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Didactic knowledge of mathematics teachers in lesson study: students' difficulties and teaching strategies

Conocimientos didácticos de los profesores de matemática en el estudio en clase: dificultades y estrategias didácticas de los estudiantes

Connaissances didactiques des enseignants de mathématiques en classe: difficultés des élèves et stratégies pédagogiques

Conhecimento didático de professores de matemática em estudo de aula: dificuldades dos alunos e estratégias de ensino

Marta Cristina Cezar Pozzobon[1]
Universidade Federal de Pelotas
Doutora em Educação
https://orcid.org/0000-0003-3069-5627

Adriana Richit[2]
Universidade Federal da Fronteira Sul
Doutora em Educação Matemática
https://orcid.org/0000-0003-0778-8198

Mauri Luís Tomkelski[3]
Secretaria da Educação do Rio Grande do Sul
Doutor em Educação – Especialidade da Didática das Ciências
https://orcid.org/0000-0001-6395-7086

Abstract

The text discusses aspects of didactic knowledge mobilized by teachers who teach Mathematics when participating in a lesson study centered on the curricular topic division. The lesson study cycle involved three teachers from the public school system in the South of Brazil. The qualitative research was guided by the following question: What aspects of didactic knowledge are mobilized/developed by teachers who teach mathematics when participating in a lesson study? The qualitative analysis, based on content analysis, showed that the dynamics of the lesson study promoted the following aspects of didactic knowledge: students' difficulties and teaching strategies. As a result, the investigation points out that the lesson study allows participants to understand different difficulties of students in mathematical topics, reflecting on the nature of these difficulties and the implications for learning. The collaboration that characterizes the lesson study favors the sharing of

[1] martacezarpozzobon@gmail.com
[2] adrianarichit@gmail.com
[3] mauriluis@gmail.com

professional experiences that expand the repertoire of teaching strategies of the participants. It is pointed out that the lesson study expands the didactic knowledge of teachers, promoting professional development.

Keywords: Lesson study, Students' difficulties, Teaching strategies, Didactic knowledge, Mathematics education.

Resumen

El texto discute aspectos del conocimiento didáctico movilizado por los docentes que enseñan Matemática al participar en un estudio de clase centrado en la división temática curricular. El ciclo de estudio de la clase involucró a tres profesores del sistema de escuelas públicas del sur de Brasil. La investigación cualitativa fue guiada por la siguiente pregunta: ¿Qué aspectos del conocimiento didáctico son movilizados/desarrollados por los docentes que enseñan matemática al participar en un estudio de clase? El análisis cualitativo, basado en el análisis de contenido, mostró que la dinámica del estudio en clase promovió los siguientes aspectos del conocimiento didáctico: las dificultades de los estudiantes y las estrategias de enseñanza. Como resultado, la investigación señala que el estudio en clase permite a los participantes comprender diferentes dificultades de los estudiantes en temas matemáticos, reflexionando sobre la naturaleza de estas dificultades y las implicaciones para el aprendizaje. La colaboración que caracteriza el estudio en el aula favorece el intercambio de experiencias profesionales que amplían el repertorio de estrategias didácticas de los participantes. Se señala que el estudio en el aula amplía los conocimientos didácticos de los docentes, promoviendo el desarrollo profesional.

Palabras clave: Estudio en clase, Dificultades de los estudiantes, Estrategias didácticas, Conocimientos didácticos, Educación matemática.

Résumé

Le texte traite des aspects des connaissances didactiques mobilisées par les enseignants qui enseignent les mathématiques lorsqu'ils participent à une étude en classe centrée sur la division thématique du programme. Le cycle d'étude en classe a impliqué trois enseignants du système scolaire public du sud du Brésil. La recherche qualitative a été guidée par la question suivante : Quels aspects de la connaissance didactique sont mobilisés/développés par les enseignants qui enseignent les mathématiques lorsqu'ils participent à une étude en classe ? L'analyse qualitative, basée sur l'analyse de contenu, a montré que la dynamique de l'étude en classe favorisait les aspects suivants de la connaissance didactique : les difficultés des

élèves et les stratégies d'enseignement. En conséquence, l'enquête souligne que l'étude en classe permet aux participants de comprendre les différentes difficultés des élèves dans les sujets mathématiques, en réfléchissant à la nature de ces difficultés et aux implications pour l'apprentissage. La collaboration qui caractérise l'étude en classe favorise le partage d'expériences professionnelles qui élargissent le répertoire des stratégies d'enseignement des participants. Il est souligné que l'étude en classe élargit les connaissances didactiques des enseignants, favorisant le développement professionnel.

Mots-clés : Étude en classe, Difficultés des élèves, Stratégies d'enseignement, Connaissances didactiques, Enseignement des mathématiques.

Resumo

O texto discute aspectos do conhecimento didático mobilizados por professores que ensinam Matemática ao participarem de um estudo de aula centrado no tópico curricular divisão. O ciclo de estudo de aula envolveu três professoras da rede pública de ensino do Sul do Brasil. A investigação qualitativa foi orientada pela seguinte questão: Quais aspectos do conhecimento didático são mobilizados/desenvolvidos por professores que ensinam matemática ao participarem de um estudo de aula? A análise qualitativa, baseada na análise de conteúdo, evidenciou que a dinâmica do estudo de aula promoveu os seguintes aspectos do conhecimento didático: dificuldades dos alunos e estratégias de ensino. Como resultados, a investigação aponta que o estudo de aula propicia aos participantes compreenderem distintas dificuldades dos alunos em tópicos matemáticos, refletindo sobre a natureza dessas dificuldades e as implicações na aprendizagem. A colaboração que caracteriza o estudo de aula favorece a partilha de experiências profissionais que ampliam o repertório de estratégias de ensino dos participantes. Aponta-se que o estudo de aula amplia o conhecimento didático dos professores, promovendo o desenvolvimento profissional.

Palavras-chave: Estudo de aula, Dificuldades dos alunos, Estratégias de ensino, Conhecimento didático, Educação matemática.

Didactic knowledge of mathematics teachers in lesson study: students' difficulties and teaching strategies

In this text, we discuss professional teaching knowledge, which is "the basis of teachers' work and their identity" (Nóvoa, 2022, p. 3); that is, it represents the knowledge necessary for professional practice (Ponte & Oliveira, 2002; Colling & Richit, 2019; Richit, 2021). Professional teaching knowledge includes "teaching practice in the classroom," "other professional roles, such as tutoring students, participating in school activities and projects, interacting with community members and working in professional associations," in addition to "the teacher's vision of his/her own professional development" (Ponte & Oliveira, 2002, p. 4).

Such ideas lead us to understand that discussions about professional knowledge have been considered since the 1980s, with studies mainly by Shulman (1986, 1987), who proposes some categories regarding knowledge, such as content, general pedagogical, curriculum, pedagogical content, the purposes and values of education, educational contexts and students. Among this knowledge, we highlight the pedagogical content knowledge, which refers to knowledge for teaching relating content knowledge to teaching practice (Ball et al., 2008), especially when considering the teacher who teaches mathematics, as he/she has specific knowledge to teach that involves content knowledge and pedagogical content knowledge, enabling the exercise of teaching (Ball et al., 2008). The teacher has knowledge that is specific to this professional group, "oriented towards a practical activity (teaching mathematics to groups of students)," in addition to other theoretical, social, and experiential knowledge (Ponte, 2012, p. 3).

Given the above, this paper aims to address the professional knowledge of the teacher who teaches mathematics, mainly didactic knowledge, which involves classroom practice and is mobilized and developed in a lesson study. We understand lesson study as a formative professional development process that originated in Japan and was disseminated to other countries (Fujii, 2018; Stigler & Hiebert, 1999). Based on Day (2001), we argue that professional development is a continuous process encompassing teachers' needs, learning, knowledge, and personal and professional growth (Richit & Tomkelski, 2022).

Some researchers have discussed lesson study and didactic knowledge, such as Martins et al. (2023), Martins et al. (2024), and Bezerra and Quaresma (2023), who address the topic in initial education. One of the articles approaches key aspects of lesson study that promote the development of didactic knowledge, such as lesson planning, lesson

management, and collaborative work (Martins et al., 2023); another text discusses how lesson studies promote preservice teachers' knowledge of tasks, student work, and classroom communication (Martins et al., 2024), and the third addresses the development of didactic knowledge in supervised practicum (Bezerra & Quaresma, 2023). Santana et al. (2020, p. 107), in turn, deal with a formative process without defining whether it involves lesson studies; however, they report that a teacher participating in a formative process shows "positive perspectives regarding mathematical and didactic knowledge," evidencing reflective movements, mainly towards didactic knowledge, classroom processes, and student learning, which leads us to consider that there is a need to invest in the theme proposed in the article, mainly to raise some tensions, considering the research we have developed in the South of Brazil.

Given these ideas, we ask: What aspects of didactic knowledge are mobilized/developed by teachers who teach mathematics when participating in a lesson study? To answer this question, we considered a lesson study developed with three mathematics teachers from a public school in southern Rio Grande do Sul, Brazil. The topic of this lesson study was the division operation with natural numbers.

We argue that didactic knowledge for teaching mathematics is important for teachers to identify students' mathematical difficulties and for the exercise of teaching, as we weigh the teacher's responsibility in choosing and proposing teaching strategies from the perspective of learning. Therefore, when approaching a lesson study focused on division, we consider the learning of how to teach the topic, the curriculum, students' difficulties, and the possibilities of expanding teaching strategies.

Theoretical basis

In this section, we discuss lesson study as a process of a collaborative and reflective nature for teachers' professional development (Murata, 2011; Richit & Ponte, 2017), focusing on the exercise of mathematics teaching centered on teachers' professional practice, mainly in the classroom. It is important to emphasize that the main focus of the lesson study is student learning. However, professional learning is provided mainly by moments of collaboration and reflection promoted by that formative process (Richit et al., 2021). In the meantime, for those processes to be triggered, teachers "[...] work together, trying to identify students' difficulties, and prepare a lesson in detail, which they then observe and analyze in depth [...]", that is, "[...] they carry out a small investigation into their own professional practice, in a collaborative context [...]" (Ponte et al., 2016, p. 869).

Professional development is understood as a process in which the teacher is the protagonist of his/her learning to achieve success in teaching (Day, 2001). Professional development involves formative processes that contemplate teachers' personal lives, policies, and school contexts, expecting changes in professional practices, leading to teaching qualifications (Richit et al., 2024). It also involves teachers' learning in the sense of "improving professional capabilities," which includes "pedagogical and management skills and cultural and leadership contexts where they work" (Day, 2001, p. 44-45). The author believes schools must be committed to the continuing development of teachers, considering their personal lives, their needs and working conditions, and their students, as teachers should have "time and opportunities, as well as dispositions and abilities [...] to learn from others in the workplace and from elements outside the school" (Day, 2001, p. 45).

Such discussions lead us to consider lesson study as a professional development process centered on professional practice, with a collaborative and reflective character (Ponte et al., 2015; Quaresma & Ponte, 2015; Vieira et al., 2022). This formative process is standard in Japan, being part of the annual planning of schools and teachers' careers and responsible for the formative process as a continuing professional development process (Fujii, 2016). However, this practice takes on different characteristics depending on the cultural context, as Stigler and Hiebert (1999) warn. It is not enough to import the lesson study developed in Japan because it is not a matter of just adapting to the routine developed in a different culture and educational system. The lesson study must be adapted "[...] to the country's culture and the teachers' educational reality" (Richit, 2020, p. 3).

Hence, many studies propose several adaptations to implement this formative process, such as Perry and Lewis (2009) and Murata (2011), who suggest four stages: studying the curriculum, defining objectives, conducting the class, and reflecting. Fujii (2018), however, describes five stages: a) Goal setting: the goals and gaps for student learning and the formulation of the lesson theme; b) Lesson planning: collaborative planning of a lesson to achieve the proposed objectives; c) Research lesson: one member of the group teaches the class, the others observe and collect data; d) Post-class discussion: sharing of the data collected about learning, the structure of the lesson and other questions about teaching and learning; e) Reflection: recording the experienced cycle, based on reports with students' data and reflections.

According to those ideas, a group of teachers meets to plan a lesson, discuss students' resolution strategies, and teach and reflect on the process developed (Richit & Ponte, 2017; Quaresma & Ponte, 2019). Lesson study conceived as a professional development process is

based on teachers' teaching practice, with a focus on teaching and learning processes (Quaresma & Ponte, 2015; Ponte et al., 2016), in which the focus is on the class, understood as a unit that needs to be analyzed, improved, and planned (Stigler & Hiebert, 1999; Ponte et al., 2015). The cycles developed by lesson studies, based on teaching practice, promote the improvement of teaching by triggering four basic elements, which are: teachers' knowledge; teachers' belief and disposition, which involve knowing students' thinking and believing in their ability to learn; teachers' community, which involves collaborative practices; and curriculum, which considers teaching tasks and materials (Lewis, 2016).

According to Shulman (1986), professional knowledge for teaching mathematics involves content knowledge, curriculum knowledge, and pedagogical content knowledge. Ball et al. (2008) discuss these categories of knowledge emphasizing mathematical knowledge for teaching, which differs from the mathematical knowledge required of other professionals. For the authors, this knowledge involves recognizing and understanding students' mistakes, understanding what is taught and how it should be taught since, in order to teach, it is important to understand what is being taught, knowing how to choose the different representations of a mathematical object and select tasks and examples (Ball et al., 2008).

When teaching mathematics, we consider that there is a "[...] set of knowledge and competencies that we can call professional knowledge" (Ponte, 1999, p. 3), which interferes with teaching practices, understood as classroom practices, which involve didactic aspects, such as "the selection and design of tasks, lesson planning, the organization of students' work during class, and the creation of learning opportunities" (Martins et al., 2024, p. 107). To this end, Ponte (1999) highlights some knowledge, such as knowledge of teaching content, knowledge of the curriculum, knowledge of the student, knowledge of the instructional process (planning, teaching, and assessment of teaching practice), in addition to the teacher's personal and informal knowledge.

Based on Shulman (1986, 1987, 2014), we understand that teachers' professional knowledge differs from academic knowledge as it involves pedagogical content knowledge (PCK). According to the author, this knowledge goes beyond the knowledge of the theme, anchoring itself in the knowledge of the subject to teach because:

> Saying that a teacher must first understand both the content and the purposes, however, does not specifically distinguish the teacher from non-teachers. A mathematician is expected to understand mathematics, and a historian is expected to understand history. But the key to distinguishing the knowledge base to teach lies at the intersection of content and pedagogy, in the capacity of the teacher to transform the content knowledge he or she possesses into forms that are pedagogically powerful

> and yet adaptive to the variations in ability and backgrounds presented by the students. (Shulman, 2014, p. 217)

Based on these ideas, we understand that the teacher's professional knowledge generates the methods of selection, organization, and proposition of teaching from the perspective of students' understanding. Thus, the teacher must understand the different representations of content, the similarities between curriculum topics, and the possibilities for exploration and explanation. In other words, beyond understanding the specific knowledge of the subject he or she will teach, the teacher must understand the organization of content and concepts in the curriculum according to the level of education and the different strategies, methodologies, and assessment forms. These components constitute the PCK (pedagogical content knowledge), which outlines the different ways of representing content that enable students to understand what is being taught, consider the ways of teaching so that it becomes easier or more difficult for students to learn, and involve students' prior knowledge.

In this sense, Ponte (2012) deals with professional knowledge, mainly that related to classroom practice, didactic knowledge, which involves four aspects or domains: a) knowledge of mathematics, which encompasses the interpretations produced by the teacher about the subject that he/she will teach in class, and includes the concepts, procedures, different representations, "the internal and external connections to the subject" (Ponte, 2012, p. 4); b) knowledge of the student and their learning processes, which involves knowledge of the students, their interests, values, culture, and learning methods; c) knowledge of the curriculum, which encompasses the teacher's knowledge of the "purposes and objectives of teaching mathematics, as well as the organization of the content, knowledge of the materials, and forms of assessment to be used" (Ponte, 2012, p. 5); d) knowledge of the practice related to planning, the lesson plan, concepts about tasks, "conducting mathematics classes, namely the ways of organizing students' work, creating a culture of learning in class," in addition to "communication and assessment of students' learning and the teacher's teaching" (Ponte, 2012, p. 5).

Such discussions lead us to highlight the need for teachers to understand students' mathematical difficulties because, as Ponte et al. (2016, p. 870) point out, in lesson study experiences, teachers can learn about the topic they teach, curriculum guidelines, among other learning about "students' reasoning processes and difficulties and the very classroom dynamics." In addition to identifying students' difficulties, teachers must propose teaching strategies that promote learning, which can be developed through professional learning produced through reflections on teaching. These enable the analysis of "the difficulties their

students face in learning a given topic, what they learned, how they responded to particular representations, questions, and tasks" (Serrazina, 2014, p. 1096).

Therefore, in this text, we deal with the didactic knowledge developed by mathematics teachers in a classroom study that involved teaching the topic of division with natural numbers. This topic is often complex for students to understand, as "learning division is much more than knowing how to use the traditional algorithm," it involves recognizing this "operation in different situations," using the "relationship between division and multiplication" and developing "a web of numerical relations that allow calculations to be made flexibly, with the underlying properties of these operations" (Mendes, 2013, p. 6-7).

Methodology

The research developed presents a qualitative approach, in which we employ strategies called "qualitative," which, in this case, are the observation and data collection through the insertion of the researcher in the school space, with a description of the investigation, with emphasis on the research process and with the production of meaning from the data (Bogdan & Biklen, 1994). With this, we approached a public school in the southern Rio Grande do Sul, which serves students from early childhood education to the 9th grade of elementary school (middle school). At this school, we developed a lesson study cycle from March to May 2023, with weekly meetings with three mathematics[4] teachers who teach middle school classes.

The teachers who participated in the lesson study are identified by fictitious names to preserve their identities, following ethical guidelines, which point out the need to guarantee the rights of participants, evaluating, preventing, and avoiding "possible harm" (Brasil, 2016, p. 44). The project was submitted to the institution's Ethics and Research Committee under Certificate of Presentation of Ethical Appreciation (CAAE) 59832922.0.0000.5317 and opinion 065686/2022. The participating teachers signed the Free and Informed Consent Form (FICF).

The three teachers –Carla, Maria, and Sandra (fictitious names)– are licensed in mathematics, and all work in middle school. Maria has been teaching mathematics since 2018, and in 2023, she was teaching two 6th-grade classes. Sandra has been working as a teacher since 2007 and was teaching 7th and 8th-grade classes. Carla has been a teacher since 2008 and was responsible for the 8th and 9th-grade classes. The researcher, who assumed the role of facilitator –understood as a member of the group who organizes, leads, and collaborates with the discussions and propositions (Clivaz & Clerc-Georgy, 2020)– and an

[4] This school has only three teachers who teach mathematics in the final years (middle school).

observer –a master's student in a postgraduate course at a federal public university in southern Brazil, also participated in the lesson study.

In this study, we present the teachers' statements, which were audio recorded, covering sessions 1 to 6 and 8; and session 7 –video recorded. The recordings and video were then transcribed and analyzed. In the table below, we describe the stages of the lesson study and sessions developed by one of the authors of the article, based on what was proposed by Fujii (2018).

Table 1.

Stages and sessions of the lesson study

Stages	Sessions
Goal Setting	Session 1: Presentation of the lesson study and definition of the topic to be taught.
	Session 2: a) Recap of the schedule; b) Discussion about teaching division and students' difficulties; c) Study of curriculum documents (Base Nacional Comum Curricular – BNCC (Brasil, 2018)) on the topic to be taught; d) Study of some materials on teaching division.
	Session 3: a) Definition of the lesson objective; b) Reading part of the article by Ponte et al. (2015) b) Studies on division teaching.
Planning the research lesson	Session 4: a) Exploration of tasks involving division; b) Organization of diagnostic tasks; c) Discussion of the lesson plan based on the article: Ponte et al. (2015); d) Ideas for lesson planning.
	Session 5: a) Resumption of some tasks involving division; b) Organization of the lesson task, considering the context proposed by teacher Maria and the adaptation of studied tasks; c) Anticipation of the students' resolution methods.
	Session 6: a) Closing the task; b) Continuation of the discussion on the anticipation of how students will solve the problem; c) Organization of the times and directions for each stage of the planning (introduction of the lesson, independent work by the students, collective discussion and closing of the lesson).
Research lesson	Session 7: The lesson was led by Maria in a 6th-grade class.
Post-class discussion and reflections	Session 8: Discussion about the lesson and reflections on the lesson study cycle.

Given the stages and sessions, we explain that the topic of division with natural numbers was chosen collectively by the teachers, mainly due to the students' difficulties highlighted since the first session. In session 1, the teachers report that one of the difficulties

highlighted in the 6th to 9th-grade classes is related to division, as it seems that "everyone needs to [revisit this concept]" (Sandra), or as Carla says: "[...] coming from the perspective of fractions, I thought about division, the relationships between multiplication and division, which is one of the operations they have the most difficulty with." Maria reinforces this aspect, saying that the students have many problems understanding it. He adds: "[Students find it hard] to put together [the operation] because if they think about division, there are three operations that they have to put together [...]."

For the analysis of the materials, we established some approximations with the content analysis proposed by Bardin (2021), in which we considered: a) Floating reading, in which we read the transcripts in full, highlighting some impressions, aspects or themes that caught our attention or that were recurrent; b) Constitution of the corpus, which consisted of the selection and choice of some excerpts related to didactic knowledge; c) Exploration of the selected materials, realizing that some excerpts were similar or appeared recurrently in the different sessions of the lesson study; d) Categories and subcategories: when exploring the materials, we noticed that some themes emerged, that is, they were recurrent in the transcripts. This led us to cut out the excerpts, and, considering the theoretical framework, we organized the categories "mathematical difficulties" and "teaching strategies." Given these categories, we organized the subcategories of the analyzed materials, which we show in the figure below.

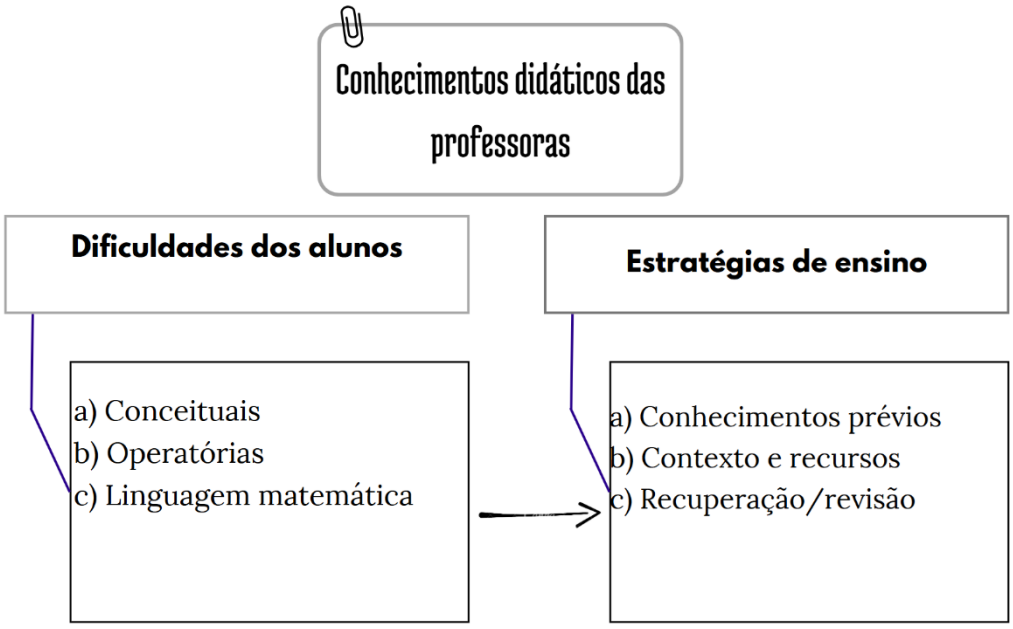

Figure 1.

Categories and subcategories

The figure explains the categories and subcategories and shows a link between the categories "student difficulties" and "teaching strategies." We consider that students' struggles affected the teaching of the teachers who participated in the lesson study in the sense that they proposed teaching strategies to help students overcome their learning difficulties. Thus, in the next section, we consider some results and discussions.

Results and discussions

Student difficulties

We describe the didactic knowledge of the teachers who participated in the lesson study, highlighting the category of students' difficulties, organized into the subcategories: conceptual, operational, and mathematical language. In this category, one of the justifications for the students' mathematical problems teacher Maria indicates: "It's because, during the pandemic, [the students] didn't do [the activities on the topic of division]. Last year all [students] were promoted. And now we always have to include the whole class" (Session 4).

Regarding the *conceptual difficulties*[5], the teacher was referring to the year 2020, when, due to restrictions from the COVID-19 pandemic, teachers sent school assignments to students via Google Forms, WhatsApp, or Facebook, which were seldom answered and returned. In 2021, the guidelines from education departments regarding teaching activities were redefined, and in 2022, even with the return of in-person classes in public schools, teachers had to promote all students. From this perspective, when referring to the difficulties, teacher Sandra indicated hardships in teaching because the students "are very dissimilar. We already had classes that had very different [learning] levels, but now, it's really made public" (Session 1). It seems that the pandemic shows what they were already facing, expanding the framework of students' non-learning, which was the focus of the lesson study since the first session. The teachers say that these difficulties are perceived in the students' records, especially regarding the mathematical concepts of subtraction, as shown in the following excerpt:

> Maria: [Currently] subtracting [has been challenging to students]. [...] when I received the students, that was before, [...] when they didn't know, at least they tried [to solve **the problem] with a ball, with each little ball, they crossed, which was the subtraction method, they crossed out**[6]. Somehow, they would try to find out. Now, not even that.
> Researcher: They don't try to solve it, do you think?

[5] The expressions in italics refer to subcategories.
[6] We use bold to highlight important parts for the analysis.

> Maria: They can't try, they can't do it. [...] To pick up and place the ball, even if it's just a small ball, **and then start cutting**. (Session 3).

The teacher points out that conceptual difficulties also involve other mathematical operations:

> Maria: Look, **I'm with the sixth grade, and I'm still working on addition and subtraction. I haven't even reached multiplication**, they still make a lot of confusion. [...] I'm still going to have to keep picking up what I'm doing, have to keep going... I'll still have to continue **taking from the fourth, from the fifth grade** (Session 2).

Maria adds that students do not try to solve the proposed tasks, even using pictorial records, especially in tasks that involve elementary mathematical operations, such as subtraction. The teacher reports that the difficulties involve other mathematical operations, such as addition, multiplication, and division. Moreover, as highlighted in the dialogue presented below, the difficulties encompass the concepts of operations.

> Maria: Then I say, let's **divide by two, the half of it**, how much is it? Until you understand that it is half. 30 divided by two, how much is it? Half of 30? Then they know.
> Carla: **Is it an "adding," is it a "taking out "**...?
> Sandra: They just ask, what account do I have to make?
> Carla: **Is it to divide or is it "times**"?
> Maria: It seems like we never get out, never get out of this (Session 2).

The teachers' reflections indicate that the conceptual difficulties they pointed out require understanding mathematical knowledge for teaching, as Ball et al. (2008) teach. Such knowledge goes beyond common mathematical knowledge, which includes knowing mathematics but is not specific to teaching. Specialized mathematical knowledge, from a learning perspective, includes understanding students' difficulties and proposing ways of teaching (Ball et al., 2008). Thus, identifying conceptual difficulties related to operations requires teachers to master mathematical knowledge and how to teach it, considering procedures, representations, and patterns... Or rather, "teachers can learn from their teaching by analyzing it, especially the difficulties their students face in learning a given topic, what they learned, how they responded to particular representations, questions and tasks" (Serrazina, 2014, p. 1056).

The above leads us to highlight the *operational difficulties* identified by teachers, mainly regarding division.

> Sandra: They don't know the division?

> Maria: [...] they once told me: That **equation with that L**, we don't remember how to do it (Session 3).
> **Maria:** It's that or **is the rest**, they asked me a lot, it isn't this one here because there the **algorithm is complex**, it works with multiplication, it works with division, with subtraction. (Session 2)
> Maria: [...]. Because their doubt is... We were putting it straight into the calculation first. Then, when you gather together the terms of the equation... So they told me: Teacher, **when we set up the equation, I don't know the result**. Which of these numbers do I...
> Carla: Where do you put it? And where do you put the result of the calculation?
> Maria: Which of these values is the result? (Session 6).

Teacher Maria believes that the students had problems using division to solve the task developed in the lesson study because they did not understand how to structure the operation algorithm, since the difficulties persisted despite the recaps during lessons since the beginning of the year (February to April).

> Maria: Well, the division is very difficult for them to understand. [...]. So, it took us several days of discussions in the classroom. And there **was always the doubt about how to set up the terms of the equation**. When I put the account on the board, they couldn't understand **how they had to set it up and make the algorithm**. So, we had several lessons addressing that division (Session 8).

Teachers point out the difficulty in the division algorithm, especially regarding "setting up the operation," problems in understanding the operational procedures for resolution, that is, mathematical notations, which involve external representation systems, which involve organized systems, which are structured through rules and properties (Martí & Pozo, 2000). According to Mandarino and Belfort (2005), the division algorithm is hard for students, as it involves understanding the numbering system, the basic facts of operations, and other operations, such as addition, subtraction, and multiplication.

In this context, we bring up the students' difficulties with *mathematical language*, mainly with mathematical terms, such as half, subtraction, and product. In Session 4, Teacher Maria says: "In the last class, [...], we played *Stop* of Mathematics, and one of the columns was to multiply by two, the **double,** and divide by two, the **half**. [...] When I said **half or double, they didn't understand** " (Session 4). In a conversation in Session 2, the teachers point out the students' difficulties with terms such as subtraction and product.

> Maria: There was also a lot of that thing about them [students] **not understanding me**. I **say subtraction, then I would have to stop...** No, **I have to say: "It is a 'take away' operation**," then, you will subtract.
> Sandra: Difficult thing.

> Maria: A ‘take away’ operation, it is a ‘take away’ operation, it is a ‘take away’ operation, guys, because I wasn’t used to saying “a ‘take away’ operation.” [...] In the fifth grade, they could subtract, so I just continued.
> [...]
> Carla: **This thing of names**, I put a question on the test: **product**? What is it?
> Maria: No way.
> [...]
> Carla: Guys, **product is multiplication**.
> Facilitator: It is the school’s role to work on this mathematical language...
> Maria: It’s just that **we couldn’t**, because yours are probably eighth and ninth [grades] (Session 2).

The statements highlight the difficulties with mathematical language and understanding mathematical terms, which are part of a system specific to the school subject. Mathematical language represents a community’s communication system, but it must be complemented by natural language so that students can access it (Menezes, 2000). Even though mathematical language has its own symbols and codes, it presents other modes of expression, such as written, oral, and pictorial language (Usiskin, 1996).

Students’ difficulties in mathematical language make teachers use other expressions or terms from their mother tongue, which raises some concerns because, even though it is teachers’ responsibility to understand students’ mathematical thinking (Ball et al., 2008), they must also teach what should have already been learned and advance what is proposed for the school year. In other words, students’ difficulties did not allow the teachers to advance in the content. For example, teacher Maria, from the 6th grade, reviewed operations with natural numbers (addition, subtraction, multiplication). During the lesson study, she could not advance to the division operation with natural numbers because students had problems understanding the use of the operation (ideas) and the algorithm.

Therefore, teachers need to mobilize and articulate different aspects of didactic knowledge to understand students’ difficulties, both conceptual ones involving mathematical operations and operational ones, mainly division and mathematical language. Teachers need mathematical knowledge of topics and procedures; however, they also need mathematics for teaching, which involves evaluating students’ answers (their errors and the reasons for making them), explaining things in a way that is understandable to students, using mathematical representations (Ball et al., 2005). In other words, the mathematical knowledge necessary for teaching is made up of both mathematical knowledge (mathematical reasoning) and pedagogical knowledge (pedagogical thinking) (Ball et al., 2005).

Teaching strategies

This section addresses the teaching strategies teachers mobilize in a lesson study cycle. Strategies are defined as teaching actions that teachers propose to plan tasks and situations that provide opportunities for the development of student learning. According to Ponte et al. (2016), in the different stages of a lesson study, teachers plan a lesson, constructing or adapting a mathematical task, and predicting and formulating different teaching strategies collaboratively to assist student learning. One of these strategies involves *mobilizing prior knowledge*, which, according to the teachers, involves starting from the simplest to the most complex, starting with what the students find easiest and going deeper.

> Maria: [...] **starting with something they know how to do**, for example... [...]. I try to make it easier, sometimes, by putting in examples with five, because they follow me on the five times table on the board. [...] We always put, which is a resolution [SMED], it always starts from the **smallest to the largest**. (Session 4).
> Carla: As they already **have this difficulty with division, they will no longer try something they have difficulty with**. They will try the things they are sure they can do: addition, multiplication.
> [...]
> Maria: Because I always worked in **increasing, multiplication always comes [with it]**. And in reducing, which would be done by division, we didn't work.
> [...]
> Carla: I was wondering, could it be **that some word that gave the idea of division**, that we put there, distribute, divide, some word that would make them realize the division [...] something that was the word, that referred to the division... (Session 8).

It is important to highlight that in lesson studies, the choice of the topic to teach is triggered by the identification of students' learning difficulties and what is proposed in the curriculum programs (Ponte et al., 2014; Richit, 2022). In other words, in the lesson study experience, students' difficulties "[...] are guiding principles when teachers consider it essential to address a topic in which students present distinct and frequent difficulties throughout their school career [...], aiming to understand and resolve such difficulties" (Richit et al., 2023, p. 116-117). As emphasized by the authors, the actions triggered by the lesson study go beyond understanding students' difficulties, anchoring themselves in promoting student learning. To this end, planning and evaluation processes and teaching intentions and interventions are essential when teaching mathematics.

Another strategy highlighted in the analysis involves the *context and resources*, which, according to the teachers, enable students to learn. In this sense, the following excerpts refer to the context for teaching the division algorithm.

Carla: Even the **algorithm can be contextualized**, for example, to explain where I put the answer, where I put what's left. They are always in doubt, they don't know where to put what.
Sandra: Which way they go.
Carla: **I contextualize this algorithm with purchasing**. So, the dividend would be how much I have, the divisor would be how much does the item I want to buy cost, and then we go, how many things we can I buy. They go through the multiplicative process and then how much I spent, so I have to discount from what I had, how much I spent and then what was left, how much was left, and our remainder. Always **giving meaning to this assembly** and then they understand it more easily. (Session 1).
Carla: I told you I talk to them about money.
Maria - Yes.
Carla: I say... I have 258 reais. I don't need to take all my money to buy something that costs 4. So, if it costs 4 reais, I might consider getting it... I can take 2 reais to buy something that costs 4. [...] Then I'll take... With 25 reais I can already buy at least one object that costs 4. Now I'll have to look for how many. How many things cost 4 reais?
[...].
Researcher: Do you do the **subtractive idea**?
Maria: Yes, **their idea**... Which makes it more real for them.
Researcher: But, in the algorithm?
Carla: Yes, **in the idea of the algorithm**. (Session 6).

They also consider the context when dealing with subtraction with the idea of difference, with the use of height comparison, as teacher Carla says: "This idea of difference for them, they will mature it much later, the difference is subtraction and then, how am I going to find out the difference between my height and yours? And then how much I am higher than you, or what is missing for us to grow?" (Session 1).

Toledo and Toledo (2010, p. 155) consider that some teachers teach the algorithm of division using successive subtractions, known as the "American process," which "is linked to the idea of 'dividing equally,' and reproduces exactly what the child does to divide a quantity of objects equally." In the context of these strategies, the long division method is mainly considered, which, according to Centurión (1994), involves the ability to estimate, which is gradually built by children until they discover how many fit into a given quantity. As Gomez (1991) discusses, the subtractive method involves the dividend, trying to reach the divisor through its multiples.

Such strategies are part of the field of didactics (Ponte, 2012), referring to the professional knowledge that constitutes the specificity of teaching work in mathematics teaching. As Shulman (1987) teaches, this dimension of knowledge, PCK, goes beyond content knowledge, enabling the proposition of teaching strategies by considering the particularities of the content, the connections in the curriculum program, and students'

learning difficulties and easiness, among other aspects. Teacher Maria describes the teaching strategy using teaching resources.

> Maria: We are working on multiplication now, we are working with little problems and with concrete material. [...] Working with concrete material so that they..., so that they can understand this function because sometimes, if there is no drawing, it is so difficult for them to understand! [...]. You know I addressed **multiplication with them last week, on EVA squares**. [...] so, some would simply put 4 times 3, then 4 in 3 piles of... I cut a few squares of colored EVA, I cut them and work with quantities with them. And some now, already, some groups have already put it in that rectangular shape, in an organized way (Session 3).

In this excerpt, the teacher refers to using resources to explore the rectangular arrangement, which facilitates the exploration of the commutative property of multiplication; that is, 4 times 5 gives the same result as 5 times 4. Representation with rectangular models can facilitate "the transition between the level of calculation by counting and the level of calculation by structuring" (Rocha & Menino, 2009, p. 130), with "linear structures being linked to repeated addition procedures" and rectangular structures "being linked to multiplicative procedures" (Rocha & Menino, 2009, p. 110). Given what the teacher proposed, we highlight the strategies involving resources and materials to improve teaching, focusing on student learning.

These difficulties presented by the students led teacher Maria to point out some teaching actions:

> Maria: As I told you last time, I had to **ask teacher Rosa to help me introduce division** last year.
> Researcher: Because it hadn't it been worked on?
> Maria: It hadn't, it was remote, and we don't know who made those little leaflets.
> Researcher: Did you get the sixth grade?
> Maria: I had to give [the content of] the **fourth, fifth, and sixth in one year**. You have to teach all of this. I had **to seek strategies with the curriculum teacher**, Rosa. She explained that she takes pens and divides them; gathers the pens; divides them by two so they can get the idea. (Session 2)

The teacher points out that learning difficulties generated some teaching actions to recover content/concepts not taught or not learned in previous years. Maria reaffirms the need to review unlearned content, pointing out that even with recovery activities, students have difficulty learning.

> Maria: Because until then, I haven't passed [the content], **I haven't recapped division yet**. Look, because multiplication is still complicated.
> Sandra: Yes, I have some [students] already coming to support classes.
> Maria: Yeah, I am **stuck in multiplication**. (Session 3)

We consider these statements bring the need for strategies to recover previous years, generating the search for ways to teach the concepts planned for the initial years, which are then deepened and formalized in the final years of elementary school. It is important to highlight that professional teaching knowledge encompasses the various knowledge necessary to teach mathematics, among them, the relevance of knowing students and their learning processes, i.e., their interests, difficulties, and ways of teaching, from the perspective of student learning (Ponte, 1999; Richit, 2020).

According to Ball et al. (2008), it is the role of teachers to familiarize themselves with students' mathematical thinking, interpreting their use of language to propose tasks that promote student learning. The authors draw attention to the fact that teaching actions, which are teachers' responsibility, involve understanding students' learning demands, proposing plans, and selecting the content and tasks that will be considered in the classroom. In other words, it is the teacher's role to choose different ways of approaching a topic to be taught, enabling different ways of representing, from the perspective of students' understanding (Ball et al., 2008).

Final considerations

The analysis indicates that lesson study enables the development of different aspects of didactic knowledge from which the teacher reflects on teaching practice and ways to promote student learning. Among these aspects, in this study, two main categories stood out: students' difficulties and teaching strategies.

Regarding *students' difficulties*, the analysis indicates that participation in the lesson study allows teachers to identify the problems and reflect on ways to overcome them. Furthermore, students' difficulties are guidelines for teachers to choose the topic for planning the research lessons (Richit et al., 2023), as well as teachers' teaching difficulties or the time of inclusion in the curriculum (Fujii, 2014). In a lesson study, there is a concern about students' difficulties and the planning processes to resolve them and promote student learning. To this end, we highlight the importance of teachers' participation in a lesson study. Teachers are encouraged to share the problems experienced in the classroom and think of alternatives for teaching increasingly committed to student learning.

As regards the *teaching strategies*, the analysis highlights that by relying on collaboration and reflection, lesson study favors sharing and discussing different teaching experiences in the classroom, thus expanding the repertoire of classroom strategies of the participating teachers. We consider that teaching strategies make up the repertoire of professional teaching knowledge, which were mobilized by teachers when proposing teaching methods aimed at recognizing students' prior knowledge, using context (algorithm with purchase and successive subtractions) and resources (rectangular layout), and recovering content/concepts from previous years. Given the appropriation of different teaching strategies, the teacher can develop lesson planning that is aligned with students' needs, contributing to student learning.

Therefore, lesson study can expand teachers' didactic knowledge from the professional development perspective. However, we emphasize that to continue thinking about the effects of students' difficulties on teaching strategies, we must discuss further the impacts of lesson study on their learning, considering more than one cycle with the same group of teachers. We also consider it relevant to expand teacher participation in other lesson studies to discuss teacher learning as students, as this formative process aims to produce changes in teachers' professional and teaching practices and promote student learning.

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