in some of these institutions are not sufficient to meet the needs of those who learn or those who teach?

[4] The process by which an element of knowledge k moves from one institution to another, with changes, is called the institutional transposition process (Chevallard & Bosch, 2020).

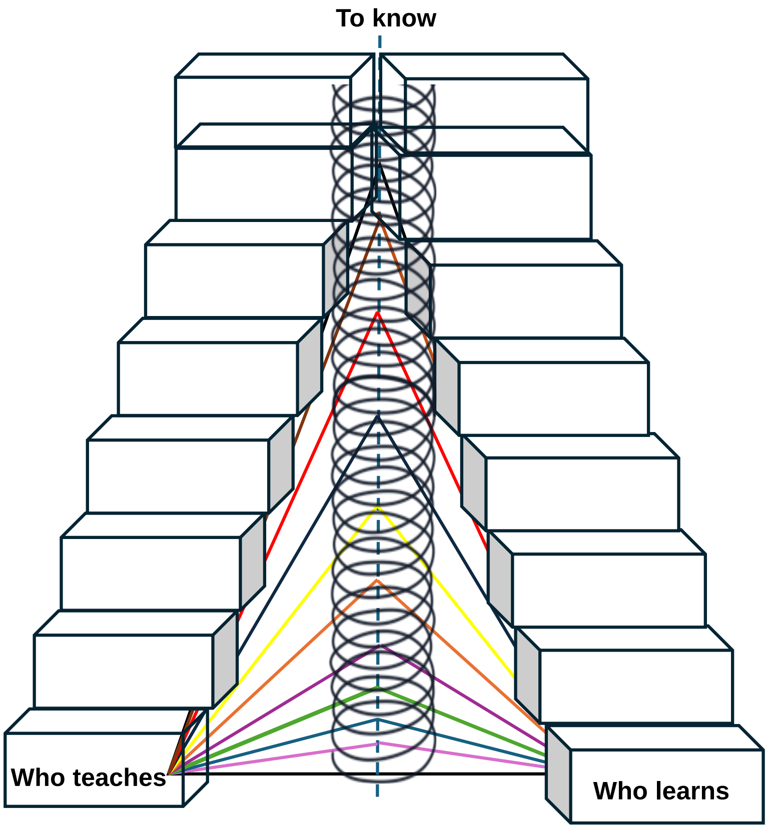

Figure 2.

Expanded didactic triangle (our own production)

Strthemskag and Chevallard (2024) state that it may happen that the actors who control an institution judge that an existing system or future activity in the institution requires meeting new praxeological needs, in which case the question is: What can the institution do under these conditions? The authors state that seeking knowledge in other institutions that may present epistemological legitimacy is necessary. Given this fact, we built the EDT structure, with the insertion of a spring in its central axis, coupled with the knowing and the center of the relationship between those who teach and those who learn. The function of the spring movement and its intensity must be determined by identifying the need for praxeological resources from other institutions.

The insertion of the spring in the EDT structure reflects the need for know-how in the context of the paradigm of questioning the world proposed by Chevallard (2009), whose oscillation of the spring opens space for mathematical paths in diverse mathematical and extra-mathematical contexts guided by study and research paths (SRPs) that can be applied in teacher education courses, thus enriching the praxeological equipment of those involved in those processes.

The EDT construct aims to establish a culture of permanent intercommunication between the institutions in which ***o*** is the vanishing point of relationships ***R(x, o)*** and $\boldsymbol{R_I(p, o)}$**.**

In this context, we realize the need to bring to the discussion the construct *archeschool* arising from the discrepancies pointed out by Strømskag and Chevallard (2024), which occur in the logos block of different institutions in the didactic transposition processes. This construct consists of a forum to expand discussions regarding the praxeological usefulness and epistemic legitimacy of the responses provided in the various teacher education institutions.

As a result, we must have answers to the following questions, which, according to Artaud and Bourgade (2022), were proposed by Chevallard in a lecture: "What should students-teachers study?" or, more precisely, "What should the praxeological equipment of the p_{wt} position be made of?" where p_{wt} is the student-teacher position.

These questions converge on another question: How can didactic organizations (DOs) be strengthened?

In this case, we infer that the EDT can also contribute to answering this question, since its structure contemplates the need for the emergence of the search for praxeologies necessary for the processes of didactic transposition to occur in a way that the "n" perspectives of knowledge are contemplated and legitimized, both from the perspective of mathematical organizations and from the perspective of didactic organizations, to strengthen personal relationships and expand the institutional relationships of the subject with knowledge in the position he/she occupies in a given institution.

Therefore, we can infer that the EDT also envisions archedidactic transposition processes, which, according to Artaud and Bourgade (2022), is a notion introduced to analyze the transposition of mathematics to a knowledge-producing institution. This notion refers to the development of a didactic infrastructure formed by the construction of models of didactic praxeologies accessible to the teacher in the position $\boldsymbol{p_t}$, teacher educator, to be studied in the position $\boldsymbol{p_{wt}}$ (student-teacher position), teaching degree students, or teachers in continuing education courses.

We can see the importance of the archedidactic transposition process when we turn our attention to the National Common Curriculum Base (Base Nacional Comum Curricular—BNCC), (Brasil, 2018). The BNCC brought several changes at the curriculum level, but research (Costin & Pontual, 2020; Jolandek, Pereira, & Mendes, 2021) reports that the questions that prevailed among teachers are: How can these changes be implemented? How can mathematical and didactic organizations be developed?

Given this scenario, we highlight that the absence of an archedidactic transposition process can be a source of difficulty in understanding most didactic problems. There is much discussion about students' difficulties, but there is an urgent need for studies addressing teachers' possible problems in perceiving students' disagreements and how to identify gaps in their praxeological equipment related to teaching infrastructure. In this case, we consider that the constructs archeschool and archedidactic transposition are essential in the context of discussions about the construction of a reference epistemological model (REM).

Based on this statement, we propose an expanded reference epistemological model (EREM) that includes the constructs mentioned above regarding the vanishing point (Figure 1) and the EDT (Figure 2). The proposed EREM was based on the following constructs: the archeschool and the archedidactic transposition.

Expanded Reference Epistemological Model (EREM)

The REM presents the epistemology and genesis of a given mathematical object based on its historical evolution and is free from influences from educational institutions. The REM can be presented as a reference praxeological model (RPM) consisting of praxeologies that play an important role in the analysis of the praxeologies that live in educational institutions. Florensa, Bosch, and Gáscon (2020) assert that the REM allows the researcher, without the influence of representatives of the school and school institutions, to propose alternative models for the knowledge to teach. From this perspective, the REM is fundamental in the analysis of transpositive didactic processes, the study of didactic phenomena, and the design of new study processes. However, these authors ask: "How can they be made available in the educational institution and to participants in the teaching process?" (Florensa, Bosch, & Gáscon, 2015, p.2640)

This question led us to propose the EREM, which expands the REM in the sense that the EREM highlights the importance of theoretical models in discussion spaces in teacher education courses to contribute to teacher education regarding mathematical and didactic knowledge. Thus, the EREM establishes a culture regarding the importance of knowing the epistemology and genesis of the mathematical objects to be studied.

From the perspective of DOs, Artaud and Bougarde (2022, p.145) emphasize that the teaching profession does not express the needs of didactics as knowledge and, consequently, there are no "didactic teachers similar to mathematical economists." This fact has consequences for the didactic transposition processes in institutions, such as the lack of questions regarding the teacher's praxeologies and how to expand their praxeological equipment to narrow the interactions between teacher-student and knowledge for the institutionalization of the knowledge in play.

The proposed EREM presents an expansion in the sense of, based on the EDT construct, acting as a driver for the confluence of an analysis model for two aspects:

a) The construct of *arch school aims* to create a discussion forum and expand the praxeological equipment of students in mathematics teaching degree courses based on answers to questions about the various “why” and “what for” regarding 𝓀, focusing on its praxeological utility and epistemic legitimacy.

b) The construct of *archedidactic transposition* aims to encourage discussions about a didactic infrastructure, with questions such as: How can we improve or develop didactic praxeologies for teaching and learning a specific object of knowledge?

Figure 3 presents the structure of the EREM, where we highlight that the study of the historical evolution of knowledge leads to the revelation of its genesis, which, in Figure 3, is presented in a region of a grid highlighting mathematics as knowledge that arises from praxeologies of various cultures, from various know-how in various institutions.

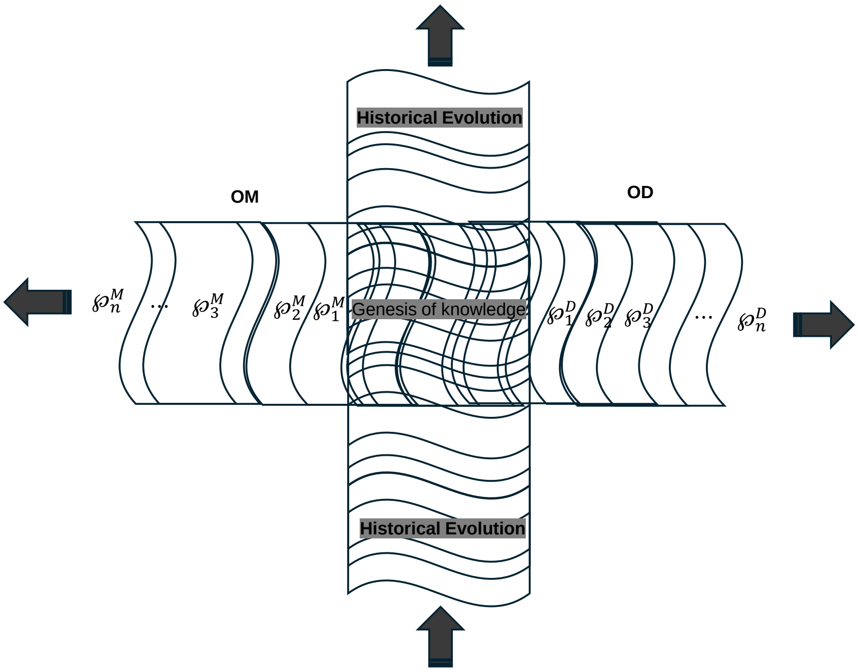

Figure 3.

EREM structure (our own production)

The construction of EREM highlights the importance of this fact being taken into consideration and serving as a reference for the study of mathematical knowledge from the perspective of the EDT, so there may be discussions about the expansion of mathematical praxeologies, $\wp_1^M, \wp_2^M, \wp_3^M, \cdots \wp_n^M$[5], related to mathematical organizations (MO) and didactic praxeologies, $\wp_1^D, \wp_2^D, \wp_3^D, \cdots \wp_n^D$[6], referring to didactic organizations (DO), represented in Figure 3, so that the EREM establishes a culture for those who teach and for those who learn in teacher education courses that points to SRPs represented by the arrows (Figure 3).

The creation of this culture may lead to several questions regarding the objects of knowledge that can contribute to answering questions regarding the didactic problems existing in institutions and that may exist due to the changes that have occurred in teaching spaces.

For Almouloud (2022), the denial of the need for scientifically based praxeological equipment constitutes one of the most important restrictions within the scope of a teacher education project since, in the teaching profession, the situation described by Chevallard (1997, apud Almouloud, 2022, p.42) is relevant:

> we must notice that no language is sufficiently rich and widely shared to allow an objective (and not simply personal) analysis of even the most common professional situations, with the consequence of a weak collective and individual capacity to communicate, debate, and even think about the objects of an activity that becomes easily trapped in the repetition of gestures and technical solipsism.

The aspects highlighted in this quote weigh heavily on teacher education and give rise to a series of difficulties that educators face, such as the difficulty of justifying to students the need to trust scientifically based tools" (Almouloud, 2022, p. 42). This author also asserts that, besides the complexity of "the issue at stake in the study of the education process, i.e., teaching praxeologies" (p.42), it is necessary to consider the "weak development of praxeologies of education and, therefore, the virtual absence of didactic infrastructures for teacher education" (p.42).

In the next section, we reflect on the construction of the EREM related to differential and integral calculus.

EREM - Differential and Integral Calculus

The key questions of the EREM are: a) What is the genesis of differential and integral calculus? b) What is the vanishing point of concepts related to functions, limits, derivatives, and integrals? c) What is the function of the EDT? d) Can the answers to these questions contribute to the differential and integral calculus teaching and learning process?

[5] M represents mathematics
[6] D represents didactic

We infer that the EREM for differential and integral calculus will contribute to accessing the answers to these questions since its foundation in the ATD implies turning our gaze to the epistemology of the differential and integral calculus to extract its essence and, consequently, its reason for being, which is directly related to the vanishing point and the EDT. In this way, we will have some answers to the previous questions.

When studying the epistemology of differential and integral calculus, we start from the conceptions of antiquity, which involve the thinking of the Egyptian and Babylonian peoples, who, according to Boyer (1949, pp.14-15), lacked scientific mathematical thinking, did not appreciate the characteristics that distinguish mathematics from science, its logical nature, and the need for deductive proofs. For these people, everything came down to numbers, which were related to their utilitarian needs. But it was they, through their empirical investigations, who revealed ways of doing mathematics and exploring mathematical objects, key elements inherent to the differential and integral calculus tools, and all the mathematics we have today.

Examples of their discoveries include the Egyptian rule for calculating the volume of a square pyramid and the rule for determining the volume of a square pyramid trunk, among others. But the result for the general case requires the use of infinitesimal considerations or limits, which, although constituting the starting point for the history of the derivative and the integral, are not found in any record before the Greek period.

Boyer (1949, pp.15-16) asserts that in Babylonian astronomy, problems involving variation were studied, but focusing only on the tabulation of values of a function (such as the brightness of the moon, for example) and of values of the argument (time) measured at equal intervals. Through this data, the maximum (intensity) of the function could be calculated.

Many results may have been obtained in this way, but their generalization involved the recognition of patterns that required tools. These tools became accessible, understandable, and accepted by society during the development of mathematics and today constitute differential and integral calculus tools.

Unlike the utilitarian vision of the Egyptians and Babylonians, which consisted of a set of techniques for measuring, counting, and calculating, Greek mathematics, according to Devlin (2002, p.8), stood out as an intellectual activity.

The Greeks were the first to systematically analyze the idea of continuous magnitude and develop concepts that lead to the integral and the derivative. Until then, mathematical objects were studied in their static form without the need to analyze phenomena that involved movement and change.

From the period of Greek mathematics, we highlight in this text Zeno's paradoxes, which, for Radice (1981, p. 44), aimed to reduce to absurdity the theses of the Pythagoreans, who had assumed that space and time can be thought of as points and instants, monads, which, for them, was indivisible. Zeno's paradoxes became necessary for the Greeks to renounce the attempt to give the phenomena of movement and variability a quantitative explanation. We can argue that Zeno's paradoxes showed that the key to studying motion and change is finding a way to describe the patterns involving infinity.

Devlin (2002, p.82) states that Zeno's paradox is only paradoxical if we think an infinite series must have an infinite value. The key to finding the value of a series is to shift attention from adding the individual terms to identifying and manipulating the overall pattern. In other words, this is the key to dealing with infinity in mathematics. However, the development of a logical theory on how to manipulate infinite series was completed by the end of the 19th century.

Strogatz (2019, p.42) assures that 200 years after Zeno reflected on the nature of space, time, movement, and infinity, Archimedes revealed the mathematical principles of balance and buoyancy in his laws on the lever and hydrostatics. However, this author states that (p.74) these laws are only aimed at situations of immobility.

Strogatz (2019, p.74) asserts that in the 17th century, Galileo and Kepler studied how things moved, and for this, a type of mathematics capable of dealing with movement at variable rates was necessary. For example, the analysis of why a ball rolls faster and faster as it slides down a ramp and why planets gain speed as they approach the Sun and lose speed as they move away from it.

Such analyses and new discoveries brought about the need to transpose what was studied in a static way to the study of situations that involved mathematical elements, subject to movement and change over time.

This transition from the study of what is static to what is transitory met with resistance of a scientific, philosophical, and theological nature.

The elements of calculus were being developed according to the need for research. In educational institutions, the study of calculus follows an order: from the study of differential calculus to the study of integral calculus.

However, from the perspective of historical development, this study happened in the reverse order since differential calculus emerged from algebra, and this took centuries to develop, i.e., to transpose from its original form in China, India, and the Islamic world, where it was verbal to a symbolic algebra (Strogatz, 2019, p.101).

The author also states that, around 1200 AC, this symbolic algebra was coupled with geometry, giving rise to analytical geometry, whose study of curves converged with the study of differential calculus.

Differentiation, as an operation for calculating the derivative, depends on the function concept. But what is the function? It can be said that a function is a ***standard*** of association between pairs of numbers: the independent variable or argument x and the dependent variable or value y.

Roque (2012, p. 344) states that, although in infinitesimal calculus courses, the definition of derivative is preceded by the sentence: “Let a function be y=f(x)”, the concept of function was only introduced after the improvement of differential techniques carried out by Leibniz and Newton. Thus,

> Until the advent of calculus, mathematics was a science of quantities. In the 17th century, work on curves related geometric quantities. From the 18th century onwards, many mathematicians began to consider that their main object was the function. Jaques Hadamard described this change: “The mathematical being, in a word, ceased to be the number: it became the law of variation, the function. Mathematics has not only been enriched by new methods but it has also been transformed into its object.” (Roque, 2012, p. 344)

In the case of differentiation, Newton's perspective involved a physics problem, the fluent function, and its derivative, which he called fluxion. Leibniz approached the issue as a geometric problem when studying the gradients of curves. Both investigated dynamic situations through successive approximations, observing numerical and geometric patterns. Newton and Leibniz arrived at the correct answer because they observed numerical and geometric patterns in this approximation process.

Devlin (2002, p. 93) observes that the fundamental patterns of differentiation are the same patterns that underlie the calculation of area and volume measurements; this is the inverse of differentiation, the foundation of integral calculus.

The advent of calculus provided an articulation between practically all mathematical objects, which, in their evolution, were considered static: numbers, points, lines, and equations, among others.

We can say that differential and integral calculus development results from the recognition of patterns that describe the study of movement and change. In this case, the vanishing point of the differential and integral calculus (Figure 4) refers to the study of movement and change (represented by the set of dashed ellipses that intersect, symbolizing the relationship between the various bodies of knowledge that constitute it), and it involves the knowledge of functions, limits, derivatives, and integrals as a set of techniques for manipulating patterns of infinity, the infinitely large and the infinitely small, present in patterns of movement and change.

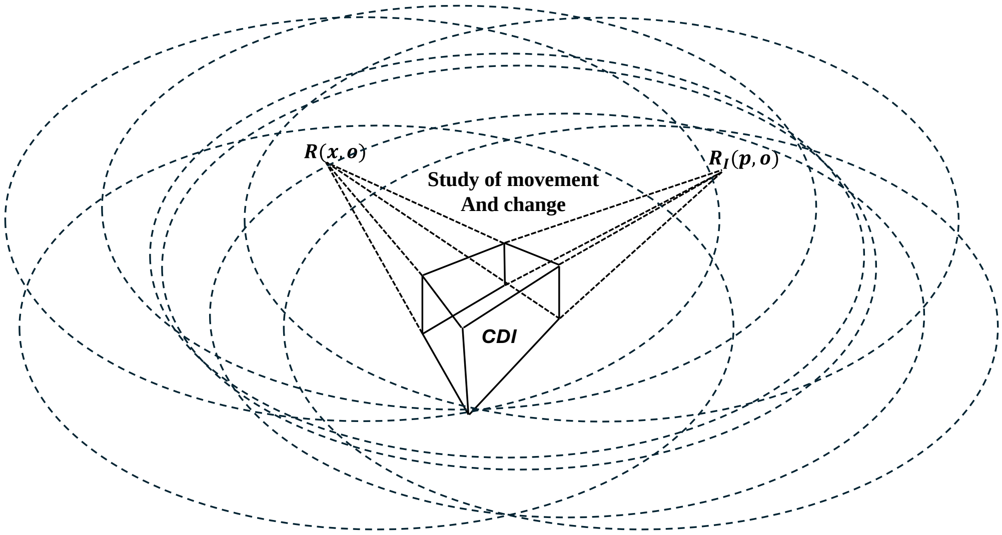

Figure 4:

Vanishing Point – Differential and Integral Calculus (our own production)

Once the vanishing point of relationships has been defined ***R(x, o)*** and ***R_I(p, o)*** from the EDT (Figure 5), it is possible to work on the different perspectives of differential and integral calculus based on its different reasons for being, i.e., based on different extra-mathematical contexts and its objects of knowledge (functions, limits, derivatives, and integrals).

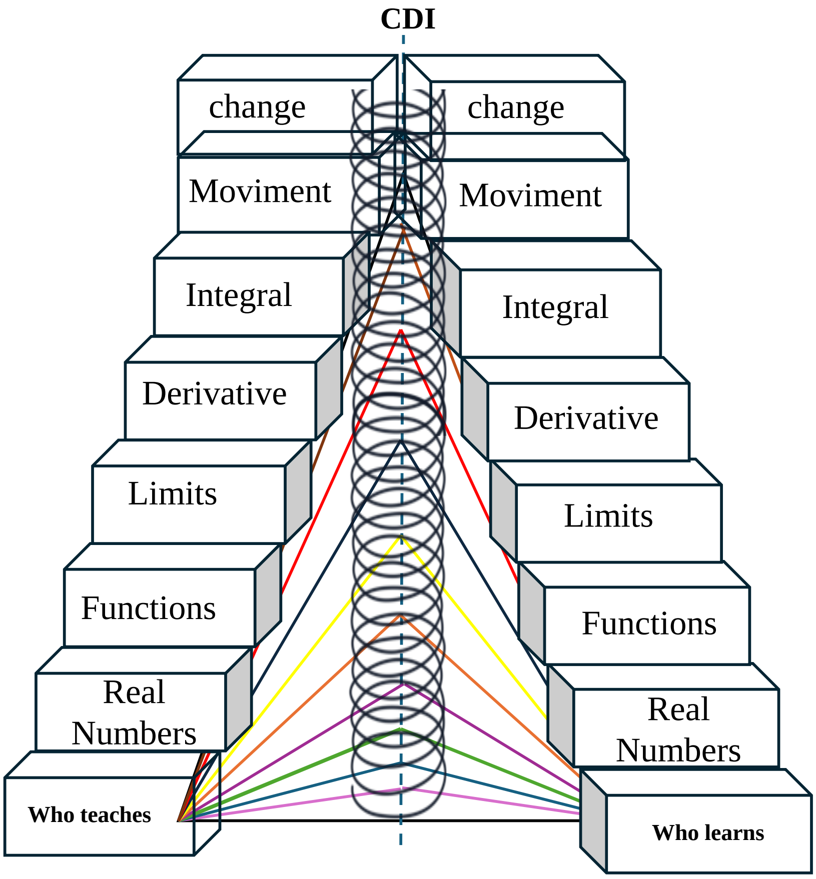

Figure 5.

EDT – Differential and Integral Calculus (our own production)

The EDT structure (Figure 5) recalls the need for connections between concepts and suggests the possibility of developing an SRP.

Figure 6 presents a diagram of the EREM for teaching differential and integral calculus based on its vanishing point, study of movement and change, and the perspective of the development of mathematical praxeologies $\wp_1^M, \wp_2^M, \wp_3^M, \cdots \wp_n^M$ and didactic praxeologies $\wp_1^D, \wp_2^D, \wp_3^D, \cdots \wp_n^D$ according to the vanishing point.

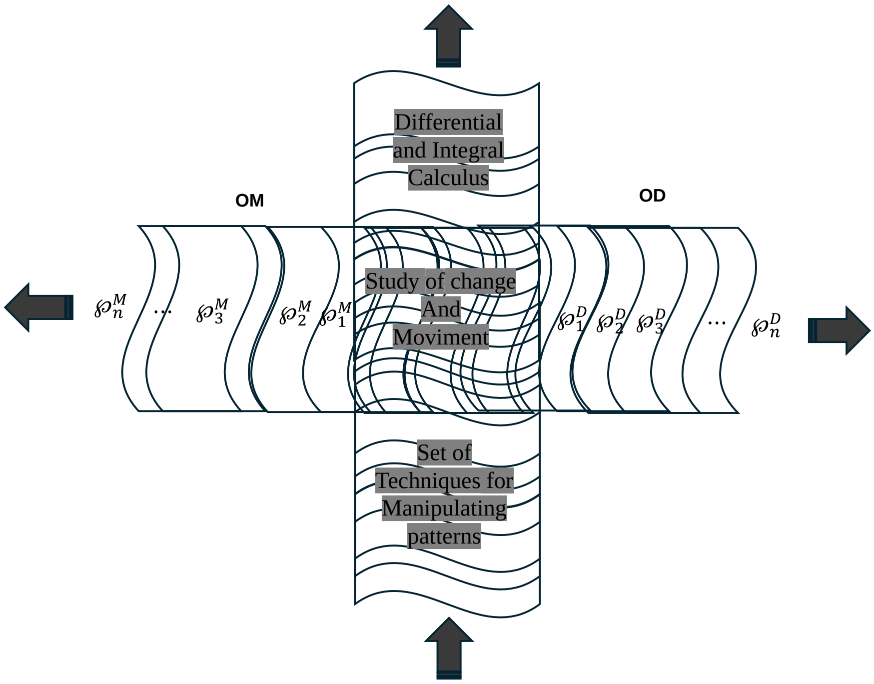

Figure 6.

EREM– Differential and Integral Calculus (our own production)

To exemplify the application of the EREM we will discuss the fundamental theorem of calculus (FTC), proposing suggestions for possible mathematical praxeologies, $\wp_1^M, \wp_2^M, \wp_3^M, \cdots \wp_n^M$ referring to mathematical organizations (MOs) and didactic praxeologies, $\wp_1^D, \wp_2^D, \wp_3^D, \cdots \wp_n^D$, referring to didactic organizations (DOs) aligned with the FTC vanishing point, the study of movement and change.

Fundamental Theorem of Calculus (FTC)

In calculus textbooks (Simmons, 1987, p.281; Leithold, 1982, p. 258; Thomas, 2005, p.358), the FTC is stated as follows:

If f is continuous in $[a, b]$,, the function $F(x) = \int_a^x f(t)dt$ is derivable at every point x in $[a, b]$ and, $\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$, (Thomas, 2005, p. 358)

According to Strogatz (2022, p. 187), students solve so many integrals with the help of FTC, but they have no idea of the importance and application of this theorem. Furthermore, the author compares this fact to a joke about the fish that asks his friend:

> "Aren't you grateful for water?" The other fish replies: "What is water?" as calculus students always swim in the fundamental theorem. Therefore, they do not give it the value it deserves. (Strogatz, 2022, p. 187)

The EREM proposal aims to contribute to changing the reality stated by Strogatz (2022, p.187) based on its constituent elements: *vanishing point and expanded didactic triangle*, which involve the constructs of *archeschool* and *archedidactic transposition*.

To explain these constructs with a focus on the FTC, the development of the EREM presents a reinterpretation of the analysis proposed by Strogatz (2022, pp. 185-202) based on the ATD, focusing on a geometric analysis of the FTC that involves an articulation between two types of problems: a direct problem and an inverse problem.

Groetsch (1999, p.1) states that direct problems are often worked on in undergraduate courses. These problems can be characterized as those in which the student is provided with sufficient information to carry out a well-defined, stable process that leads to a unique result. In science, the process is often called a model, with the input labeled as the cause and the output as the effect.

The direct problem suggests two types of inverse problems, i.e., a) the causality problem, given a model and an effect, find the cause of the effect; b) the model identification problem, given the cause-effect information, identify the model (Groetsch, 1999, p. 3).

In the study of derivatives, the following type of direct problem is usual:

Given a curve *y*, determine the slope of the tangent line to the curve *y* at a given point.

This direct problem is the task type (*T*) in calculus books. We can cite as an example task t, from Thomas's (2005, p. 150) book

t: Find an equation for the tangent line to the equation curve $y = x^3 - 4x + 1$ at point (2, 1).

This task is subdivided into two subtasks:

t_1: Determine the derivative function

τ_1: Calculus by definition

θ_1: Theory on limits of functions

t_2: Determine the angular coefficient of the tangent line

τ_2: Calculation of the formation law of the derivative function at a given point

θ_2: Theory about the value of a function at a given point

Therefore, the direct problem results from one of the applications on derivatives of functions.

However, from the point of view of the development of the history of mathematics, from Newton's perspective, the FTC corresponds to solving an inverse problem that involves area measurements from a dynamic perspective. The question is: What is the importance of determining these area measurements?

From this perspective, according to the EREM structure, some questions arise regarding the didactic organization Q_i^D and mathematics Q_i^M referring to the view under the study of FTC in undergraduate courses, whether in mathematics teaching degree courses, as well as in other courses, i.e.:

Q_1^D : Is the FTC proposed in calculus books as a task that explains its reason for being?

Q_1^M: How to propose this type of task when approaching an inverse problem?

To answer the question Q_1^D, we searched the following calculus textbooks: Simmons (1987, p. 278-283); Anton (2005, p. 363), Leithold (1982, p. 256-259).

These books present the FTC, its mathematical proof, and direct application exercises. However, the authors do not comment on its reason for being.

Given this perspective, what is at stake in EREM is the *vanishing point* from the study of calculus (Figure 4), i.e., the study of problems involving patterns of movement and change. The EREM for calculus proposes that the study of movement and change be instituted as a link between personal and institutional relationships in a process of didactic transposition so that the tasks proposed for teaching calculus exist around its vanishing point to contribute to the institutionalization of its reason for being for all those involved in its study.

To answer the question Q_1^M, we present the task proposal from the perspective of the inverse problem to be studied in calculus classes.

According to Strogatz (2022), from Newton's perspective, the direct and inverse problems consist respectively of:

> a) Given the fluents, how do we find their fluxions? Or: Find the slope of a given tangent line to a curve or find the rate of change or derivative of a known function.
> b) Given the fluxions, how do we find their fluents? How can a curve be inferred from its slope or an unknown function from its rate of change? (Strogatz, 2022, p. 202)

Newton's proposition b) is explicit in calculus textbooks through tasks of the type T_1: Let be $\frac{dA}{dx} = y$, what is y?

The EREM proposes the reinterpretation of T_1 to make the reason for FTC's existence explicit since the EDT aims to institutionalize knowledge based on a set of previously learned knowledge so that relationships are established between them, which are revisited according to the need to advance to another level in the process of knowledge expansion that encompasses previous knowledge. Because of this, the spring propels the revisiting of previous knowledge based on the relationships between teacher, student, and knowledge, according to the need for new knowledge to be established.

According to the EREM proposal, for this to happen effectively, we require the construct of *archeschool*, which establishes the need for questions pertinent to the study of a given object of knowledge among those who teach something to someone, such as: What is the reason for the FRC? Why and for what purpose should we study the fundamental theorem of calculus?

After answering these questions, the ideal fundamental situation (Brousseau,1990) involves examining an expanded didactic triangle (Figure 5) to establish an SRP that contributes to obtaining answers that explain the praxeological usefulness and epistemic legitimacy of the FTC.

Given these constituent elements of the EREM, the proposed task for the study of the EDT, which involves the study of an inverse problem, is the proposal of mathematical praxeologies $\wp_1^M, \wp_2^M, \wp_3^M, \cdots \wp_n^M$ associated with area measurement problems, and from the perspective of the *archedidactic transposition*, didactic praxeologies are suggested $\wp_1^D, \wp_2^D, \wp_3^D, \cdots \wp_n^D$, which can be discussed among professors and researchers working in higher education institutions.

Below is an example based on the proposition and analysis of tasks related to mathematical praxeologies (t_i^M) and tasks related to didactic praxeologies (t_i^D) in which $i = 1, 2, \cdots$

t_1^D: How to introduce the FTC from an inverse problem associated with a problem involving area measurement?

This type of task requires the technique (τ) of analysis and research, based on technology (θ) proposed by the EREM, i.e., the epistemology of the object under study, which establishes the culture of a research professor who lives in a process of study and research and who disseminates this culture in his personal and institutional relationships.

The following tasks result from this process of research and study in the history of mathematics books, such as Devlin's (2002, pp. 102-103) and Strogatz's (2022, pp. 193-198) books.

t_1^M: Observe the graph (Figure 7), obtained by sketching a curve with the equation y=*f*(*x*) and the region of area *A(x)*, delimited above by this curve, as well as imagining that this region expands to the right. In other words, imagine the movement of the dotted line to the right and, consequently, the change in position of *x* along the abscissa axis. Can we denote *A(x)* as an area function?

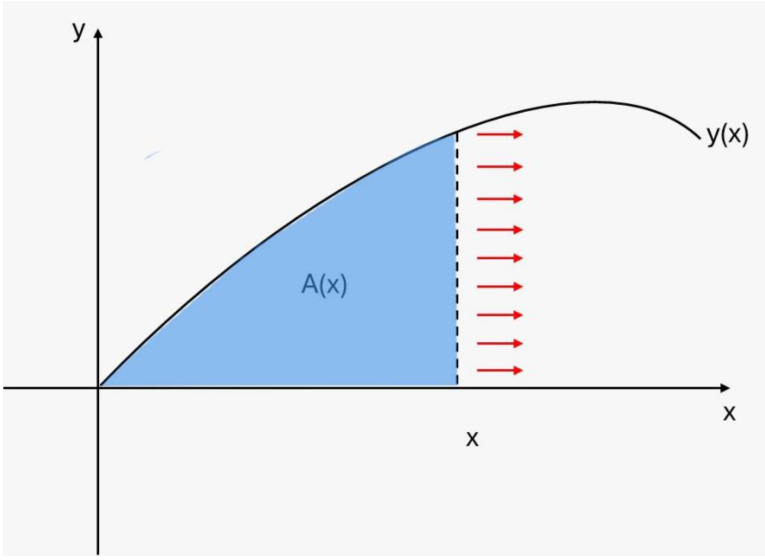

Figure 7.

Graph of y=f(x) (our own production)

This task requires the technique of observation and recognition of calculus elements present in the *EDT*: real numbers, functions, limits, derivatives, integrals, and, amid all these elements, the abstraction that the articulation between all of them is the study of movement and change, based on technology (θ) established by the EREM, i.e., for the student to develop this ability of observation and abstraction, the study of these elements must have a reason for being, which does not depend solely on algebraic calculations.

The answer to task question t_1^M results from the observation that for any point x, there will be a corresponding area. Perhaps the question arises: But who is A(x) at the level of algebraic registration?

t_2^M: Discuss what mathematical tool can describe the instantaneous rate of expansion of area as *x* moves to the right. Or, what is the rate at which this area under the curve expands for a given x-position?

The development of this task depends on the result of the previous task. It is essential for discussion and reflection between students and the teacher about the knowledge at stake.

Ideally, students should identify the derivative as the mathematical tool that allows them to describe the instantaneous rate of expansion of the area as *x* moves to the right.

t_3^M: Show through a sketch how to obtain this instantaneous expansion rate of the area.

This task is challenging, requiring students' imagination and abstraction skills.

Research, such as that by Nurcahyono et al. (2019) and Ibrahim et al. (2024), indicates that the study of mathematics requires these characteristics since imagination was necessary to create ideas and prove them for the development of the history of mathematics. Furthermore, the authors reinforce the importance of analyzing the existence of a relationship between ideas formed from empirical experiences and those formed using imagination.

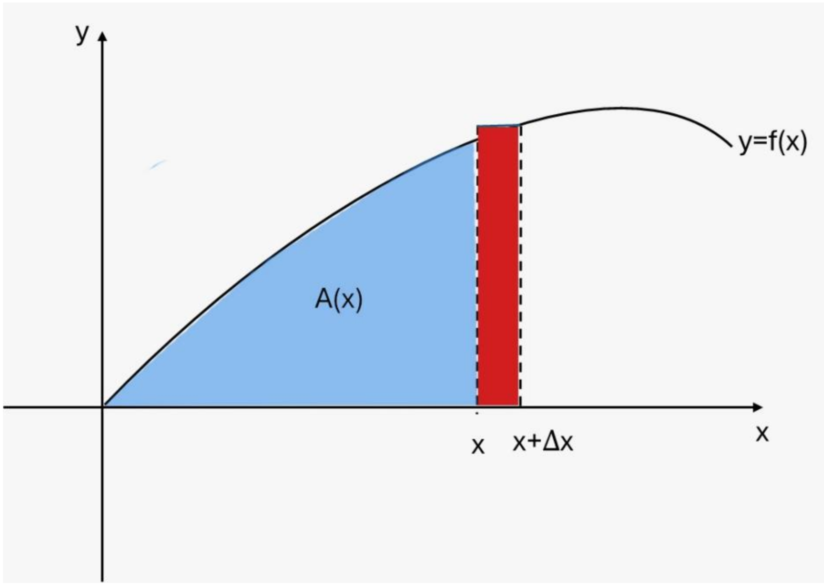

Figure 8.

Expansion of A(x) (our own production)

Figure 8 presents a technique (τ) that favors carrying out the task t_3^M, i.e., the graph sketch, whose technology (θ) involves the concept and understanding of the instantaneous rate of change based on a geometric perspective.

t_4^M: Express this rate mathematically.

The technique needed to perform this task comes from the answers obtained when performing the previous tasks (t_1^M, t_2^M, t_3^M), and the ability to express oneself mathematically, from the perspective of algebra.

In "Metasynthesis of research on the role of language in didactics of mathematics," Almouloud and Figueroa (2021, p.284) state that:

> We agree with Perrin-Glorian and Bosch (2013) when they observe that speech, like symbols, graphs, or gestures, is part of the ostensive tools of mathematical activity. As such, they require specific distribution, use, and maintenance conditions. They add that we must learn to verbalize symbolic writings to be able to comment on and organize them; use, choose, or invent appropriate words; produce specific discourses; articulate the ostensive aspects of the different registers; reduce and restore the ostensive thickness of praxeology, etc.

Thus, the objective is that students, based on previous tasks, have observed the variation of measurement A(x) of the area A of the region under the graph of the function when x moves from a distance Δx. From this change we have a new area measurement $A(x + \Delta x)$, written algebraically as follows:

$$A(x + \Delta x) = A(x) + f(x)\Delta x$$

By manipulating this equation algebraically to determine the expansion rate, we obtain:

$$\frac{A(x + \Delta x) - A(x)}{\Delta x} = f(x)$$

However, this equality refers to a "rough" approximation, hence the need for the next task to reflect on this fact.

t_5^M: As Δx approaches zero, what does this rate mean?

Performing this task involves analyzing function limits. Therefore, for equality:

$$\frac{A(x + \Delta x) - A(x)}{\Delta x} = f(x)$$

to be accurate, an analysis at the function limit level is necessary to Δx tending to zero. Hence,

$$\lim_{\Delta x \to 0} \frac{A(x + \Delta x) - A(x)}{\Delta x} = A'(x) = f(x)$$

Could completing this task lead to questions as to why A'(x) is equal to f(x)? Something may have been lost in the process of solving the task. Therefore, it is fundamental to ensure this fact.

The answer to the previous question depends on an analysis of what happens close to the infinitesimal interval as *x* changes to x+Δx, considering this distance Δx tending to zero, i.e., infinitesimal, therefore its length, given by *f(x)*, remains almost constant. In this case, the measure of the infinitesimal area is *dA=f(x)dx*.

We notice that tasks $t_1^M, t_2^M, t_3^M, t_4^M, t_5^M$ were planned from the perspective of the *EDT* involving the *archeschool* and the *archedidactic transposition*, which consists of a spring, whose oscillation and intensity of movement is given by the need for praxeological resources from other institutions.

This work intends that students and teachers discuss this proposal to reveal more elements for the EREM, thus expanding their praxeological equipment in didactic transposition processes.

In the books researched (Simmons, 1987; Anton, 2005; Leithold, 1982; Tan, 2004), we observed that the reason for being of the FTC is linked to the process of transposing wise knowledge to the knowledge established by textbooks, without it being an object of study.

t_2^D: How to institutionalize the FTC concept?

The tasks presented so far have led to an important conclusion: that *A'(x) = f(x).*

Through this result, the FTC was studied based on determining the formation law of the function *A(x),* i.e., find a function whose derivative is *f(x)*. However, that needs a search for more abstract patterns involving derivatives of functions and calculation of area measurements.

t_5^M:How important it is to determine the area of the region under an arbitrary curve? What does this area mean?

This task is challenging because it requires students to abstract from analysis of mathematics through mathematics. Area measurements that change at a rate of variation and/or expansion refer to area measurements that accumulate as a function of the variation in the x positions on the abscissa axis, as observed in task t_3^M (Figure 8).

The direct and inverse problems, i.e., determining slopes of tangent lines and area measurements, are important problems for any area of science since each area involves problems that require determining rates of change and the way to determine these accumulations of changes, which can be indicators for process control around project planning and development.

The purpose of the EREM is to contribute to developing praxeologies that focus on establishing the reason for being of the knowledge at stake.

We suggest that, based on the institutionalization of the FTC, problems of the type proposed by Strogatz (2022, pp. 201-202) be explored, such as problems about the fluctuation of the inflow of resources into a bank account and the accumulated balance in it; problems about the growth rate of the world population and the number of people on Earth, problems about the variable concentration of a chemotherapy drug in a patient's blood and the accumulated exposure to this drug over time. These are problems that involve future predictions in a world that operates in movement and change over time.

Final Considerations

The REM is fundamental for researchers in didactics of mathematics and should be known by teacher educators to increase discussions on epistemology and the genesis of knowledge.

We infer that the idea of the vanishing point highlights the importance of the narrowing between ***R(x, o)*** and ***R_I (p, o)***, which allows for increased discussions that turn institutions into spaces for debate, using reference epistemological models (REM) as a reference.

In our view, EREM expands the spaces for these discussions based on EDT, which considers teaching through relationships ***R(x, o)*** and ***R_I(p, o)***. These relationships influence science practice with its countless questions and destabilizations. As we question them, we uncover other perspectives, frameworks, records, cultures, and possible connections required in a world full of unexpected catastrophes and continuous technological evolutions.

However, for this to occur, based on the research evidence, the importance of the EREM being a reference for constituting and instituting in teaching spaces the constructs of *archeschool* and *archedidactic transposition*.

This need is related to expanding the praxeological equipment of teacher educators and pre-service teachers.

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