EMP Educação Matemática Pesquisa
ISSN 1983-3156
Qualis A1
http://dx.doi.org/10.23925/1983-3156.2025v27i1p155-185

Implications of mathematical knowledge for teaching using mathematical modelling: a look at the discussions of teachers in continuing education

Implicaciones del conocimiento matemático para la enseñanza utilizando la modelación matemática: una mirada a las discusiones de los profesores en formación continua

Implications des connaissances mathématiques pour l'enseignement utilisant la modélisation mathématique : un regard sur les discussions des enseignants en formation continue

Implicações do conhecimento matemático para o ensino usando modelagem matemática: um olhar para as discussões de professores em formação continuada

Karina Alessandra Pessoa da Silva[1]
Federal University of Technology - Paraná
Doctorate in Science Teaching and Mathematics Education
http://orcid.org/0000-0002-1766-137X

Emerson Tortola[2]
Federal University of Technology - Paraná
Doctorate in Science Teaching and Mathematics Education
https://orcid.org/0000-0002-6716-3635

Abstract

In this paper we bring reflections to the question What implications of mathematical knowledge for teaching using mathematical modelling emerge from discussions of teachers in continuing education? To this end, we focus on the discussions of teachers in the role of modellers in a mathematical modelling activity. The Content Analysis, of a qualitative nature, was supported by audio and video recordings of nine teachers who participated in a Mathematical Modelling course in a professional master's program in Mathematics Teaching at a Brazilian federal university in 2022. The results indicated that the implications of teaching using modelling were mainly evidenced in what we call pedagogical discussions, and mathematical knowledge for teaching was mainly revealed in the mathematical discussions carried out jointly between the teacher educator and the teachers in continuing education. The intentional questions undertaken by the teacher educator were the key to elucidate the specialized content knowledge, provoking implications regarding the anticipation of mathematical content that could be employed in

[1] karinasilva@utfpr.edu.br
[2] emersontortola@utfpr.edu.br

practices when teaching using modelling. These results elucidated the importance of sharing ideas among teachers in continuing education so that, in collaboration, improvements in knowledge are present when discussing a modelling activity within the scope of continuing teacher education.

Keywords: Mathematical knowledge for teaching, Mathematical modelling, Mathematical model, Second degree polynomial function, Function defined by two sentences.

Resumen

En este artículo aportamos reflexiones sobre la pregunta ¿Qué implicaciones del conocimiento matemático para la enseñanza mediante modelación matemática emergen de las discusiones de los docentes en la educación continua? Para hacer esto, analizamos las discusiones de los profesores sobre el papel de los modeladores en una actividad de modelación matemática. El Análisis de Contenido, de naturaleza cualitativa, fue respaldado por grabaciones de audio y video de nueve profesores que participaron en una disciplina de Modelación Matemática de un programa de maestría profesional en Enseñanza de Matemáticas de una universidad federal brasileña, en el año 2022. Los resultados indican que las implicaciones de la enseñanza usando modelación fueron evidenciadas, principalmente, en lo que llamamos discusiones pedagógicas, y el conocimiento matemático para la enseñanza se reveló, principalmente, en discusiones matemáticas llevadas a cabo de manera conjunta entre la profesora formadora y los profesores en formación. Las preguntas intencionales emprendidas por la profesora formadora fueron la clave para elucidar el conocimiento especializado del contenido, provocando implicaciones con respecto a la anticipación de contenidos matemáticos que podrían ser empleados en prácticas al enseñar usando modelación. Estos resultados elucidan la importancia del intercambio de ideas entre los profesores en formación continua para que, en colaboración, mejoras en los conocimientos estén presentes al discutir una actividad de modelación en el ámbito de la formación continua de profesores.

Palabras clave: Conocimientos matemáticos para la enseñanza, Modelación matemática, Modelo matemático, Función polinómica de segundo grado, Función definida por dos oraciones.

Résumé

Dans cet article, nous apportons des réflexions à la question Quelles implications des connaissances mathématiques pour l’enseignement utilisant la modélisation mathématique émergent des discussions des enseignants de la formation continue ? Pour ce faire, nous avons regardé les discussions des enseignants sur le rôle des modélisateurs dans une activité de modélisation mathématique. L'Analyse de Contenu, de nature qualitative, a été soutenue par des enregistrements audio et vidéo de neuf enseignants ayant participé à une discipline de Modélisation Mathématique d'un programme de master professionnel en Enseignement des Mathématiques dans une université fédérale brésilienne, en 2022. Les résultats indiquent que les implications de l'enseignement en utilisant la modélisation ont été principalement mises en évidence dans ce que nous appelons des discussions pédagogiques, et que les connaissances mathématiques pour l'enseignement se sont révélées principalement dans des discussions mathématiques menées de manière conjointe entre la formatrice et les enseignants en formation. Les questions intentionnelles posées par la formatrice ont été la clé pour élucider les connaissances spécialisées du contenu, provoquant des implications concernant l'anticipation des contenus mathématiques qui pourraient être abordés dans les pratiques d'enseignement à travers la modélisation. Ces résultats ont mis en lumière l'importance du partage d'idées entre les enseignants en formation continue afin que, en collaboration, des améliorations des connaissances soient présentes lorsqu'ils discutent d'une activité de modélisation dans le cadre de la formation continue des enseignants.

Mots-clés: Connaissances mathématiques pour l'enseignement, Modélisation mathématique, Modèle mathématique, Fonction polynomiale du deuxième degré, Fonction définie par deux phrases.

Resumo

Neste artigo trazemos reflexões para a questão: Quais as implicações do conhecimento matemático para o ensino usando modelagem matemática emergem das discussões de professores em formação continuada? Para isso, nos debruçamos nas discussões dos professores no papel de modeladores em uma atividade de modelagem matemática. A Análise de Conteúdo, de natureza qualitativa, foi subsidiada por gravações em áudio e vídeo de nove professores participantes de uma disciplina de Modelagem Matemática de um programa de mestrado profissional em Ensino de Matemática de uma universidade federal brasileira, no ano de 2022. Os resultados indicam que as implicações no ensino usando modelagem foram evidenciadas, principalmente, no que chamamos de discussões pedagógicas, e o conhecimento matemático para o ensino se revelou, principalmente, nas discussões matemáticas procedidas de forma conjunta entre a professora formadora e os professores em formação. Os questionamentos intencionais empreendidos pela professora formadora foram o mote para elucidar o conhecimento especializado do conteúdo, provocando implicações com relação à antecipação de conteúdos matemáticos que poderiam ser empreendidos em práticas ao se ensinar usando modelagem. Esses resultados elucidaram a importância do compartilhamento de ideias entre os professores em formação continuada para que, em colaboração, aprimoramentos nos conhecimentos se façam presentes ao se discutir uma atividade de modelagem no âmbito da formação continuada de professores.

Palavras-chave: Conhecimento matemático para o ensino, Modelagem matemática, Modelo matemático, Função polinomial do segundo grau, Função definida por duas sentenças.

Implications of mathematical knowledge for teaching using mathematical modelling: a look at the discussions of teachers in continuing education

Research and theorization on teacher education date back to the last century when theories on knowledge, learning, motivation, curriculum, and assessment began to be disseminated. These theories, focused on students or teaching resources, were encouraged by the view of teachers as professionals who reflect, think, and need to construct their own practice. Thus, it becomes relevant for teacher education programs to "understand what and how teachers think and know, and especially how they act" (Curi & Pires, 2008, p. 153).

Becker (2012), for example, discusses the implications for teaching based on how teachers conceive mathematical knowledge. Shulman (2004) argues that teaching is one of the most complex and challenging activities created by humanity and dedicated himself to investigating how teachers can address such difficulties. Regarding teacher knowledge, Shulman (1986) structured it into three kinds: subject matter content knowledge, pedagogical content knowledge, and curricular knowledge. In general terms, the first refers to knowledge specific to their field, the second to how they manage the teaching of content and classroom situations, and the third to curricular programs and the means by which they can be addressed.

With the emergence and dissemination of new theories and pedagogical alternatives for teaching Mathematics, teacher education, as indicated by Curi and Pires (2008), needs to establish itself as a research field within education. Regarding mathematical modelling, teacher education has been a concern for educators and researchers from different perspectives, who work toward integrating modelling into school settings and recognize its contributions to shaping students into more critical, creative, and reflective individuals regarding the use of Mathematics and its applications (Maaß et al., 2022).

The dynamics of approaching Mathematics through modelling involve organizing problem situations presented to students as an invitation to investigation; providing guidance for discussions that assist in understanding the problem context and the intended mathematical exploration; offering support in problem-solving by systematizing concepts based on students' discussions; and encouraging mathematical communication. This approach, therefore, requires teachers to master content and, above all, understand how to teach it, as Shulman had already highlighted in the 1960s regarding a change in focus present in some teacher education programs from "what is taught" to "how it is taught" (Curi & Pires, 2008).

To discuss which formative aspects shape the pedagogical practices of teachers engaged in continuing education in modelling, Mutti and Klüber (2018, p. 104) reveal a multidimensional movement that includes "traces of these teachers' experiences as students, their experiences during their undergraduate studies, and [...] those associated with teaching practice, readings, and peer support."

These aspects align with the three formative axes proposed by Almeida and Silva (2015), who consider, in a teacher education context, the need to learn about modelling, learn through modelling, and teach using modelling. The authors propose that, in addition to providing theoretical knowledge about modelling and its development, teachers in education should be encouraged to develop modelling practices in the classroom.

This approach considers different aspects of teacher knowledge, as does Shulman (1986). Thus, examining modelling from this theoretical framework can provide insights into what should be considered in teacher education in this context. In particular, this paper focuses on discussions held by teachers in education as they learn through modelling, that is, while developing a modelling activity as modellers but also debating the mathematical content that emerges during problem-solving to consider what can be taught using modelling.

To this end, we analyze a mathematical modelling activity carried out by nine Basic Education teachers, working in the final years of Elementary School and in High School, who attended, in the first semester of 2022, a course on Mathematical Modelling from a Teaching Perspective as part of a master's program in Mathematics Teaching at a federal university in southern Brazil.

This is a qualitative study conducted in an educational context, in which data were produced, as guided by Lüdke and André (2014), from an investigative curiosity sparked by a problem revealed in educational practice, in which the researchers are immersed and actively involved: What implications of mathematical knowledge for teaching using mathematical modelling emerge from the discussions of teachers in continuing education?

In this context, we begin the text with an exploration of mathematical knowledge for teaching, followed by an examination of teacher education in mathematical modelling. Subsequently, we discuss the research context and analysis methodology. Next, we present a description of the problem-solving process and an analysis of teachers' discussions. We conclude with our final considerations and references.

Mathematical knowledge for teaching

We understand that the construction of knowledge in the educational field is largely supported by teachers' actions and by the way they introduce and develop content with students. In this sense, the approaches taken in the classroom depend on teachers' knowledge.

In an effort to understand how teachers' knowledge is developed and how new knowledge is integrated with pre-existing knowledge, Shulman (1986) structured the knowledge base for teaching into three categories: subject knowledge matter, pedagogical knowledge matter, and curricular knowledge.

Subject matter content knowledge, or content knowledge, refers to what the teacher understands about the structure of the discipline and the knowledge that will be taught, in relation to its production, representation, and epistemological validation. This requires teachers to understand the structure of the discipline in terms of the attitudinal, conceptual, procedural, representational, and validation domains of content. However, as Shulman (1986, p. 8) asserts, this knowledge "[...] is likely to be as useless pedagogically as content-free skill." According to the author, an effort must be made to bring into the teacher's practice not only content knowledge but also its connection to a didactic dimension, thus enabling the transformation of content into pedagogically powerful forms, referred to as pedagogical content knowledge.

Pedagogical content knowledge includes teachers' knowledge of their students and their characteristics, knowledge of educational contexts, and knowledge of educational purposes, values, and philosophical and historical foundations. To this end, it is necessary to employ examples, analogies, illustrations, and demonstrations to assist in the learning of a particular content, while also considering students' possible conceptual errors and their implications for learning, that is, "the ways of representing and formulating the subject that make it comprehensible to others" (Shulman, 1986, p. 9).

Curricular knowledge, in turn, refers to knowledge of teaching programs, emphasizing the use of teaching resources, the relationship between content and contexts presents in the curriculum, and the connections among different topics within the same subject that were studied in previous years or will be studied later. It is important to highlight that "a professional teacher to be familiar with the curriculum materials under study by his or her students in other subjects they are studying at the same time" (Shulman, 1986, p. 10).

The categories established by Shulman (1986) were expanded in later studies (Shulman, 1987) and serve as the foundation of the knowledge base for teaching in any field of study. Based on Shulman's framework (1986, 1987), some studies have specifically focused on teachers' knowledge for teaching Mathematics (Hill et al., 2005; Ball et al., 2008), leading to the concept of Mathematical Knowledge for Teaching (MKT). According to Ball et al. (2008), it is essential for teachers to have knowledge of Mathematics that is directly related to the act of teaching.

In this regard, teachers must be familiar with the content they teach in a way that helps students learn. According to Ball et al. (2008), Content Knowledge (CK) involves understanding mathematical concepts and their definitions, the sequencing of mathematical concepts, proofs, demonstrations, and approaches to developing and generating mathematical knowledge. However, the authors emphasize that knowing the content alone is not sufficient for teaching it.

Thus, regarding Pedagogical Content Knowledge (PCK), Ball et al. (2008) consider aspects related to learning Mathematics, such as students' conceptions and what might be challenging or interesting for them. Additionally, they consider teachers' knowledge in planning lessons, in conjunction with students' prior knowledge, errors and difficulties, different representations of mathematical objects, and the characteristics of teaching tasks.

In this way, Mathematical Knowledge for Teaching is based on the professional practice of Mathematics teachers and can be seen as a redefinition of the knowledge these teachers need to develop for their work as educators. Through investigations into the classroom practices of teachers who teach Mathematics, Ball et al. (2008) subdivided CK and PCK into six domains of mathematical knowledge for teaching: common content knowledge, horizon content knowledge, specialized content knowledge, knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum.

According to Ball et al. (2008), common content knowledge refers to general mathematical knowledge and skills; horizon content knowledge is related to the interconnections among terms or topics across the entire curriculum; specialized content knowledge pertains to the mathematical skills and knowledge specific to a teacher's work, characteristic of their pedagogical practice; knowledge of content and students combines knowledge of students and Mathematics; knowledge of content and teaching integrates knowledge about content and the teaching of that content; and knowledge of content and curriculum encompasses educational goals, standards, assessments, and the grade levels in which certain topics are typically taught.

These six domains were organized into an illustrative framework, as represented in Figure 1.

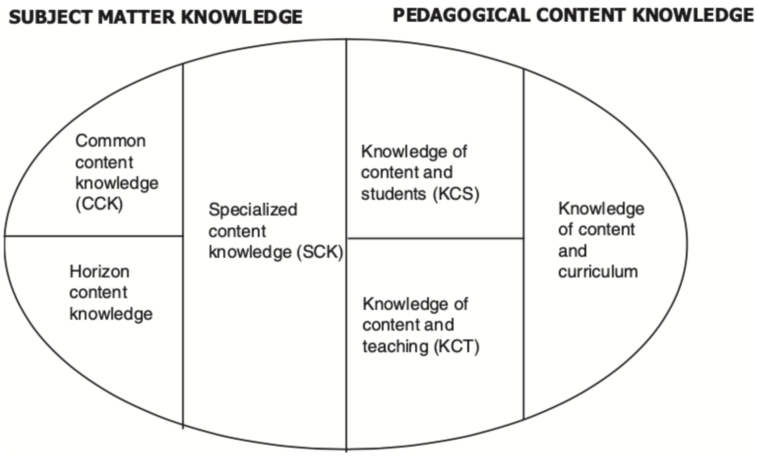


Figure 1.

Domains of Mathematical Knowledge for Teaching (Ball et al., 2008, p. 403)

In light of these domains, we highlight the multiplicity of knowledge required for Mathematics teachers to implement in their teaching practices, given that the focus lies on understanding “how teachers reason about and deploy mathematical ideas in their work” (Ball et al., 2008, p. 403). To achieve this, it is necessary to employ strategies that effectively bring such domains to light.

One approach closely linked to the Domains of Mathematical Knowledge for Teaching that has been developed is Lesson Study, which is strongly connected to the practice of teachers in continuing education. In this approach, lessons are collaboratively planned, and reflective discussions are conducted to improve them. In a systematic literature review, Cardoso et al. (2024) found that knowledge of content and students, knowledge of content and teaching, knowledge of content and curriculum, as well as specialized content knowledge, are the most emphasized in research published in Brazilian theses and dissertations and in national and international papers. The authors note that “there is still a scarcity of studies that analyze the activities carried out by teachers, how these activities mobilize the domains of mathematical knowledge for teaching, and how this knowledge evolves” (Cardoso et al., 2024, p. 10).

In the process of creating a Professional Learning Task (*Tarefa de Aprendizagem Profissional* - TAP), Elias et al. (2022), drawing on the Lesson Study approach and the Domains of Mathematical Knowledge for Teaching, highlighted that, in a continuing education program for teachers who teach Mathematics in the early years of primary education, opportunities were created for the development and refinement of one participating teacher’s mathematical knowledge for teaching—Maria. The authors structured “a way to produce an artifact for teacher education, with the potential to bring aspects of practice into formative processes” (Elias et al., 2022, p. 191).

The concern with practice in the teacher education process is evident in research related to both initial and continuing teacher education when trends in Mathematics Education are considered. In the context of lesson planning by future Mathematics teachers from a Problem-Solving perspective, in the Mandatory Curricular Internship course of a Degree in Mathematics, Bonato (2020) investigated the mobilization of mathematical knowledge for teaching. According to Bonato (2020, pp. 89-90):

> Lesson planning for teaching through Problem Solving played a significant role in the participants' development, particularly in the transition from CCK to SCK, as the interns sought justifications for the resolution procedure of the Compound Rule of Three through the Problem-Solving methodology.

With the aim of identifying domains of Mathematical Knowledge for Teaching based on the teaching practices of educators in a professional master's course in Mathematics Teaching, Shiinoki et al. (2024) planned and implemented a task in two 5th-grade classes using Exploratory Teaching. According to the authors, "the collaborative work carried out during the lessons in the course was fundamental for reflection and the manifestation of these subdomains at each established stage, as well as for the teacher, who reflected on her teaching practice" (Shiinoki et al., 2024, p. 17).

To infer the knowledge necessary for a teacher to teach Mathematics, Bisognin and Bisognin (2021) developed a modelling activity within continuing education teachers enrolled in a Master's course in Mathematics Teaching. The authors concluded that there is evidence that mathematical modelling supported the construction of both common and specialized content knowledge, as well as knowledge of teaching and students, which are essential for Mathematics teaching.

Following Bisognin and Bisognin (2021) and considering the need to address the gap identified by Cardoso et al. (2024) regarding the activities developed by teachers and the mobilization of the domains of mathematical knowledge for teaching, our research focus is the teacher education in mathematical modelling. In other words, mathematical modelling serves as a space where domains of mathematical knowledge for teaching can be emphasized.

Teacher education in mathematical modelling

In general terms, "mathematical modelling can be described as an activity involves *transitioning back and forth between reality and mathematics*" (Borromeo Ferri, 2018, p. 13, emphasis added). Thus, mathematical knowledge becomes relevant in order to move from an initial situation (the problem from reality) to a final situation (the solution to the problem from a mathematical perspective). To make this transition, the literature has relied on the construction and validation of mathematical models (Bassanezi, 2002; Borromeo Ferri, 2018; Almeida et al., 2012; Niss & Blum, 2020).

According to Niss and Blum (2020, p. 6), a mathematical model is "a representation of aspects of an extra-mathematical domain by means of some mathematical entities and relations between them," structured through mathematical schemes or structures that reveal aspects of the situation. The deduction of a mathematical model requires mathematical procedures and techniques, which allow the modellers' mathematical knowledge to be highlighted. Given this purpose, we understand mathematical modelling as a pedagogical alternative to introduce or apply mathematical content where the aim is to teach Mathematics (Almeida et al., 2012).

Teaching mathematics through modelling can have diverse objectives, both for the teacher and for the students. In general, in a modelling activity, the student's goal is to present a solution to the problem; the teacher, on the other hand, may also aim to envision the teaching of mathematical content.

Considering mathematical modelling in the educational context may present some unpredictability concerning mathematical content, as "each problem is different, and mathematical content is addressed according to the characteristics of each situation investigated, as well as being aligned with the knowledge of the students acting as modellers" (Silva et al., 2021, p. 42-43). Thus, it can cause some insecurities, leading to tensions and challenges for some teachers (Ceolim, 2015), and in some cases, hindering the implementation of modelling activities in the classroom.

What often concerns teachers in implementing mathematical modelling practices is that the content to be addressed is not necessarily defined in advance, but rather depends on the discussions that arise during the activity. However, some studies signal the possibility of anticipating and preparing for unforeseen events in a modelling activity through planning (Braz, 2017; Pinto, 2020; Pinto & Araújo, 2021; Tortola et al., 2023). The question, however, is what to consider in this planning without proper education in mathematical modelling.

Planning, according to Takinaga and Manrique (2023, p. 193), is "the moment when considerations primarily directed at the target audience and the encounter with mathematical knowledge are intertwined," when learning objectives, materials, and methods to achieve them are defined. Thus, within the scope of modelling activities, it is necessary to reflect on what is required to achieve the defined objectives for their development.

Doerr and English (2006) argue that implementing activities like modelling in classrooms requires teachers to learn about mathematical content, how students develop and represent their ideas, and, additionally, new methods of interacting with students, focusing on listening, observing, and asking questions. Thus, our efforts have been directed at structuring a teacher education design in mathematical modelling based on three axes that, to some extent, support the teacher's preparation to meet these demands: learning about modelling, learning through modelling, and teaching using modelling (Almeida & Silva, 2015).

Learning about modelling involves understanding the theoretical contributions to this approach in Mathematics Education. Learning through modelling refers to the development of modelling activities as modellers. Teaching using modelling is the implementation of one or more experiences with modelling activities in the classroom. These three axes, in a way, guide our perspectives, aligning with Borromeo Ferri's (2018, p. 1) statement that "learning and teaching go hand in hand and so they often were and are investigated together, in order to get a whole picture of the inter-connected processes."

Tortola et al. (2023, p. 190), following the guidelines of Almeida and Silva (2015) for teacher education in modelling, when analyzing "the reports of teachers who experienced a practice with mathematical modelling in Basic Education," highlighted "how these teachers (re)signify the way they teach and, in particular, how they teach Mathematics." The results led to two categories for the (re)signification of the teaching practices of the teachers participating in a Mathematical Modelling course in a Master's in Mathematics Teaching: "a teaching practice different from the usual, which involved the teacher's role in guiding and assisting students in the activity; and a practice that emphasizes student involvement, in terms of the possibilities offered by the mathematical approach to extra-mathematical situations" (Tortola et al., 2023, p. 168).

Within the scope of a research project on formative pathways and Mathematical Modelling in the New High School, Malheiros et al. (2024) investigated the process of designing a modelling activity in a collaborative space for teacher education in modelling. In this context, they highlighted that "collaboratively designing an activity, elements such as dialogue, listening, and collaboration were essential, in a process in which we are being formed and are forming" (Malheiros et al., 2024, p. 15). Furthermore, the authors indicated that mathematical modelling can be seen as a form of "continuing education, as it stems from the teacher's classroom experience, combining theory and practice, toward praxis" (Malheiros et al., 2024, p. 15).

Listening and dialogue can reveal aspects of the understanding of those involved in mathematical modelling activities. When developing a modelling activity, the modellers (students or teachers) follow "Individual Modelling Routes" (Borromeo Ferri, 2018).

According to Borromeo Ferri (2018), modellers may focus on a specific phase of the modelling activity along the route and ignore others. However, we only have access to "*visible modelling routes,* as one can only refer to verbal utterances or external representations for the reconstruction of the starting-point and the modelling route" (Borromeo Ferri, 2007, p. 265, emphasis added).

Considering the verbal utterances (discussions) present in the directions of a modelling activity by students, as well as the characterizations of knowledge (mathematical, technical, and reflective) addressed by Skovsmose (1990), Barbosa (2010) outlined objectives for mathematical modelling, organizing the discussions into three kinds: mathematical, technical, and reflective, as shown in Table 1. For the author, the three kinds of discussions among students are present in the construction of a mathematical model and can be components of the modelling routes.

Table 1.

Objective for modelling and kinds of discussions highlighted (Barbosa, 2010, p. 370)

Objective for Mathematical Modelling	Highlighted Discussion
To develop a mathematical concept	Mathematical
To promote skills and competences in modelling real-world problems	Technical
To analyze the nature of mathematical models and their uses in society	Reflective

In addition to the three kinds of discussions, Barbosa (2010) identified parallel discussions that involve mathematical ideas or procedures but may not have clear relevance to the approach of the problem situation and might not even be considered in the solution.

When teachers, in the context of professional development, engage in modelling activities as modellers, besides the discussions outlined by Barbosa (2010), there are notably pedagogical discussions that intersect with the others. The word "pedagogical" comes from the Greek *paidagogikós*, which refers to what is suitable for the teacher in their professional practice–that of teaching. Thus, we understand that the adjective "pedagogical," used in the discussions (verbal utterances) of teachers in education, is intrinsic to pedagogy, which is the art of instructing and educating people at different stages of life. In this way, we agree with Silva et al. (2022, p. 131), who state that pedagogical discussions concern "attention to students' assignments," meaning what is appropriate for the teacher to develop in their professional practice so that students are instructed in the situation being studied.

Therefore, we consider pedagogical discussions as verbal utterances that refer to the act of teaching, to teachers' concerns regarding an activity, and that relate to their students' knowledge. In alignment with pedagogical content knowledge, teachers' pedagogical discussions involve considering aspects of their students, educational contexts, and educational purposes. In this scenario, when teachers learn about and through modelling, they connect aspects that can be implemented when teaching using modelling.

Research Context and Analysis Methodology

The data collection took place in a Mathematical Modelling course as part of a master's program in Mathematics Teaching at a federal university in southern Brazil. During the first semester of 2022, a total of nine Basic Education teachers, working in the final years of Elementary School and in High School, participated in 14 sessions, each lasting three hours.

The modelling activities in the course were planned and developed following guidelines related to the three axes—learning about, learning through, and teaching using mathematical modelling—which shaped a teacher education design envisioned in a research project approved in the CNPq Universal Call for Proposals in 2021. From the first session of the course, the course instructor (the author of this paper) informed the teachers about the structure of both the course design and the ongoing research project. She also requested the teachers' consent to record audio and video of some discussion moments during the course and to use the written records in future publications, while maintaining anonymity, to present results of the implemented design. The teachers then signed a free and informed consent form, authorizing the production and use of the data.

In three sessions, texts were discussed, and activities were analyzed to familiarize the teachers with the theoretical contributions related to mathematical modelling in Mathematics Education, structured by the modelling cycles (Borromeo Ferri, 2018). The objective was to show that there are different configurations for characterizing modelling in the literature, thereby structuring the "learning about modelling" axis. When faced with different configurations and activities developed and reported in the literature, the teachers incorporated the educational context in which they were situated into their discussions. As a result, in addition to focusing on the theoretical and mathematical approaches presented in the texts, they also explicitly addressed classroom implementation.

In the fourth session of the course, the teachers, organized into three groups of three, developed modelling activities based on topics chosen by the teacher educator and shared their solutions on a Moodle Wiki, a virtual platform for sharing texts, images, and videos among participants, so that everyone had access to each other's solutions. The groups had one week to analyze their colleagues' solutions and pose questions during the fifth session. These two sessions, to some extent, constituted the "learning through modelling" axis.

Considering that the teachers would report their classroom experiences based on the topics and solutions they had access to, we expected that discussions related to teaching practices and revealing mathematical knowledge for teaching would also be present. In this context, the discussions in the fifth session were recorded in audio and video and fully transcribed. The recording was done using a camcorder mounted on a tripod, positioned at an angle that allowed capturing both the image and audio of all course participants. We used square brackets to describe gestures made by the teachers that were necessary for understanding the discussions. To maintain anonymity, we referred to the nine teachers as Prof_1, Prof_2, ..., Prof_9, and the course instructor as Prof. Table 2 presents the distribution of the groups along with the respective themes of the modelling activities developed.

Table 2.

Distribution of the teacher groups (Research data, 2022)

Activity Themes	Teachers in charge
Parking meter in Londrina	Prof_2, Prof_5, Prof_7
Cell phone battery charging	Prof_3, Prof_8, Prof_9
Trash in Arapongas	Prof_1, Prof_4, Prof_6

Considering the purposes of our study regarding the research question—What implications of mathematical knowledge for teaching using mathematical modelling emerge from the discussions of teachers in continuing education?—we chose to analyze the activity with the theme "Cell phone battery charging," as it was the one that emerged with different kinds of discussions (Barbosa, 2010) among the activities investigated, allowing us to highlight mathematical knowledge for teaching.

For addressing the theme, in addition to an informational text, pre-existing data on the percentage of a cell phone battery's charge (Figure 2) were provided. Using this information and data, the teachers formulated the problem: How long will it take for the cell phone battery to be fully charged (100%)?

Cell phone battery charging

The cell phone battery is a kind of generator that produces electrical energy through chemical reactions occurring inside it.
In general, batteries have a limited lifespan. The substances involved in the chemical reactions inside the battery degrade over time, leading to a decrease in electrical energy production.
However, a cell phone battery can be recharged before reaching the end of its lifespan.

Starting the charging process of a cell phone battery after it had turned off, the percentage of its total capacity was recorded every 15 minutes as time *t* progressed. The collected data is presented in Table 1.

Table 1: Percentage of the cell phone battery's full charge as a function of time

Time (in minutes)	0	15	30	45	60	75
Percentage of full charge (in %)	0	27	48	61	77	92

Source: Empirically collected data.

Figure 2.

Cell phone battery charging activity (Research data, 2022)

The records of the activity resolution and the presentation of a solution to the problem were provided by the trio of teachers Prof_3, Prof_8, and Prof_9. The other teachers were in charge of analyzing this resolution and providing notes, starting the discussions with questions and classroom implications. These discussions were alternated with responses or explanations from the teachers in charge, as well as mediations from the teacher educator.

The data analysis—transcriptions of the spoken statements and written records of the resolution—followed the guidelines of Content Analysis (Bardin, 2011), involving three procedures: pre-analysis, material exploration, and processing of the results obtained and interpretation.

The main goal of the first procedure, pre-analysis, is to provide the researcher with an initial exposure to the information that will be analyzed. According to the author, in this phase, emerging hypotheses are identified, adjusted to theories, and analytical techniques are envisioned (Bardin, 2011). This phase of the research involved reading all the transcriptions of the recordings for the analyzed activity, coupled with the written records of the teachers.

The second procedure, material exploration, involved connecting the data with aspects presented in the literature regarding the discussions that arise during the development of a mathematical modelling activity (Barbosa, 2010)—mathematical, technical, and reflective discussions—in addition to the pedagogical ones. In this phase, segments of the transcriptions that highlight the kinds of discussions in progress were identified, in order to classify them.

In the final phase of Content Analysis—the processing of the results obtained and interpretation—inferences were made based on the analyzed information to provide insights into the research question, specifically related to mathematical knowledge for teaching using modelling in the classroom , where implications were highlighted from the collective notes, according to an analytical approach that considered the theoretical framework we relied on—mathematical knowledge for teaching and discussions in teacher education in mathematical modelling.

Description of the activity resolution and analysis of the teachers' discussions

The trio of teachers in education—Prof_3, Prof_8, Prof_9—in charge of investigating the cell phone battery charging situation, hypothesized that the data followed a quadratic polynomial function. To deduce the mathematical model, they performed curve fitting using Microsoft Excel software, obtaining the algebraic expression $F(t) = -0{,}0063t^2 + 1{,}6629t + 1{,}5714$, where $F(t)$ represents the cell phone battery charging percentage as a function of time t, in minutes. The group validated the deduced mathematical model by comparing the calculated percentage values with those presented in the table in Figure 2. After validation, they presented an answer to the problem—After how long will it take for the cell phone battery to be fully charged (100%)? —concluding that, after 90 minutes connected to the power source, the cell phone battery would be fully charged. Figure 3 shows the resolution provided by the group on the Moodle Wiki.

Variables: Time (in minutes)
Percentage of full charge (in %)

Hypothesis: The percentage of charge increases over time.

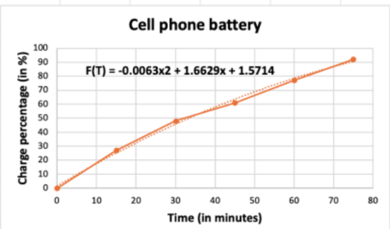
Deduction of the mathematical model: $F(t) = -0{,}0063t^2 + 1{,}6629t + 1{,}5714$

To obtain the result, the data points from the table were used in EXCEL. A scatter plot was then created, and by applying a trendline, it was possible to observe that the function that best fit the original graph was a second-degree polynomial, given by the following equation: $F(t) = -0.0063t^2 + 1.6629t + 1.5714$, as illustrated in the figure below.

Curve fitting

Image 1: Graph construction in EXCEL

Time	Percentage
0	0
15	27
30	48
45	61
60	77
75	92



Validation

Table 2: Percentage of the cell phone battery charge as a function of time + Percentage of the battery charge as a function of time using the obtained mathematical model

Time (in minutes)	0	15	30	45	60	75
Percentage of full charge (in %)	0	27	48	61	77	92
Percentage of full charge (in %) using the mathematical model	1.55714	25.09	45.79	63.64	78.66	90.85

By comparing the data, it can be seen that the results obtained are satisfactory. For this reason, this model will be used to solve the problem.

Answer to the problem

Problem to be studied: After how much time will the cell phone's battery reach full charge (100%)?

$$F(t) = -0.0063t^2 + 1.6629t + 1.5714$$

$$100 = -0.0063*t^2 + 1.6629*t + 1.5714$$

$$t \approx 90$$

Therefore, after 90 minutes, the cell phone's battery will reach full charge (100%).

Figure 3.

Resolution of the Cellphone Battery Charging Situation (Report of the Trio of Teachers, 2022)

In light of the resolution provided in the Wiki, during the meeting dedicated to discussing the solutions, some comments were made by the trainee teachers and the instructor. In the material exploration phase, which corresponds to the transcriptions of the approaches, the second stage of Content Analysis, we identified connections related to the kinds of discussions presented by Barbosa (2010), as well as pedagogical discussions in three episodes: comments from other teachers; comments from the trio in charge of the solution; and comments from the instructor.

Table 3 presents segments of the transcriptions related to the episode "**notes from other teachers**," highlighting the kinds of discussions underway based on their interaction with the record provided for analysis.

Table 3.

Discussions in the episode ***notes from other teachers*** *(Research Data, 2022)*

Excerpts from notes from other teachers	Kinds of discussions
Prof_4: The deduction of the model that they [referring to the teachers in the trio] made using Excel—we were debating this afterward... If this were done by hand, maybe the students wouldn't be able to arrive at a quadratic function. Looking at the graph there in Excel, it even seems like a logarithmic function [gesturing with their hand].	Technical Pedagogical Mathematical
Prof_4: The first deduction we thought of, that the students would think if it were done by hand, there. [...] That it's linear.	Pedagogical
Prof_4: In the classroom with my students. I would first see if they would arrive at the linear function, and then I would demonstrate in Excel that there could be another function there as well [referring to the quadratic polynomial function].	Pedagogical
Prof_6: And maybe propose a research activity for them, right? Using their own phones, their own battery charging data, how it works and how they could come up with a solution.	Technical
Prof: And how long did it take to charge? **Prof_1:** 90 hours [reviewing the solution]. No... it's 90 minutes! Sorry, I got mixed up here!	Reflective
Prof: If a student responds, "Oh, but how long? 90 hours? Is that really it?"—It's important for us to discuss specifying the unit of time being considered.	Pedagogical
Prof_6: [...] Then we even asked ourselves whether there's any possibility that this percentage stabilizes?	Reflective
Prof_6: And another question we actually had—he [referring to Prof_4] mentioned it, but I'm not sure if it was clear—is: "How could we arrive at this deduction of the model and this result without using Excel [...]? So how could we reach this result with them without using technology?".	Mathematical

In the cited episode, the discussions centered on the approaches taken by the trio of teachers in charge of the activity, leading to considerations for classroom applications through pedagogical discussions, particularly regarding the use of Microsoft Excel. It is evident that the teachers analyzing the resolution relied exclusively on what was presented to them, indicating evidence of **common content knowledge** concerning the behavior of the data, as analyzed with the support of the software. This became apparent as the teachers examined the trio's resolutions, focusing on the mathematical approach taken. Notably, a discussion about adjusting for a linear curve arose from the visualization of the points on the Cartesian plane, making mathematical discussions recurrent.

The pedagogical discussions were directed toward **knowledge of content and students**, as the teachers reflected on how their students—those in the final years of elementary school and high school—would interpret the behavior of the points represented on the Cartesian plane and the mathematical concepts they would associate with them. For Prof_4, in this context, the development of the modelling activity, from a didactic-pedagogical perspective, would take as its starting point the content already familiar to students, making an approach through a first-degree polynomial function possible— "The first deduction we thought of, that the students would think of if it were done by hand, there. [...] That it's linear."—, at the same time, a second-degree polynomial function would not initially be considered, requiring teacher intervention to clarify **knowledge of content and teaching**, as the teacher recognized the need to support students' work.

In general, in the classroom context, it is natural that "students' initial models (or conceptual systems) are often not very sophisticated or useful" (Ärlebäck & Doerr, 2018, p. 189), requiring teacher intervention. This process is inherent to the development of a modelling activity. However, we can infer that Prof_4 believed that only with the use of technology would students be able to arrive at the algebraic expression of the quadratic function— "[...] I would demonstrate in Excel that there could be another function there as well"—thus limiting the approach to a specific educational level.

Through the pedagogical discussion initiated by Prof_4, it became evident that there was an attempt to bring into the teacher's practice not only specific content knowledge but also a didactic dimension attached to it (Shulman, 1986), as they envisioned a way to implement the modelling activity incorporating technology.

Prof_6 suggested having students collect empirical data themselves, through a technical discussion, with the aim of fostering real problem-modelling competencies (Barbosa, 2010)—"Using their own phones, their own battery charging data, how it works."—This approach could enable modellers, whether teachers or students, to engage in the modelling activity and later participate in the mathematical discussions that may emerge. As a result, modellers can "describe the situation, allow for the analysis of relevant aspects of the situation, answer the formulated questions [...] and, in some cases, even enable the making of predictions" (Almeida et al., 2012, p. 16). In this context, **knowledge of content and teaching** is evident, as Prof_6 envisioned a way to articulate knowledge related to the content—whether a first- or second-degree polynomial function—and its teaching through mathematical modelling.

Thus, Prof_6 revealed an understanding of the need to establish connections between mathematical content with a didactic dimension, thereby transforming content into pedagogically powerful forms (Shulman, 1986), considering that "knowledge is constructed by students throughout the activity itself, assuming an active role, while the teacher essentially acts as an organizer and facilitator of learning" (Ponte & Serrazina, 2000, p. 111).

Given Prof_1's mistake regarding the unit of time, the teacher educator emphasized the importance of teachers, in a pedagogical discussion, being attentive to how students interpret the unit assigned to each variable, ensuring that the interpretation of the modelling activity results remains coherent. The teacher educator was inclined to highlight the importance of explaining this approach, as they aimed to "better explicate how this knowledge is used in teaching effectively" (Ball et al., 2008, p. 404).

The trio in charge of the resolution appeared to struggle with deducting the mathematical model represented by a second-degree polynomial function without the use of technology, as evidenced by excerpts from the transcriptions in the episode **notes from the trio in charge of the resolution** (Table 4).

Table 4.

Discussions in the episode notes from the trio in charge of the resolution (Research Data, 2022)

Excerpts from the notes of the trio in charge of the resolution	Kinds of discussions

Prof_9: We did it by hand. When we did it by hand, we said, “No, it’s linear,” right? Then we said, let’s check in Excel. When we put it in Excel, “No, it’s not linear.”	Mathematical Technical
Prof_9: We started solving it using a system [referring to a system of linear equations], right? Prof_8: Yes! Prof_9: We tried, and then we saw that the “b” [referring to the coefficient of the function] always tended... It was zero. Then there came a point where we got stuck, and that’s when we decided to use Excel.	Mathematical Mathematical Technical
Prof_3: Mainly the formula, the real problem is the formula [referring to the algebraic expression]. The graph is easy for them to find, even a curve, right? But for them to get to that formula there… [pointing to the Excel slide projection]	Mathematical
Prof_8: I think they could even come up with the linear one, you know? But not the quadratic one, especially because looking at the graph, it doesn’t seem like a quadratic function!	Mathematical

The excerpts presented in Table 4 seem to reveal, through the mathematical discussions undertaken in the justifications of the trio of teachers, that although they recognized the possibility of obtaining a quadratic function by solving a system of linear equations, they did not pursue this approach. What we can observe is that, initially, the **specialized content knowledge** related to the mathematical procedures for solving a system of linear equations was not demonstrated by the teachers who taught in high school. However, the trio remained engaged in a mathematical discussion linked to **knowledge of content and students**.

On the other hand, the other teachers, when analyzing the resolution, did not mention procedures for manually solving the situation in a way that would reveal **specialized content knowledge**. This is evident, as two teachers (Prof_4 and Prof_6) questioned whether it was possible to obtain the model represented by the quadratic function without the use of a technological tool.

However, the teacher educator, considering mathematical modelling as a pedagogical alternative (Almeida et al., 2012)—both in teaching mathematics through modelling and in teacher education in modelling—primarily raised questions that led to mathematical discussions, highlighting mathematical knowledge for teaching. Compared to the other episodes, the episode **notes from the teacher educator** (Table 5) was the one that most prompted mathematical discussions.

Table 5.

Discussions in the episode notes from the teacher educator (Research Data, 2022)

Excerpts from the notes from the teacher educator	Kinds of discussions
Prof: [...] Can't we reach this mathematical model manually? Prof_9: Yes!	Mathematical
Prof_5: Well, I think that with the teacher's help to remind them, right... I think the teacher needs to be there offering some help [...]. The data here doesn't form a curve, so I don't think they'll see that.	Pedagogical
Prof: [...] If it's quadratic, how many points do we need to determine the coefficients? Prof_4: Three! Prof: Why? What's the general expression for the quadratic model? Prof_1: $y = ax^2 + bx + c$. Prof: There are three points, you set up the system, solve the system of three equations and three unknowns. You can solve it manually. Then we can use matrices, scaling. Prof_9: This is High School, right?	Mathematical
Prof: Then Prof_7 said: "Oh, I think they would say it's linear." So, this is one model, one kind of model that could be deduced. It increases linearly. You did it here, it increases through a quadratic model, and then even Prof_4 said here, it could be an exponential model.	Technical
Prof: When it reaches ninety minutes, it stabilizes and stays constant. Prof_5: So it's not entirely a second-degree function. [...] Prof: Why? Prof_5: Because the second-degree function, it isn't, for example, like this? [gestures drawing a parabola in the air]. It's not like this and then becomes... [gestures the increasing part of the parabola and then a constant function]. Part of it is a second-degree function and then part of it is a constant linear... Prof: What function is this? Prof_5: Ah, okay, a mixture of these two? I don't remember. Prof_1: Yes, in parts! We call it a piecewise function.	Mathematical
Prof: What is the characteristic of this function? [referring to the mathematical model deduced by the trio] Prof_5: It's... with concavity facing downwards... Prof: So this "a" is less than zero ($a < 0$). [...], with concavity facing downwards. Can this model be considered over the entire real domain for this situation? Prof_1: No, it can only be positive, right? Prof: [...] the concavity facing downwards, so what does that mean? What does that mean here? What will happen at this point [pointing to the vertex of the parabola]? Prof_9: It will decrease!	Mathematical
Prof: Is that what happens when we leave the phone charging? Prof_5: Charging, no, but if you unplug it, it starts to drop! Prof: That's another situation we need to analyze considering decay! So, this model is valid until...? Prof_9: Until 100%.	Reflective

Prof: Was it exactly ninety minutes that it reached 100%?	Mathematical
Prof_8: It was 89. something. Ah, it wasn't exactly 90!	
Prof: One thing we need to be very careful about when using a function defined by two pieces, or piecewise, this content is from High School,	Mathematical
[...] A piecewise function is something that students have a lot of difficulty with when building the graph, and I asked you if it was exactly ninety because we make rounding [...]. Leave it in fractional form. [...]	Pedagogical
Now comes the domain! A piecewise function is a function that changes the sentence according to the domain, so we're going to consider the time, greater than or equal to zero. And less than... Prof_3: 89.61!	Mathematical
Prof: So, okay! 89.61. And the constant function equals 100 if "*t*" is greater than... We write if it's equal here or equal to here [writing the mathematical model on the board]. Can we write both? Prof_9: If you write both, it's not a function! A domain can't map to two images for it to be a function. Prof: So, we need to determine when this equals 100! And this is the value when it reaches 100, and from that point on, it's always 100. This, in a piecewise function, so... Set this equal to 100 and solve the quadratic equation and leave it in fraction form! It's better to leave it in fraction form.	Mathematical

Using technology, it is possible to deduce a mathematical model. However, considering the teacher's objective of determining the coefficients of a quadratic model, this could support mathematical discussions that elicit the introduction of new content through the modelling activity. With this action, "the emphasis is on promoting the independence that the knowledge constructed in a modelling activity gains from the model" (Almeida, 2022, p. 136-137). The approach of using a system of linear equations with three equations and three unknowns became the discussion undertaken by the teachers, supported by the notes of the teacher educator, which even included the procedures for solving it using matrices and scaling.

The review of procedures to deduce an algebraic expression for the second-degree polynomial function from a set of points allowed the teacher educator to create an educational environment where **specialized content knowledge** emerged. The teachers then noticed and recalled procedures for obtaining the coefficients when using three points, as well as the characteristics and specifics of the behavior of the second-degree polynomial function when the coefficient "a" is negative. From the mathematical discussions, it became evident that Prof_9 recognized this as an approach generally used in High School, providing evidence of **knowledge of content and curriculum**.

Through reflective discussions initiated by the teacher educator—“Is that what happens when we leave the phone charging?”—the mathematical discussions took a new direction, branching from the deduction of a second-degree polynomial function via a system of linear equations to the approach of a piecewise function. This led to an expansion in solving the problem, considering the need for the implementation of the constant function in the mathematical model. Such an approach even raised some teachers’ lack of familiarity with this mathematical object—the piecewise function—leading the teacher educator to resort to its concept in a simplified manner (A piecewise function is one that changes the sentence according to the domain), as well as revisiting **common content knowledge** related to the definition of a function (A domain can’t map to two images for it to be a function – Prof_9).

It became evident that through the mathematical discussions prompted by the notes from the teacher educator, the understanding emerged that “to teach, it’s not enough to use Mathematical Modelling to construct a model and confront it with the real situation; it’s necessary to build other knowledge, beyond the common content knowledge” (Bisognin & Bisognin, 2021, p. 16). The mathematical discussions made **specialized content knowledge** emerge as the teachers could consider the problem situation, even for teaching mathematical content different from what had been evidenced in the resolution of the problem and even in the notes from other teachers. Moreover, **knowledge of content and students** were highlighted through Prof_5’s pedagogical discussion— “Well, I think that with the teacher’s help to remind them, right...”

Under the guidance of the teacher educator, through mathematical discussions, the teachers concluded that a piecewise function could represent the phenomenon, revisiting **specialized content knowledge** and paying attention to the domain that expresses each sentence—**common content knowledge**—they wrote a “new” mathematical model: $F(t) = \begin{cases} -0{,}0063t^2 + 1{,}6629t + 1{,}5714, if\ 0 < t \leq \frac{1882}{21} \\ 100, \qquad if\ t > \frac{1882}{21} \end{cases}$, where $F(t)$ represents the percentage of the cell phone battery charge as a function of time (t), in minutes.

With the processing of the obtained results, the interpretation in the analysis, to some extent, corroborated Malheiros et al. (2024) on the importance of dialogue and listening. For our research context, by engaging the teachers in dialogue and listening, different discussions emerged from the development of a mathematical modelling activity. Using the written records of a trio of teachers on the theme—cell phone battery charging—, the participants in a master's course engaged in mathematical, technical, reflective, and pedagogical discussions that revealed mathematical knowledge for teaching.

Initially, the mathematical discussions from the episode notes from other teachers began by revealing the common content knowledge, where it became evident that the initial intention and concern were focused on the knowledge and skills considered by the trio of teachers, validating what they had done. However, in light of such knowledge, the teachers pointed out the need to address the problem in the classroom, so technical discussions, regarding the use of Microsoft Excel, and pedagogical discussions led to the emergence of knowledge of content and teaching, as well as knowledge of content and students. This led to a pause to analyze the mathematical model deduced by the trio's members, as well as the mathematical content that supported it.

In the episode notes from the trio in charge of the resolution, the mathematical discussions remained centered on knowledge of content and students, with the members justifying the use of the software considering what students could use to solve the problem, as they might not follow steps where a mathematical model, considering a second-degree polynomial function, could be manually deduced. In this case, the mathematical discussions were confined to the approach taken, not indicating specialized content knowledge as seen in Prof_6's statement— "How could we arrive at this deduction of the model and this result without using Excel [...]?"

With the limitation of mathematical knowledge for teaching revealed in the momentary mathematical discussions, the teacher educator made interventions from which the episode notes from the teacher educator emerged. In this episode, due to the teacher educator's requirements, mathematical, technical, reflective, and pedagogical discussions emerged. The mathematical discussions, in particular, provided an approach regarding specialized content knowledge for solving a system of linear equations, as well as the delineation of a piecewise function and considerations for defining its domain. The teacher educator's intention in this episode was to expand the approach to the situation for other educational levels, whose knowledge of content and curriculum were also elucidated.

However, even though the teacher educator sought to encompass mathematical discussions within the scope of specialized content knowledge, common content knowledge and knowledge of content and students permeated the discussions.

In summary, the discussions that emerged within the framework of the axis learning through modelling, through the lens of the Domains of Mathematical Knowledge for Teaching, had implications for teaching using mathematical modelling, specifically concerning the anticipation that can be inserted, for example, in lesson planning when considering the problem situation of cell phone battery charging. Depending on the objectives for the classes, different mathematical approaches can emerge—polynomial functions, solving systems of linear equations, matrices, scaling, curve fitting using computational software, and defining the domain of a function. Planning can alleviate uncertainties, causing tensions and challenges for some teachers (Ceolim, 2015) in implementing modelling practices, as they may intertwine considerations directed at the target audience (Takinaga & Manrique, 2023).

Final considerations

Mathematical modelling education has primarily been designed to address pragmatic aspects, focusing on how modelling activities can be implemented in the classroom (Tortola et al., 2023), as well as theoretical-epistemological elements aimed at structuring modelling cycles to guide practice (Almeida, 2022; Borromeo Ferri, 2018). Both approaches aim to ensure the development of specific competencies and learning outcomes.

Within the scope of a research project funded by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (*National Council for Scientific and Technological Development*) (CNPq), we have investigated and structured educational programs organized around three axes: learning about modelling, learning through modelling, and teaching using modelling.

In the first semester of 2022, nine teachers participating in a Master's course in Mathematics Teaching engaged in modelling activities, and their discussions, in general, fostered an approach focused on the educational context. Upon analyzing these discussions, we observed similarities with the characteristics highlighted by Barbosa (2010), who categorized students' mathematical, technical, and reflective discussions in the construction of mathematical models based on the concept of modelling routes. It is important to emphasize that we found a pedagogical character present in the discussions, in which the teachers, as they learned through modelling, related aspects that could be implemented to teach using modelling.

Such implementations provided evidence of mathematical knowledge for teaching, especially in relation to pedagogical content knowledge. While content knowledge—whether common or specialized—as well as horizontal knowledge, is inherent to the teacher teaching Mathematics, we understand that it is pedagogical content knowledge that reveals how the teacher handles the knowledge intrinsic to their profession, considering the peculiarities of the students they work with. In the context of mathematical modelling education, which is the focus of our investigation, the teachers paid attention to the mathematical content as they solved and analyzed the resolution of a modelling activity. While solving an activity as modellers, a trio of teachers revealed knowledge of content and students as they were satisfied with the curve fitting presented by software when considering the data as points on the Cartesian plane. To some extent, this knowledge was sufficient for them to present a solution to the problem they dedicated themselves to investigating.

However, under the analysis of other teachers and the notes from the teacher educator, it also emerged, through different kinds of discussions conducted jointly, common content knowledge, knowledge of content and teaching, specialized content knowledge, and knowledge of content and curriculum. In general, these areas of knowledge intertwined as questions were raised, revealing that the modelling activity developed seemed promising for implementation in classroom practices, including the possibility of having students collect empirical data.

In this sense, the implications of mathematical knowledge for teaching in the axis of teaching using mathematical modelling were revealed through the sharing of ideas, mainly in what we refer to as pedagogical discussions articulated with mathematical discussions. Specifically, this allowed the anticipation of mathematical approaches that could emerge in the implementation of the activity in the classroom through planning. It should also be noted that aspects of the modelling process were revealed from the intentional questions, pedagogically speaking, raised by the teacher educator. In this context, envisioning actions that can contribute to the development of didactics for teaching using mathematical modelling becomes essential, particularly when teachers reflect on planning based on what they have experienced. This involves examining the actions and discussions of teachers in this process to highlight mathematical knowledge for teaching, a line of research that could be pursued in the future within the context of Lesson Study, for example. To this end, questions such as: What are the implications of mathematical knowledge emerging from discussions of teachers in continuing education when designing a lesson plan for teaching using mathematical modelling? and even: What mathematical knowledge is revealed in the implementation of practices with mathematical modelling by teachers in continuing education? could serve as a continuation of the research.

Acknowledgements

We thank the Brazilian National Council for Scientific and Technological Development (CNPq) for their financial support. Grant # 409309/2021-4.

References

Almeida, L. M. W. (2022). Uma abordagem didático-pedagógica da modelagem matemática. *Vidya*, 42(2), 121-145. https://periodicos.ufn.edu.br/index.php/VIDYA/article/view/4236

Almeida, L. M. W., & Silva, K. A. P. (2015). Práticas de professores com Modelagem Matemática: algumas configurações. *Educação Matemática em Revista*, 46, p. 6-15. https://www.sbembrasil.org.br/periodicos/index.php/emr/article/view/498

Almeida, L. W., Silva, K. P., & Vertuan, R. E. (2012). *Modelagem Matemática na educação básica*. Editora Contexto.

Ärlebäck, J. B., & Doerr, H. M. (2018). Students' interpretations and reasoning about phenomena with negative rates of change throughout a model development sequence. *ZDM-Mathematics Education*, 50(1–2), 187-200.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content Knowledge for Teaching: what make it special? *Journal of Teacher Education*, 59(5), 389-407.

Barbosa, J. C. (2010). The students' discussions in the modeling environment. In R. Lesh, P. L. Galbraith, C. R. Haines, & A. Hurford. (orgs.), *Modeling students' mathematical modeling competencies* (pp. 365-372). Springer.

Bardin, L. (2011). *Análise de Conteúdo*. Edições 70.

Bassanezi, R. C. (2002). *Ensino-aprendizagem com modelagem matemática: uma nova estratégia*. Editora Contexto.

Becker, F. (2012). *Epistemologia do professor de Matemática*. Vozes.

Bisogin, E., & Bisognin, V. (2021). Modelagem Matemática: uma análise do conhecimento matemático para o ensino. *Revista de Ensino de Ciências e Matemática*, 12(2), 1–19. https://doi.org/10.26843/rencima.v12n2a06.

Bonato, G. V. (2020). *Conhecimento Matemático para o Ensino mobilizado em um planejamento de aula na perspectiva da Resolução de Problemas*. [Dissertação de Mestrado em Ensino de Ciências e Educação Matemática, Universidade Estadual de Londrina]. https://repositorio.uel.br/items/1347861b-1816-4870-b1fb-1ff503e5848b/full.

Borromeo Ferri, R. (2007). Modelling from a cognitive perspective: individual modelling routes of pupils. In C. Haines, P. Galbraith, W. Blum, & S. Khan (eds.), *Mathematical modelling: education, engineering and economics* (pp. 260–270). Horwood Publishing.

Borromeo Ferri, R. (2018). *Learning How to Teach Mathematical Modeling in School and Teacher Education*. Springer International Publishing.

Braz, B. C. (2017). *Aprendizagens sobre modelagem matemática em uma comunidade de prática de futuros professores de matemática* [Tese de doutorado em Educação para a Ciência e a Matemática, Universidade Estadual de Maringá]. http://repositorio.uem.br:8080/jspui/handle/1/5728.

Cardoso, M. B., Barreto, M. C., & Pinheiro, J. L. (2024). Conhecimentos matemáticos para o ensino na perspectiva do Lesson Study: uma revisão sistemática de literatura. *Conexões, Ciência e Tecnologia*, 18, e022002, 1-12. https://doi.org/10.21439/conexoes.v18i0.2853.

Ceolim, A. J. (2015). *Modelagem Matemática na Educação Básica: obstáculos e dificuldades apontados por professores* [Tese de doutorado em Educação, Universidade Federal de São Carlos]. https://repositorio.ufscar.br/handle/ufscar/7523?show=full.

Curi, E., & Pires, C. M. C. (2008). Pesquisas sobre a formação do professor que ensina matemática por grupos de pesquisa de instituições paulistanas. *Educação Matemática Pesquisa*, 10(1), 151-189. https://revistas.pucsp.br/index.php/emp/article/view/1655/1065.

Doerr, H. M., & English, L. D. (2006). Middle grade teachers "learning through students" engagement with modeling tasks. *Journal of Mathematics Teacher Education*, 9(1), 5-32. https://doi.org/10.1007/s10857-006-9004-x.

Elias, H. R., Ribeiro Rodrigues, S., & Trevisan, A. L. (2022). Análise do Conhecimento Matemático para o Ensino em um Estudo de Aula: um caminho para produzir Tarefas de Aprendizagem Profissional. *Educação Matemática Pesquisa*, 24(1), 156–193. https://doi.org/10.23925/1983-3156.2022v24i1p156-193.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematics knowledge for teaching on student achievement. *American Education Research Journal*, 42(2), 371-406.

Lüdke, M., & André, M. E. D. A. (2014). *Pesquisa em Educação: abordagens qualitativas* (2a. ed.). E.P.U.

Maass, K., Sorge, S., Romero-Ariza, M., Hesse, A., & Straser, O. (2022). Promoting Active Citizenship in Mathematics and Science Teaching. *International Journal of Science and Mathematics Education*, 20, 727-746. https://doi.org/10.1007/s10763-021-10182-1.

Malheiros, A. P. S., Souza, L. B., & Forner, R. (2024). Elaboração de uma Atividade de Modelagem em um Espaço Colaborativo de Formação. *ACTIO*, 9(3), 1-20. https://doi.org/10.3895/actio.v9n 3.18849.

Mutti, G. S. L., & Klüber, T. E. (2018). Aspectos que constituem práticas pedagógicas e a formação de professores em modelagem matemática. *Alexandria*, 11(2), 85-107. https://doi.org/10.5007/1982-5153.2018v11n2p85.

Niss, M. & Blum, W. (2020). *Learning and Teaching of Mathematical Modelling*. England.

Pinto, T. F. (2020). *A elaboração de planejamento de atividades de modelagem matemática por professores com pouca vivência em modelagem* [Dissertação de Mestrado em Educação, Universidade Federal de Minas Gerais]. http://hdl.handle.net/1843/45939.

Pinto, T. F., & Araújo, J. L. (2021). Um estudo sobre planos de atividades de modelagem matemática. *Revista de Ensino de Ciências e Matemática*, 12(2), 1-25. https://revistapos.cruzeirodosul.edu.br/rencima/article/view/2933.

Ponte, J. P., & Serrazina, M. L. (2000). *Didáctica da matemática do Ciclo I*. Ed. Universidade Aberta.

Shiinoki, V. G., Rocha, Z. F. D. C., & Probst, M. M. A. (2024). Conhecimento Matemático Evidenciado por Professores da Educação Básica por meio do Ensino Exploratório. *Perspectivas da Educação Matemática*, 17(46), 1-19. https://doi.org/10.46312/pem.v17i46.19672.

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4–14.

Shulman, L. (1987). Knowledge and teaching: foundations of the new reform. *Harvard Educational Review*, 57(1), 1-22.

Shulman, L. (2004). *The Wisdom of Practice: Essays on Teaching, Learning, and Learning to Teach*. Jossey-Bass.

Silva, A. L. S., Moura, P. R. G, & Pino, J. C. (2022). Atividade Experimental Problematizada (AEP): discussões pedagógicas e didáticas de uma asserção de sistematização voltada ao ensino experimental das ciências. *Contexto & Educação*, 37(116), 130–144. https://doi.org/10.21527/2179-1309.2022.116.12491.

Silva, K. A. P., Tortola, E., Koga, M. A., & Koga, M. F. (2021). Abordagem Documental do Didático no planejamento e na implementação de uma atividade de modelagem matemática. *Educação Matemática Pesquisa*, 23(3), 33-69. https://revistas.pucsp.br/index.php/emp/article/view/56150

Skovsmose, O. (1990). Reflective knowledge: its relation to the mathematical modelling process. International Journal of Mathematical Education in Science and Technology, 21(5), 765-779.

Takinaga, S. S., & Manrique, A. (2023). L. Contribuições da teoria da objetivação para a análise do planejamento de tarefas de um professor de matemática envolvendo alunos com transtorno do espectro autista. *Educação Matemática Pesquisa*, 25(2), 189-210. https://doi.org/10.23925/1983-3156.2023v25i2p189-210.

Tortola, E., Silva, K. A. P., & Dalto, J. O. (2023). Professores em ação: (res)significando o ensino por meio da Modelagem Matemática. *Bolema,* 37(75), 168-193. https://doi.org/10.1590/1980-4415v37n75a09.