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Mathematics teacher's specialized knowledge of pre-service initial years teachers

El conocimiento especializado del profesor que enseña matemáticas a los futuros profesores de los años iniciales

Les connaissances spécialisées de l'enseignant qui enseigne les mathématiques aux futurs enseignants des premières années

O conhecimento especializado do professor que ensina matemática de futuros docentes dos anos iniciais

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Abstract

The objective of this research is to analyze evidence of the Specialized Knowledge of the Teacher who teaches Mathematics to future teachers of the initial years. Based on an empirical, qualitative and descriptive perspective, we developed research with 6 students of the Pedagogy Degree course of a public university in the northwest of Paraná. Our main results reveal that the participants of the research show evidence of having the Knowledge of Characteristics of Mathematical Learning, since they identified the facilities and difficulties of the division process. Regarding the Knowledge of Mathematics Teaching, they showed themselves efficient in indicating how to use teaching materials and resources to improve the teaching of Mathematics, although they did not explain their limitations. Regarding the Knowledge of Mathematical Learning Patterns, only a third knew how to relate the mathematical skill that deals with the idea of equality for the sentences of addition and subtraction of two natural

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numbers to its thematic unit. In the Knowledge of Topics, they did not present any evidence, since they did not explain divisibility or the possibility of using the decomposition method. In the Knowledge of the Structure of Mathematics, they demonstrated an understanding of the process of simplifying fractions. Finally, the students demonstrated evidence of Knowledge of Mathematical Practice by recognizing the equivalence between two fractions. The results point to the need for improvement in the mathematical training of pedagogues, especially about understanding the limitations of teaching resources, didactic sequencing, and knowledge of mathematical procedures and rules.

Keywords: Specialized knowledge of mathematics teachers, Teachers teaching mathematics, Teaching-learning, Bachelor's degree in pedagogy.

Resumen

El objetivo de esta investigación es analizar la evidencia del Conocimiento Especializado del Profesor que enseña Matemáticas a futuros docentes en los años iniciales. Basándonos en una perspectiva empírica, cualitativa y descriptiva, desarrollamos una investigación con 6 estudiantes de la carrera de Pedagogía de una universidad pública del noroeste de Paraná. Nuestros principales resultados revelan que los participantes de la investigación muestran indicios de poseer Conocimiento de las Características del Aprendizaje Matemático, ya que identificaron las facilidades y dificultades del proceso de división. Respecto al Conocimiento de la Enseñanza de las Matemáticas, fueron eficientes al señalar cómo utilizar los materiales y recursos didácticos para mejorar la enseñanza de las Matemáticas, aunque no explicaron sus limitaciones. Respecto al Conocimiento de los Estándares de Aprendizaje Matemático, sólo un tercio supo relacionar la habilidad matemática que trata la idea de igualdad para oraciones de suma y resta de dos números naturales con su unidad temática. En Conocimiento de Tópicos no presentaron ninguna evidencia, ya que no explicaron la divisibilidad ni la posibilidad de utilizar el método de descomposición. En Conocimiento de la Estructura de las Matemáticas (KSM), demuestran una comprensión del proceso de simplificación de fracciones. Finalmente, los estudiantes mostraron señales de Conocimiento de la Práctica Matemática al reconocer la equivalencia entre dos fracciones. Los resultados apuntan a la necesidad de mejorar la formación matemática de los pedagogos, especialmente en lo que se refiere a la comprensión de las limitaciones de los recursos didácticos, la secuenciación de la enseñanza y el conocimiento de los procedimientos y reglas matemáticas.

Palabras clave: Conocimiento especializado del profesor de matemáticas, Profesores que enseñan matemáticas, Enseñanza-aprendizaje, Licenciada en pedagogía.

Résumé

L'objectif de cette recherche est d'analyser les preuves des connaissances spécialisées de l'enseignant qui enseigne les mathématiques aux futurs enseignants dans les premières années. Sur la base d'une perspective empirique, qualitative et descriptive, nous avons développé une recherche auprès de 6 étudiants du cursus de Licence en Pédagogie d'une université publique du nord-ouest du Paraná. Nos principaux résultats révèlent que les participants à la recherche montrent des signes de connaissance des caractéristiques de l'apprentissage mathématique, puisqu'ils ont identifié les facilités et les difficultés du processus de division. En ce qui concerne les connaissances en enseignement des mathématiques, ils ont été efficaces pour indiquer comment utiliser le matériel et les ressources pédagogiques pour améliorer l'enseignement des mathématiques, même s'ils n'ont pas expliqué leurs limites. En ce qui concerne la connaissance des normes d'apprentissage mathématique, seulement un tiers savait comment relier la compétence mathématique qui traite de l'idée d'égalité pour les phrases d'addition et de soustraction de deux nombres naturels à son unité thématique. Dans Connaissance des sujets, ils n'ont présenté aucune preuve, car ils n'ont pas expliqué la divisibilité ou la possibilité d'utiliser la méthode de décomposition. Dans Connaissance de la structure des mathématiques, ils démontrent une compréhension du processus de simplification des fractions. Enfin, les élèves ont montré des signes de Connaissance de la Pratique Mathématique en reconnaissant l'équivalence entre deux fractions. Les résultats soulignent la nécessité d'améliorer la formation mathématique des pédagogues, notamment en ce qui concerne la compréhension des limites des ressources pédagogiques, le séquençage de l'enseignement et la connaissance des procédures et règles mathématiques.

Mots-clés : Connaissances spécialisées du professeur de mathématiques, Enseignants qui enseignent les mathématiques, Enseignement-apprentissage, Diplôme en pédagogie.

Resumo

O objetivo desta pesquisa é analisar indícios do Conhecimento Especializado do Professor que ensina Matemática de futuros docentes dos anos iniciais. Pautada em uma perspectiva empírica, qualitativa e descritiva, desenvolvemos uma pesquisa com 6 estudantes do curso de Licenciatura em Pedagogia de uma universidade pública do noroeste do Paraná. Nossos principais resultados revelam que os participantes da pesquisa apresentam indícios de possuir o Conhecimento de Características da Aprendizagem Matemática, uma vez que, identificaram as facilidades e dificuldades do processo da divisão. A respeito do Conhecimento de Ensino da

Matemática, mostraram-se eficientes em apontar como utilizar os materiais e recursos didáticos para melhorar o ensino da Matemática, embora não tenham explicado as suas limitações. Quanto ao Conhecimento dos Padrões de Aprendizagem Matemática, apenas um terço soube relacionar a habilidade matemática que trata da ideia de igualdade para as sentenças de adição e subtração de dois números naturais à sua unidade temática. No Conhecimento de Tópicos, não apresentaram indícios, uma vez que não explicaram a divisibilidade nem a possibilidade de utilizar o método da decomposição. No Conhecimento da Estrutura da Matemática, demonstram compreender o processo de simplificação de frações. Por fim, os estudantes evidenciaram ter indícios do Conhecimento da Prática Matemática ao reconhecerem a equivalência entre duas frações. Os resultados apontam para a necessidade de aprimoramento na formação matemática dos pedagogos, especialmente no que tange à compreensão das limitações dos recursos didáticos, no sequenciamento didático e no conhecimento dos procedimentos e regras matemáticas.

Palavras-chave: Conhecimento especializado do professor de matemática, Professores que ensinam matemática, Ensino-aprendizagem, Licenciatura em pedagogia.

Mathematics teacher's specialized knowledge of pre-service initial years teachers

Introduction

The National Common Curricular Base (BNCC) is a document that regulates the curriculum of Brazilian basic education and structures the first and last years of the *Ensino Fundamental* and *Ensino Médio*[4]. This document delineates a set of skills and competencies that encompass all the content students need to learn, structured into thematic units and learning topics which aim to organize the specific knowledge of each subject (Brasil, 2018).

In basic education, mathematical content typically follows a particular sequence, creating a linear process for learning construction where prior knowledge is essential for building new understanding. Consequently, students learn based on the contents of previous grades. Nogueira et al. (2016) point out that "the initial years of schooling are decisive for the construction of the cornerstone that supports the later contents" (Nogueira et al., 2016, p. 5; authors' translation). Thus, teachers who work in the initial years of schooling hold significant importance in the process of forming the introductory concepts that will support new knowledge.

According to the Law of Guidelines and Bases of National Education (LDBEN), teachers with higher education degrees in Pedagogy and other teaching disciplines are eligible to work in basic education. However, specific degrees are required for teaching in the final years of *Ensino Fundamental*[5] and *Ensino Médio*. The National Curriculum Guidelines for the Pedagogy education program, article 4, states that "the Pedagogy course is designed to train teachers to perform teaching functions in early childhood education and the first years of *Ensino Fundamental*[6]" (Brasil, 2006, p. 6).

In addition, CNE/CP No. 5/2005 establishes that in the first years of *Ensino Fundamental* students should be taught written and mathematical language (Brasil, 2005). Hence, this legislation recognizes the importance of Mathematical studies for prospective teachers enabling them to facilitate learning in their professional practice. Therefore, initial grade educators need to develop and utilize specialized knowledge to provide Mathematical instruction at this educational level.

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[4] *Ensino Médio* refers to the last three grades of Brazilian basic school education. It is close to the education level of High School students in the U.S.

[5] Also called *Ensino Fundamental II*, it refers to the period between the 6th and 9th grades of Brazilian basic school education. It is close to the education level of Middle School students in the U.S.

[6] Also called *Ensino Fundamental I*, it refers to the period between the 1st and 5th grades of Brazilian basic school education. It is close to the education level of Elementary School students in the U.S.

Carrillo et al. (2018) described a model named Mathematics Teacher's Specialized Knowledge (MTSK) characterizing the unique expertise of teachers of this discipline. Within the context of work focused on the MTSK model in pedagogy courses, Conceição (2019) investigated third-year students of a Pedagogy course at a public university in São Paulo focusing the research on this model for teaching parallelism and perpendicularism in Plane Geometry. In this regard, Conceição (2019, p. 121) noted that this work prepared the prospective teachers with knowledge that would allow them to "provide their students with learning that helps them understand the reasoning underlying the mathematical content, rather than just memorizing formulas and concepts".

Similarly, Meireles (2021) investigated the teaching of nets focused on the MTSK model with sixth semester's Pedagogy undergraduates. The author emphasized the model as relevant for identifying and organizing the specialized knowledge revealed by prospective teachers, because the MTSK allows for more specific identification of these understandings. In a broader manner, Moriel, Junior, and Wielewski (2017, p. 131) state that the MTSK, besides guiding teachers in Mathematical instruction, serves as a methodological tool for analytical exploration of this knowledge. Nonetheless, Santos and Denardin (2022) reviewed the literature studying theses and dissertations on MTSK in undergraduate courses and concluded that further research is necessary, thus indicating a gap in the discussion that underscores this paper's relevance.

In light of these considerations, the present study questioned which MTSK subdomains the pre-service teachers, who have previously studied Mathematics disciplines, demonstrated signs of understanding? Hence, this research aims to analyze the Mathematics Teacher's Specialized Knowledge of prospective educators for the initial years.

The term "signs" is used, because the MTSK is fairly complex and wide-ranging, rendering unfeasible to verify, by means of a single questionnaire, whether a person has mastered all knowledge related to the subdomains. However, it is possible to identify specific signs of each of them.

The Mathematics Teacher's Specialized Knowledge and its correlations with undergraduate students of Pedagogy

Mathematics Teacher's Specialized Knowledge consists of two domains. The first, the **Pedagogical Content Knowledge (PCK)** addresses the following subdomains: *Knowledge of Features of Learning Mathematics (KFLM)*, *Knowledge of Mathematics Teaching (KMT)*, and *Knowledge of Mathematics Learning Standards (KMLS)*. The second domain is the

Mathematics Knowledge (MK), that covers the subdomains: *Knowledge of Topics (KoT)*, *Knowledge of the Structure of Mathematics (KSM)*, and *Knowledge of Practices in Mathematics (KPM)*.

Therefore, the MTSK is a model of knowledge that is acquired/built and utilized by teachers who teach Mathematics. This kind of expertise enables the realization of challenges and possibilities in the teaching-learning of Mathematics (Montes et al., 2024). Carrillo et al. (2018) states that:

> The group made an interpretation of the approach to explore the specialized knowledge of Mathematics teachers, with the objective of developing a model that allows this knowledge to be analyzed in depth (Carrillo et al., 2018, p. 237; authors' translation).

According to Ferreti, Martignone, and Rodríguez-Muñiz (2021), this knowledge is defined by skills a teacher must have to develop their teaching practice. Acquiring the MTSK can enhance the mathematics instruction processes.

Regarding the **Pedagogical Content Knowledge (PCK)**, Carrillo et al. (2018) affirm that it represents the Mathematics teacher's practical knowledge in the classroom. This domain has three subdomains. The first is the *Knowledge of Features of Learning Mathematics (KFLM)* which refers to the importance of the teacher understanding the process of thinking and building mathematical knowledge. Carrillo et al. (2018) explain:

> It includes understanding the process that students must go through to be familiarized with different contents, items, and peculiar characteristics of each item that may offer learning advantages or, conversely, present difficulties. [...] More specifically, the subdomain includes the awareness of where students present difficulties and, conversely, where they show strengths, both in general and regarding specific content (Carrillo et al., 2018, p. 246; authors' translation).

For Delgado-Rebolledo and Zakaryan (2020), this knowledge deepens the teacher-student relationship, expressing when the teacher analyzes the students' difficulties, aptitudes, and emotions. Ferreti, Martignone, and Rodríguez-Muñiz (2021) expands the MTSK by targeting the Pedagogical Content Knowledge and dividing the KFLM into two parts – one focused on the teacher and the other on the student. Nevertheless, these scholars emphasize that more empirical research on this expansion should be conducted. Thus, Carrillo et al. (2018) reason that professionals teaching Mathematics should understand the origins, timing, and resolution of student learning difficulties to fulfill their educational needs.

Knowledge of Mathematics Teaching (KMT) involves various methods for Mathematical instruction, employing differentiated strategies and activities to teach a particular topic. Furthermore, KMT encompasses teaching resources and materials. Rather than just addressing their mere comprehension and usage, this subdomain relates to the ways resources can improve teaching as well as limitations (Carrillo et al., 2018).

The research by Escudero-Ávila and Carrilo (2020) reveals that KMT is evident when teachers also work with technologies and technological instruments. These authors highlight the importance of the teacher understanding how to use multiple approaches and strategies to address Mathematical content to enhance learning.

The *Knowledge of Mathematics Learning Standards (KMLS)* refers to the understanding of official curriculum documents, such as BNCC and the *Referenciais Curriculares*. According to Carrillo et al. (2018, p. 248; authors' translation) "the relevance of this subdomain is the matter of sequencing topics and demands placed on students in terms of knowledge and skills". This gives teachers a content overview of what the students have previously learned and which content will be covered in the future.

The second MTSK domain is **Mathematical Knowledge (MK)**, which consists of all systemic knowledge of Mathematics. Through this understanding, teachers can relate their mathematical knowledge to that of their students (Zakaryan & Ribeiro, 2019). Santos and Oliveira (2022) highlight that the greatest focus of research in MTSK is investigating **Mathematical Knowledge**. This domain has three subdomains: *Knowledge of Topics (KoT), Knowledge of the Structure of Mathematics (KSM)*, and *Knowledge of Practices in Mathematics (KPM)*.

As stated by Carrillo et al. (2018, p. 242), *Knowledge of Topics (KoT)* “describes what and how mathematics teachers know the topics they teach”. In other words, this subdomain indicates how deeply the professional comprehends the mathematical content and their meanings (concepts, algorithms, rules, theorems, etc.). Moreover, KoT involves comprehending mathematical properties, their fundamentals, representations, definitions, and procedures. Policastro and Ribeiro (2022, p. 5; authors' translation) emphasize that KoT "refers to the teacher's knowledge of the different mathematical definitions associated with the same topic – when there is more than one definition –, including their distinct forms of presentation (through symbolic and/or verbal language”. As an example, these scholars underscore this subdomain regarding division:

> Within the scope of division, KoT includes, for example, knowledge associated with the mathematical meaning of what it is to divide; with the two meanings of division; with the different procedures associated with the division operation, including the Euclidean algorithm, but not only; with the distinct types of representation for a division and the

> relations between them (for example, pictorial, numerical, verbal), which contribute to give sense to the presumed meaning (equitable sharing or measuring); with the types of problems that can be formulated in correspondence with each of the division meanings, among other aspects (Policastro & Ribeiro, 2022, p. 6; authors' translation).

Knowledge of the Structure of Mathematics (KSM) regards the knowledge of Mathematical structure as the teacher's ability to establish relationships between the topics they teach. Furthermore, this subdomain refers to the teacher's capacity to look to the horizon, specifically in terms of understanding both prerequisite and subsequent content. Carrillo et al. (2018) argue that the teacher with this expertise can transcend the simple curriculum sequence, favoring the topic's simplification or complexity.

While assessing textbooks from the 4th grade of *Ensino Fundamental*, Piccoli and Alencar (2021) identified that they facilitate the KSM by allowing teachers to link content from previous and subsequent grades. Hence, the study demonstrated the significance of KSM in the work of elementary school teachers. In-depth understanding of Mathematics is of great importance for professionals teaching in initial years of elementary school, enabling them to establish connections with future subjects. This view is also expressed by undergraduate students of Pedagogy who participated in Zero's (2021) research. These participants indicated which MTSK they believed to be the most important for their professional role. The KSM was highlighted for its possibility of integrating contents.

The subdomain of *Knowledge of Practices in Mathematics (KPM)* can be defined as any systematic activity. In other words, the teacher is familiar with Mathematics strategies (demonstrations, definitions, counterexamples, etc.) and can organize their class based on them. For Carrillo et al. (2018, p. 244; authors' translation), "the object of this practice is Mathematics itself". Thus, the focus is on the mathematical content itself, rather than how to teach it. Moreover, teachers possessing KPM can use heuristic strategies to prove a topic or solve problems, facilitating their mathematical reasoning.

Moriel Junior and Camacho (2022) investigated the Mathematics Teacher's Specialized Knowledge in fractions teaching. This subject starts in the 5th grade, which implies the need for the elementary school teacher to know it. In their research, the authors showed that the Knowledge of Practices in Mathematics is relevant to fraction teaching. This is because the teacher can, on one hand, illustrate the practical application, such as adding fractions, and on the other hand, provide a clear definition of the underlying mathematical concept. Moriel Junior and Camacho (2022, p. 141, authors' translation) emphasize that “constructive processes of definition assist in the systematization of algorithms for fraction operations in the classroom”.

Educators with concept comprehension or mathematical definitions of the content taught demonstrate signs of KPM.

Based on this model proposed by Carrillo et al. (2018), the MTSK of prospective teachers who will be teaching mathematics in elementary school will be analyzed.

Methodology

The qualitative approach was chosen for this research because "it develops in a natural situation, is rich in descriptive data, has an open and flexible plan, and focuses on reality in a complex and contextualized manner" (Lüdke & André, 1986, p. 18; authors' translation). According to Bogdan and Biklen (1994):

> 1. In qualitative research, the direct data source is the natural environment, and the researcher is the main instrument; 2. Qualitative research is descriptive; 3. Qualitative researchers are more interested in the process than simply in the results or products; 4. Qualitative researchers tend to analyze their data inductively; 5. The meaning is of vital importance in the qualitative approach (Bogdan & Bilken, 1994, p. 47-50; authors' translation).

Moreover, this research is characterized as exploratory since it "intends to provide greater familiarity with the problem, aiming to make it more explicit or build hypotheses" (Gil, 2007, p. 41; authors' translation). Specifically, the aim was to broaden the discussion around the knowledge of teachers who teach Mathematics in the first years of elementary school.

Undergraduate students from the Pedagogy program of a public university in the northwest of Paraná state were invited to participate in the research. The inclusion criteria were students finishing the subject "Methodologies of Mathematics Teaching II - 1st to 5th grade of *Ensino Fundamental*." Also, it should be noted that the undergraduate students had already concluded the subject "Methodologies of Mathematics Teaching I - 1st to 5th grade of *Ensino Fundamental*."

The rationale for participant selection stems from the number of studies that have shown weaknesses in mathematics education in this field (Minhoto et al., 2022; Nogueira et al., 2016). In addition, Baumann (2009) and Curi (2020) indicate the necessity of specialized knowledge in mathematics for initial years teachers in Pedagogy courses to train educators who are aware of their pedagogical training and proficient in Mathematics instruction. Hence, Pedagogy students enrolled in the subject "Methods of Teaching Mathematics II" were conveniently selected. Moreover, when the questionnaire was conducted, these students were on their last day of class and had already completed the entire subject. In a class of 30 students, six completed the questionnaire and participated in the study.

Prior to data collection, participants were provided with a presentation elucidating the research, the Informed Consent Form (ICF), and the methodology for data acquisition. The questionnaire's link was sent to the students via their institutional e-mail. They had 12 calendar days to submit their responses (October 16, 2023 to October 28, 2023). The form included 15 questions about the six MTSK subdomains, organized into open-ended and closed-ended questions.

It is worth mentioning that KoT, KSM, and KPM had fewer questions. This reality indicates a research limitation due to the hardship of formulating questions relating these three knowledge areas with the Pedagogy program as the target. Furthermore, the main objective was pedagogical knowledge.

The questionnaire answers were qualitatively analyzed to determine which MTSK subdomains the participants exhibited. Thus, this paper's definition of a questionnaire is based on Gil (2008, p. 121; authors' translation) as "the investigation technique composed of a set of questions that are submitted to people for the purpose of obtaining information about knowledge, beliefs, feelings, values, interests"[7].

Based on the type of question, data analysis was conducted in two distinct ways. Descriptive statistics were used to present the data for the close-ended questions. Conversely, Content Analysis (Bardin, 2016) was employed with open-ended questions. This approach follows the stages of: Pre-analysis, Material Exploration, and Processing of Results and Interpretation. They were described in summary:

Pre-analysis involves organizing the elements that will be considered in the analysis, from selecting submitted documents and formulating hypotheses to preparing materials. Pre-analysis is divided into five topics. "Floating" reading consists of becoming familiar with the documents that will be studied, knowing the text, and noting impressions and guidelines. The choice of documents, which will be examined, is the step where the analysis universe is demarcated. The formulation of hypotheses and objectives, according to Bardin (2016, p. 128) is “to raise a hypothesis is to ask ourselves questions”, while the objective is the general purpose of the analysis.

Indexing and the elaboration of indicators constitute another topic. Bardin (2016, p. 130) attests that the index can be the explicit mention of a theme in a message. Once the indices have

[7] The research was approved by the Research Ethics Committee on Human Beings (COPEP) with certification by Certificate of Presentation for Ethical Consideration (CAAE), number 73143523.9.0000.0104.

been chosen, reliable indicators are established. These indicators can correspond to the theme frequency during the content analysis.

Material preparation occurs before the analysis and involves arranging the gathered material. This “editing” can range “from the alignment of untouched statements, proposition by proposition, to the linguistic transformation of syntagms, for standardization and classification by equivalence” (Bardin, 2016, p. 131).

Material Exploration is the step in which the decisions made during Pre-analysis are employed – coding, decomposing, or enumerating the answers received in the documents. Finally, the results are classified and validated based on data comparisons during the Processing of Results and Interpretation. Therefore, the questionnaire had a mixed format, convening open-ended (descriptive, argumentative) and closed-ended questions.

The data analysis was conducted following all the steps outlined by Bardin (2016). The "floating" reading was performed shortly after collecting the answers, followed by the documents choosing. Indexing and the elaboration of indicators were carried out based on the number of times each type of response appeared, as recommended by Bardin (2016). Nevertheless, the examination transcended quantitative measures and encompassed a consideration of the process, consistent with the recommendations of Bogdan and Biklen (1994). Moreover, the collected answers were grammatically revised during the material preparation process.

Finally, it should be noted that the categories of analysis mirror the MTSK subdomains which are presented together with their respective questions in the subsequent section.

Analysis of the Results

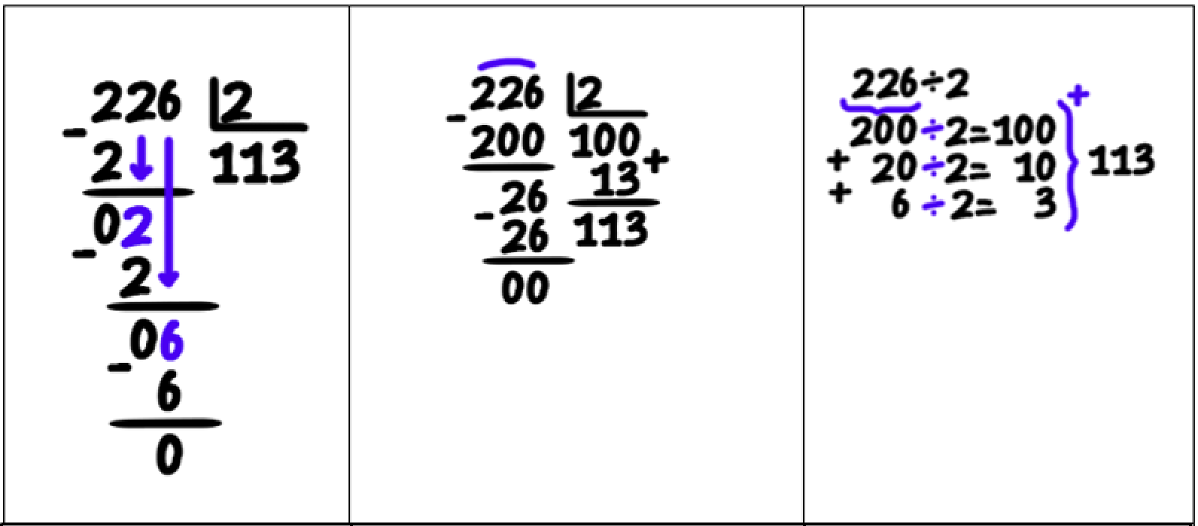
This section analyzes the results from the six subdomains, each addressed separately in its respective questions. First, **Pedagogical Content Knowledge (PCK)** and its subdomains are addressed; subsequently, the three subdomains of **Mathematical Knowledge (MK)**.

Knowledge of Features of Learning Mathematics (KFLM)

Concerning the KFLM, it was proposed that the pre-service initial years teachers answer three questions about division methods. This decision was based on Carrillo et al. (2018):

> More specifically, the subdomain includes the awareness of where students present difficulties and, conversely, where they show strengths, both in general and regarding specific content (Carrillo et al., 2018, p. 246; authors' translation).

Hence, in the first question three division methods were shown (Figure 1):



A) Division Algorithm	B) Partial quotients method	c) Decomposition Method

Figure 1.

Division methods presented in question 1 to identify KFLM (Authors)

The following question was presented: *"1) Among the listed techniques, choose the one(s) you believe your student would best understand when you are teaching. Write their letter and explain why"*. This question aimed to identify whether the prospective teachers realized that not all pupils would express the same ease or difficulty with a division method. The answers received are described in Table 1.

Table 1.

Verification of ease and/or difficulty pointed out by the undergraduate students (Authors)

Category	Respondents' response	Ease/Difficulty
Division algorithm	P1 — A. Because **it is more didactic** and visually easier to understand. P2 — A. Because this operation **step-by-step** is **easier** for the student to understand. P5 — A. Because the result is being written as we solve it. There is **less room for error**. And in the partial quotients' method, the student **may not even understand** what an estimate is.	P1 — Ease. P2 — Ease. P5 - Ease and difficulty.
Partial Quotients Method	P3 — B.	P3 — The participant did not give a reason.
Decomposition Method	P4 — C. It demonstrates all parts of the decimal numbering system, **favoring the construction of logical** and abstract thinking. P6 — C. I believe that by the decomposition method the student can have a **better visualization** of what is being divided and the numbers are whole, which most of the time **is easier to divide**, sometimes even by logical reasoning.	P4 — Ease. P6 — Ease.

As shown in Table 1, respondents P1, P2, P4, P5, and P6 indicated at least one aspect of the teaching process they found easier to navigate, whereas P3 did not mention any. Carrillo et al. (2018, p. 246; authors' translation) asserts that "KFLM refers to the need for the teacher to be aware of how students think and build knowledge when approaching mathematical activities and tasks".

Therefore, five of the six participants presented awareness of the knowledge construction process when indicating potential for ease in working with division algorithms. To achieve this goal, the pre-service educators described didactic methods such as the step-by-step approach, which minimizes errors, and division by decomposition, which promotes logical thinking and better visualization of the mathematical process. Possible difficulties were pointed out by P5 when the student might not understand the algorithm.

The second question asked students to explain at what point they believed their students would have difficulty working with the Division Algorithm in class: *"2) When solving a problem that requires the use of the division algorithm (figure above), at what point do you think your student will have difficulty?"* The focus here is ensuring that pre-service initial years teachers recognize the potential challenges of the division algorithm. Table 2 displays the responses received.

Table 2.

Verification of possible challenges in the division algorithm (Authors)

Category	Respondents' response
Numbers position	P1 — In **the position of the numbers,** maybe. P2 — They will probably have difficulty when it comes to putting **the numbers that are in the quotient**. P6 — At this moment, I believe it to be more difficult to understand how to **bring down the next algorithm [digits]** since they need to be brought down in the correct order.
When multiplying	P4 — At the time of **multiplication**.
Understanding the result	P5 — Perhaps in **understanding that the end result** is half of 226. Because using an algorithm may not be intuitive at first.

Table 2 reveals that the pre-service initial years teachers P1, P2, and P6 stated difficulties regarding the numbers' position in the division algorithm process. P4 indicated that the challenge happens during the multiplication process. This statement suggests that, for example, when dividing 4 by 2, a student should identify the number that, when doubled, equals 4. Additionally, P5 identified that the hardship lies in comprehending the final result. In this case, the participant refers to the example in Figure 2 in which, at the end, the pupil should understand that 113 is half of 226.

The prospective elementary school teachers indicated possible difficulties that students would have in using the division algorithm, as shown by the responses of P1, P2, P4, P5, and P6. Therefore, they have demonstrated signs of KFLM.

The third question read: "*3) Consider the examples below. It is known that of a class of 30 students, 20 understand the example of division by sharing better than division by grouping. Therefore, which of the two examples would you use to teach the topic of division? a) Division by sharing: In a classroom there are 35 desks and every row has 5 desks. How many rows of desks are there in the classroom? b) Division by grouping: How many rows of 5 desks can we form in a classroom with 35 desks?"*

This question aimed to identify which division type the prospective teachers would present in the classroom (sharing or grouping). The rationale here is that "students tend to be more familiar with situations involving the sharing of items equally than those involving the grouping of items" (Carrillo et al., 2018, p. 247; authors' translation). Consequently, the undergraduate students who indicated the example by sharing proved to know about the students' difficulties and exhibit signs of KFLM. By establishing connections with previous questions, P2 illustrated possible difficulties in a number of moments, such as when placing the numbers in the quotient during the division algorithm.

Escudeiro-Ávila (2022) points out that the main source of KFLM is teaching practice. The undergraduate students of this study were in the middle of the program and had not yet been teaching classes. Nonetheless, this research concluded that the participants, through the undergraduate course's theoretical classes, demonstrated KFLM signs concerning the challenges and ease of teaching division. This finding aligns with Carrillo et al.'s (2018) assertion that this subdomain involves identifying students' strengths and weaknesses relating to particular specific content.

Knowledge of Mathematics Teaching (KMT)

Questions were formulated to explore the utilization of didactic resources in Mathematics to examine the KMT. The first question was: *"4) Among the resources presented below, which ones do you know how to use?"* The options described were: Board, Projector, Textbook, Graphics Software, Golden Beads, Abacus, Cuisenaire Scale, Place Value Poster, Fraction Disc. It is important to note that participants were able to select multiple options and provide additional didactic resources in the "Other" category. Question 4 replies are shown in Figure 4:

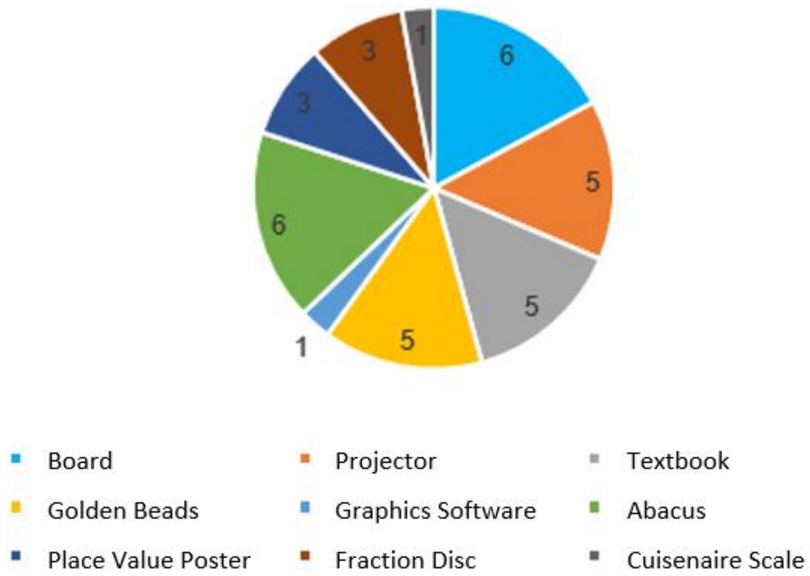


Figure 2.

Answers to question 4 on didactic resources that the undergraduate students know how to use (Authors)

As illustrated by the graphic in Figure 2, among the items listed in the question, all six participants indicated that they knew how to use the Board and the Abacus. Five of them were acquainted with the Textbook, Projector, and Golden Beads (P1, P2, P4, P5, and P6). Three respondents understand the usage of the Fraction Disc and the Place Value Poster (P3, P4, and P5). And P5 attested they know how to use the Graphics Software and the Cuisenaire Scale. None of the participants mentioned any other didactic resource. This question assesses pre-service educators' knowledge of materials and resources for teaching Mathematics.

However, as Carrillo et al. (2018) assert, KMT signs go beyond understanding the materials; it requires recognizing their limitations and possibilities to improve pedagogy. Hence, the next question asked participants to select one of the didactic resources they indicated to know how to use, in the previous question, and describe how this resource could enhance Mathematics teaching: "*5) From the resources you chose in the previous question (question 4), elect one and explain how it can improve the Mathematics teaching of a topic of your preference.*" The results are displayed in Table 3:

Table 3.

Answers to question 5 about enhancing Mathematics teaching with a chosen resource as sign of KMT (Authors)

Category	**Respondents' response**
Golden Beads	P3 — Golden Beads. P4 — I believe the golden beads **allow us a wide range of activities**, **both in material and symbolic representation, and to exemplify numbering systems, arithmetic problems, etc.** P5 — Golden Beads help in **the teaching of number comprehension, adding and subtracting, and place value of ones, tens, and hundreds.** P6 — Golden beads **make it much easier for the student to have something concrete in hand and see that those beads they make, whether of division or another characteristic, are real.** With the beads, they identify and understand what is happening in solving the operation.
Projector	P1 — It is a material that **can be worked on in several ways, bringing a more productive class with explanatory videos, etc.**
Place Value Poster	P2 — **Helps the student to better understand the position of each digit** (Unit, Ten, Hundred...).

As presented in Table 3, four of the six undergraduate students selected the Golden Beads; however, one of them (P3) did not specify the advantages of using this resource in the classroom. Advantages include the material's symbolic representation, number comprehension, and facilitation of concrete operational processes.

P1 chose the Projector, which enables a more productive lesson, as well as video presentations. P2 elected the Place Value Poster, which promotes the understanding of the numbers' placement. According to Carrillo et al. (2018), knowing the resources available for working on Mathematics is insufficient; educators need to critically evaluate how the material/resource will improve teaching. Fiorentini and Miorim (1992) affirm that

> The teacher is not always clear on the fundamental reasons why materials or games are important for the teaching-learning of Mathematics, and are usually necessary, and when they should be used (Fiorentini; Miorim, 1992, p. 34; authors' translation)

Contrary to Fiorentini and Miorim’s (1992) statement, this research reveals that Pedagogy undergraduate students show signs of understanding how to use some didactic resources, particularly the Golden Beads, for the instruction of various topics within the first years of the *Ensino Fundamental* curriculum. This is true for five students (P1, P2, P4, P5, and P6), who noted the materials' advantages/importance as part of their Knowledge of Mathematics Teaching.

The other part of this type of knowledge lies in knowing the limitations of the material. Question 6 explored the expertise regarding the resources' restrictions: *“6) Concerning the same resource chosen in question 4, what are the limitations of its use in teaching?”* The replies are reported in Table 4.

Table 4.

Answers to question 6 about the limitations of chosen resources as sign of KMT (Authors)

Category	Respondents' response
Golden Beads	P3 — Golden Beads. P4 — [**In]sufficient availability for use** in the classroom in the public education system. P5 — They are only means for children to use numerical representation; in the adult world **it is not useful to calculate always using the golden beads.** P6 — At the moment, I don't recall.
Projector	P1 **— Presenting wrong and unnecessary information** to children in class.
Place Value Poster	P2 - I cannot tell.

Although the respondents knew the advantages of the picked teaching materials/resources, only P5 suggested a restriction concerning the Golden Beads usage. Therefore, the others did not know these materials' limitations. This fact led to the conclusion that pre-service initial grades educators must improve their Mathematics Teaching Knowledge regarding didactic resources' limitations. Upon completion of the Mathematics coursework, it was anticipated that prospective teachers would be able to identify at least one limitation of their chosen resource.

Moreover, an image of a didactic resource for teaching geometry called *Geoboard* (Figure 3) was presented to the participants, aiming to further the KMT verification. The purpose was to investigate whether the research subjects could recognize the limitations of this widely used tool for teaching plane geometric figures.

The pre-service teachers should select which figures (Square, Equilateral Triangle, Right Triangle, Isosceles Triangle, Rectangle, Parallelogram, and Trapezoid) could be constructed on the *Geoboard*, ignoring those that are impossible, in this case, the Equilateral Triangle. The question read: "*7) Geoboard (resource indicated in the image below) is a tool used to teach plane geometric figures online or with string/elastic, where each elastic band is placed on the board pins, thus forming geometric figures. Select which of the figures below can be created on the Geoboard.*"

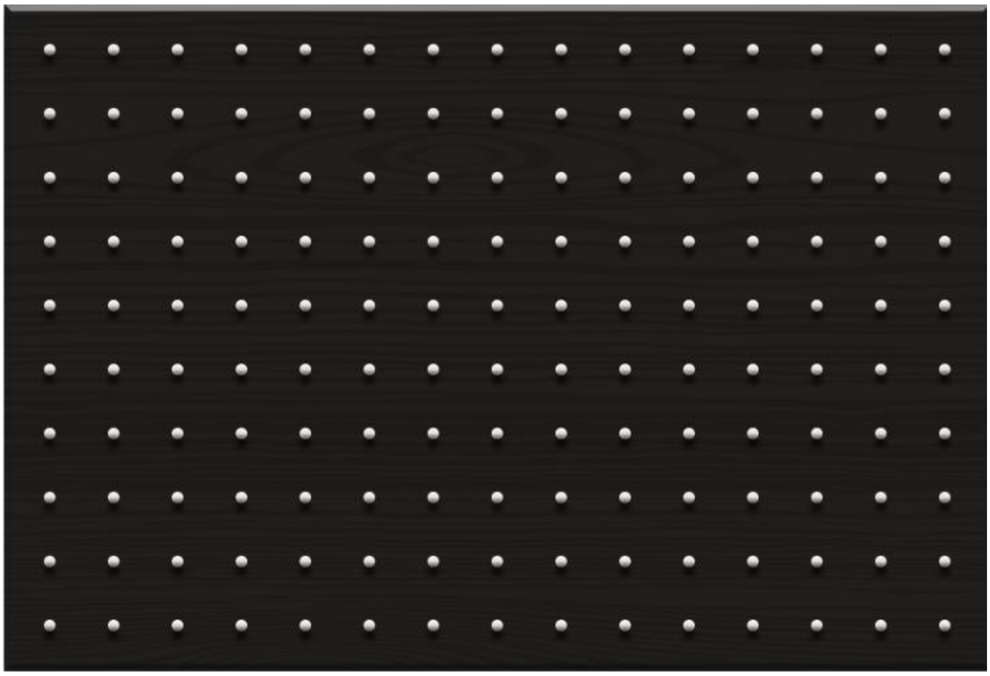


Figure 3.

Geoboard (Math Learning Center)

Of the responses from question 7, P4 and P5 marked all the alternatives, including the incorrect choice of the equilateral triangle. The other undergraduate students (P1, P2, P3, and P6) did not select all the options, but they also indicated the equilateral triangle. This data reveals that none of the pre-service teachers identified the Geoboard's limitation when constructing an equilateral triangle, a construction not feasible due to the fixed grid points on the board.

Instruction on plane figures occurs in the third year of *Ensino Fundamental* and can be done employing the Geoboard. Furthermore, this grade level is among those in which pre-service elementary teachers may be assigned. For this purpose, the teacher must understand that an equilateral triangle is not a figure that can be represented using Geoboard. Hence, the evidence suggests that the participants lack Knowledge of Mathematics Teaching about this resource.

Lastly, the respondents were asked to note how they could explain the subtraction operation using metaphors or everyday situations. This approach is justified by Carrillo et al.'s (2018) assertion that it could indicate a sign of KMT. The question read: *"8) Cite which ways you could explain the subtraction operation using metaphors or everyday situations."* Table 5 conveys the replies to this question.

Table 5.

Answers to question 8 regarding the exemplification using metaphors or everyday situations as sign of KMT (Authors)

Category	**Respondents' response**
Did not present a metaphor or everyday situation	P2 — Set up the operation, start subtraction by the unit, then by the ten. P3 — Using fingers. P6 — [Left blank].
Presented a metaphor or everyday situation	P1 — I like to explain using objects: **so-and-so has 6 lollipops and gave you 3, how many lollipops were so-and-so left with?** I think it is a good and didactic example so the child can understand this process. P4 — Using **metaphors involving food, toys, school supplies, etc.** P5 — **A situation where a person loses a certain quantity of objects, or even consumes them. Spending on a purchase.**

As Table 4 displays, P1, P4, and P5 indicated a metaphor or everyday situation. In this regard, Alves (2016) points out that:

> The isolated use of manipulable materials and games does not develop the student's ability to learn Mathematics, as the teacher must be a mediator of this learning, and for this, the teacher must be willing to research the usage of these materials so that there is mastery at the time of application (Alves, 2016, p. 7, authors' translation)

In this case, the employment of metaphors by P1, P4, and P5 demonstrates conceptual proficiency. P3 has also reported a strategy. P2 presented the algorithmic usage, which is a traditional strategy that can be used after conceptual comprehension. On the theme, Guerrero and Camacho (2022) examined the use of KMT by teachers and emphasized that it is not enough for educators to simply be aware of the didactic materials available for teaching specific content. According to these scholars, it is essential for teachers to have a deep understanding of how to implement these materials effectively to facilitate deep learning among students.

Based on the analysis of the collected data and considering the statements, it can be concluded that the research participants exhibit superficial signs of KMT. While analyzing the teaching resources, they recognize the advantages of using the didactic materials. However, these undergraduate students do not understand the materials' limitations in the classroom. Thus, they did not know how to correctly explain the *Geoboard* usage, and some were unfamiliar with incorporating everyday situations or metaphors in Mathematics teaching.

Knowledge of Mathematics Learning Standards (KMLS)

Martins (2018, p. 51; authors' translation) defined didactic sequencing as "the importance of having knowledge about the content of previous years/series to be successful in subsequent contents and/or in which they are prerequisites". Hence, KMLS is the teacher's knowledge of "topic sequencing," as defined by Carrillo et al. (2018), which combines didactic sequencing with an understanding of the student's learning objectives. Therefore, this knowledge is directly linked to familiarity with the documents that underpin education.

Respondents were asked to analyze topics related to the skills outlined in the BNCC, establishing a relationship between these skills and the thematic units to identify signs of KMLS. The first question was: "9) The skill EF03MA11 of the National Common Curricular Base (BNCC) talks about 'Understanding the idea of equality to write different addition or subtraction sentences of two natural numbers that result in the same sum or difference.' With which thematic unit is this skill related to?" The pre-service initial years teachers' responses are described in Table 6.

Table 6.

Answers to question 9 about mathematical ability and its respective thematic unit (Authors)

Category	Respondents' response
Answered the Thematic Unit Correctly	P2 — Algebra. P5 — Algebra.

Answered the Thematic Unit Incorrectly	P1 — Numbers.
Did not answer or the answer is inconsistent	P3 — Fractions. P4 — I do not know how to answer/apply this question without prior consultation and study, both of the BNCC and the content itself. P6 — [Left blank].

The question aimed to verify if the undergraduate students knew the connection between the ability and its corresponding thematic unit. The results determined that P2 and P5 understood this association, evidencing knowledge about official education documents, which serve as a sign of their KMLS. Nonetheless, P1 indicated the wrong thematic unit, and the others (P3, P4, and P6) could not assert the connection.

Moreover, the participants were asked to correlate knowledge objects of the first years of *Ensino Fundamental* and their specific grade: *"10) In the table below, relate the objects of knowledge with the grade in which it is learned, referring to the initial years of Ensino Fundamental."* Table 7 illustrates the alignment between the selected knowledge objects and their corresponding grade levels as specified in the BNCC.

Table 7.

Knowledge objects and their corresponding grades levels according to BNCC (BNCC, 2018)

Addition	First grade
Composition and decomposition of natural numbers of up to two orders	
Recognition and notions of flat geometry	
Problems involving double, half, triple, third part	Second grade
Multiplication	
Recognition and notions of spatial geometry	
Composition and decomposition of natural numbers of up to four orders	Third grade
Fraction	Fourth grade
Reflection symmetry	
Decimals	Fifth grade
Equivalent fraction	
Percentage	
Area and perimeter	

Therefore, Figure 4 presents a graph illustrating respondents' errors and successes in the association task described in Table 7.

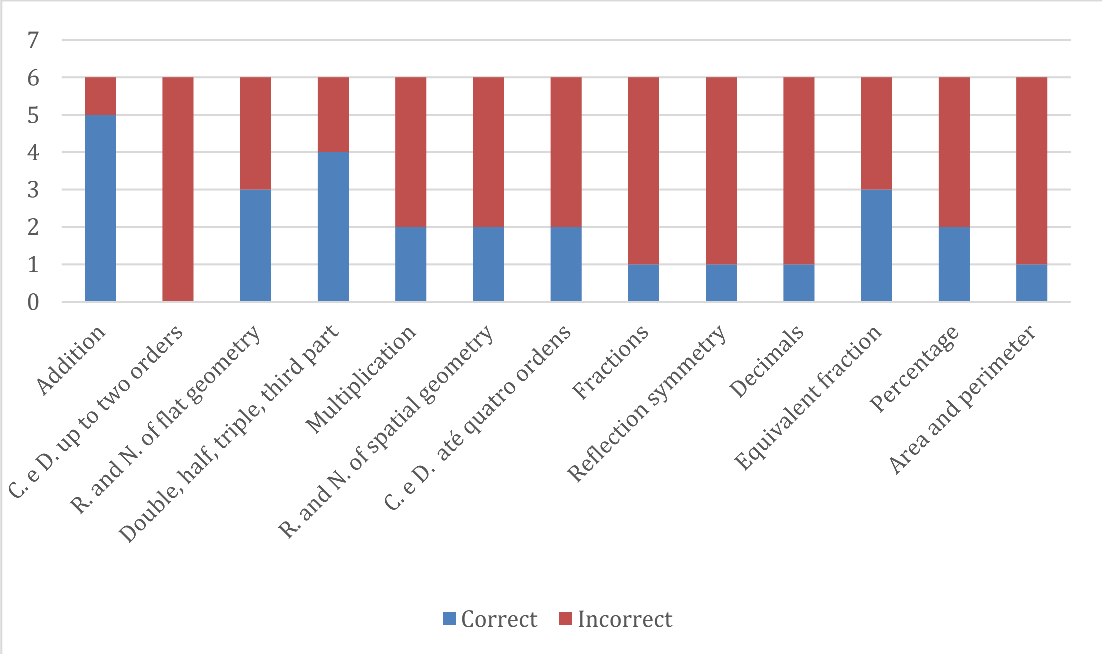


Figure 4.

Answers to question 10 on the relationships between knowledge objects and grade levels in which they are presented (Authors)

Based on the data in Figure 4, it is possible to observe that the addition and the problems involving double, half, triple, and third part had more correct answers than mistakes (5 and 4, respectively). The recognition and notions of plane geometry and equivalent fraction had the same number of successful and unsuccessful attempts (3 each). Conversely, the other nine topics had more incorrect responses than correct ones.

While this information could not be included in the graphic, it is noteworthy that P1, P3, and P5 identified the first grade as the grade level for introducing decimal numbers. Furthermore, P4 and P5 stated that this topic is learned in the second grade. Nevertheless, BNCC outlines that decimal numbers should be taught in the fifth grade. These data indicate that the pre-service initial years teachers do not demonstrate signs of the subdomain KMLS.

The final KMLS question asked whether respondents thought division, taught in the elementary school, was reviewed during middle school. In case of affirmative answer, the undergraduate students should mention during which moment this would occur: "*11) Considering the topic of division, presented in the first years of Ensino Fundamental, do you think this content is reviewed in final years of Ensino Fundamental? If so, at what point does this happen?*"

This question sought to determine whether prospective initial years educators know topic sequencing. The objective was to attest whether they can recognize where and when the content that was introduced in the first years will be reviewed during the last years. The following responses were gathered:

Table 8.

Answers to question 11 on whether the division content is reviewed in the last years of Ensino Fundamental (Authors)

Category	**Respondents' response**
Unable to answer	P1 — I can't reply. P2 - I cannot tell. P3 — I believe so, I don't know how to answer you. P6 — I don't know.
Consider it will not be reviewed	P4 — No.
Consider it will be reviewed	P5 — It can be considered as reviewed if you understand that division is permeated with other contents.

Table 8 indicates that P5 affirmed that the content is reviewed, but did not specify the timeframe. The other undergraduate students did not know, which suggests that they have no knowledge of when the content is revised. This fact implies the lack of KMLS signs regarding this theme. In the same vein, Gutierres and Flores (2016) conducted research with 52 undergraduate students of Pedagogy, analyzing content taught to the equivalent of the fifth grade in Brazil. Through statistical methods, the researchers revealed a significant connection between participants with KMLS and KFLM. According to Gutierres and Flores (2016), this relationship is evident when undergraduate students understand the content at its most difficult level; they demonstrate confidence in their ability to teach it. However, if the teachers indicate that they do not understand these concepts, the research shows that they also believe they will not be able to teach the most challenging topic.

On this matter, Vieira and Drigo (2022) explain that

> mathematics, when translated as a school subject, presents the subjects in a linear way, that is, it is considered that each concept or definition is derived from previous ones and, therefore, it is not possible to understand it without having understood the previous ones [...] This implies a teaching process in which the teacher must follow such a sequence (Vieira & Drigo, 2022, p. 326; authors' translation).

Hence, the data collected allows the conclusion that, in the context where understanding the sequence of division topics is essential, the pre-service initial years teachers lack signs of KMLS. Thereupon, the following analyzes relate to the results gathered about the subdomains of **Mathematical Knowledge (MK)**.

Knowledge of Topics (KoT)

The present subsection sought signs that the prospective educators had KoT from the questions applied. This knowledge aims to deepen the content taught by the teacher, intending to further students' understanding and obtain a deeper knowledge of the concepts, procedures, rules, algorithms, theorems, etc., on a given subject (Carrillo et al., 2018).

Two questions were formulated focusing on division to accomplish this goal. The first one assessed the participant's acquaintance with underlying mathematical concepts: *"12) Still about division, do you know what it means 'one number being divisible by another'? If so, explain."* Table 9 depicts the undergraduate students' replies.

Table 9.

Answers to question 12 regarding KoT (Authors)

Category	**Respondents' response**
Did not answer or answered incorrectly	P1 — It would be half, I think. P3 — I cannot tell. P6 — [Left blank]
Partially answered	P2 — When a number can be divisible by another number, in addition to being divisible by itself. P5 — Every number is divisible by another; I am totally against using this term. Except cases with division by zero. Daily the term is used when the dividend is greater than the divisor.
Answered correctly	P4 — Yes. For the first years [of *Ensino Fundamental*] we can explain that one number is divisible by another when the rest is equal to zero.

As the table indicates, only P4 was familiar with the concept, especially correlating it with the first years of *Ensino Fundamental*. P2 and P5 pointed out ideas of the division but did not correctly describe this mathematical concept, which shows signs of KoT. Nonetheless, P5 has shown an in-depth knowledge of using the concept on a daily basis. This reality highlights a gap in their training concerning these concepts, given that the focus of the course "Methodologies of Mathematics Teaching I - 1st to 5th grade of *Ensino Fundamental*," which the participants have already attended, is specifically on the four operations.

In this panorama, the research by Policastro and Ribeiro (2022) investigated the KoT in nine teachers working in elementary school about their division knowledge. Among the topics covered in the training, the questions related to the definitions, properties, and mathematical foundations of KoT were the least evidenced. The scholars emphasize that initial training is of fundamental importance to enable concept construction that educators will teach in the profession.

The second question concerning KoT presented to the pre-service teachers focused on the knowledge of algorithms, procedures, rules, etc. This inquiry sought to determine if respondents understand the procedures related to a topic, specifically if they know how, when, and why to apply or omit them. The question was: *"13) The division below was performed by the Decomposition Method. Is it always possible to apply this method? Explain."* The image below was presented as well (Figure 5):

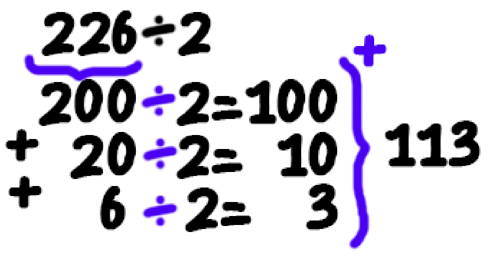


Figure 5.

Decomposition method, picture that followed question 13 (Authors)

In this case, since 226 is a number divisible by 2, the decomposition method works. However, when applying the same method (of decomposing the number decimally) to 225 divided by 3, it is impossible to perform the division. The reason is that when decomposing 225 into a sum the result is $(200 + 20 + 5) \div 3$. However, 3 does not divide 200, 20, or 5 (although 225 is divisible by three). In this case, the decomposition method is feasible for the division if 225 were decomposed into 180 + 45.

Nevertheless, the answers to this question were not clear. From the six replies collected, none of the participants explained why applying the Decomposition Method to the division may — or not — always be viable in elementary school. Table 10 describes the prospective teachers' responses to question 13.

Table 10.
Answers to question 13 regarding KoT about Decomposition Method

Category	Respondents' response
Unable to answer	P1 — I do not know how to explain.
Believe that it is not always possible, but did not explain why	P2 — I believe not, especially if we are going to insert operations with children in the 1st grade, it is something complex for them to understand. P3 — No, there are several other methods. P4 — No, I believe that every method has limitations to the same extent that it is subordinate to others' ability to understand. P5 — The short answer is no, because in the case of division by zero, no division would be possible.
Believe that it is always possible, but did not explain why	P6 — I believe so.

Regarding P2's response, the decomposition method, while not a standard first-grade content, offers a valuable introduction to division concepts for children. The pre-service teacher's reasoning may have resulted from several factors, including an implicit lack of curriculum awareness and the fact that they had no teaching experience in the initial grades.

The research by Policastro and Ribeiro (2022) also presented procedures that graduated teachers were familiar with to teach division into equal parts. Thus, it can be concluded that the prospective initial years teachers were unable to construct this notion. According to Gudmunsdottir (1991, p. 265 *apud* Marcelo, 1993, p. 8; authors' translation), "teachers do not teach the subject to their students (except at higher level education) as they studied or as they know it". Curi (2020) explains how to analyze analogous situations, emphasizing that teachers, as professionals who study and work within the same environment (school), may be strongly influenced in their teaching by their experience as students, which can diverge from the specialized knowledge needed to teach mathematics. Furthermore, McDiarmid et al. (1989) state that the teacher

> [e]xplains, asks questions, answers their students, develops and selects assignments, and evaluates what students have understood. These activities arise from a bifocal consideration of the content and the students, framed by the teacher's own understandings and beliefs about their role in teaching, as well as their knowledge and assumptions about a certain content (McDiarmid et al., 1989, p. 8; authors' translation).

Hence, teachers' understanding of a particular subject, concept, or content influences how they transmit knowledge in class. Therefore, not all pre-service initial years teachers showed signs of KoT regarding division. This fact is due mainly to a limited understanding of divisibility concepts and the possibilities of the decomposition method.

Knowledge of the Structure of Mathematics (KSM)

Regarding the KSM, the respondents were asked: *"14) About the fractions content, what other forms of representation can be used to teach fractions to students beyond the standard* $\left(\frac{n}{m}\right)$*?"* This question aspired to verify whether prospective educators know that a fraction can be expressed in decimal form – decimal numbers – percentage. The reason is because this fact involves new mathematical concepts beyond those of elementary school. Moreover, the question sought to determine if the participants knew fractions can be represented in the form of a linear fraction, division, or graphical representation.

The graph in Figure 6 shows both responses about increasing and simplifying complexity:

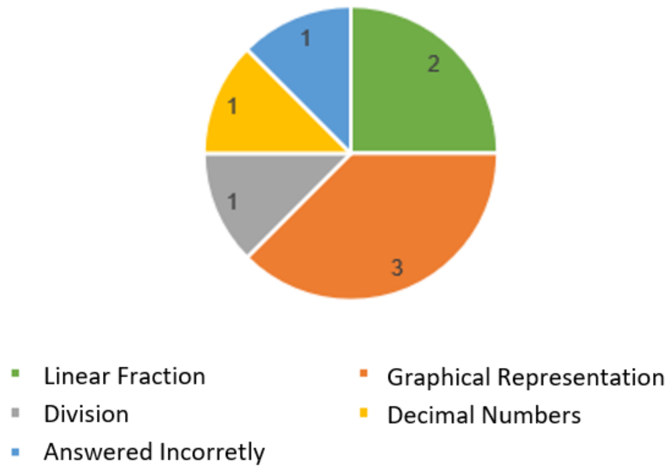


Figure 6.

Answers to question 14 about increasing or simplifying complexity regarding KSM (Authors)

Figure 6 portrays more answers than the number of respondents because some wrote more than one option. As can be observed, most of the participants (P1, P4, P5, and P6) could work with the concept's simplification by presenting examples with linear fractions, division, and graphical representation. Nevertheless, only one undergraduate student increased the topic complexity with decimal numbers (P5). These data reveal that pre-service teachers use KSM mainly to simplify their teaching process. On this matter, Piccoli and Alencar (2021) underscore that this is a vital role of teachers since by connecting previous and future content, they enhance students' content comprehension.

Furthermore, it is important to note that KSM includes not only forms of representation but also teaching strategies. For instance, one must recognize when to avoid oversimplifying scientific terms and knowledge. At the same time, the goal of complexity is to unite and face the challenge of uncertainty. While contrasting with simplified thinking, complex thinking does not reject the first (Levy & Santo, 2010, p. 136). This is precisely why teachers must know how to value the students' first language and use it to develop scientific reasoning (KSM and KPM).

Knowledge of Practices in Mathematics (KPM)

KPM can be defined as any activity performed systematically by the teacher. In other words, teachers are familiar with Knowledge of Practices in Mathematics (demonstrations, definitions, counterexamples, etc.) and can organize their classes based on them. Teachers can didactically transmit the necessary topics to the students based on their own knowledge.

The participants were proposed the following question to establish if they showed signs of KPM: *"15) To calculate the value of the expression* $\frac{18}{24} + 2$ *it is possible to simplify the* $\frac{18}{24}$ *fraction dividing both numerator and denominator by* 6*, resulting in the fraction* $\frac{3}{4}$*. Thus, the expression we are going to calculate becomes* $\frac{3}{4} + 2$*.* ***Question:*** *When calculating the expression* $\frac{3}{4} + 2$*, would we have the same result as if we calculated* $\frac{18}{24} + 2$*? Among the alternatives below, choose the one that best justifies your thinking."* The options were:

A) Yes, because $\frac{3}{4}$ is equivalent to $\frac{18}{24}$, so these fractions represent the same number.

B) Yes, because $\frac{3}{4}$ is a proper fraction of $\frac{18}{24}$, so these fractions represent the same number.

C) No, because $\frac{3}{4}$ does not represent the same part that $\frac{18}{24}$, so we would have different results.

The next graph depicts the responses gathered:

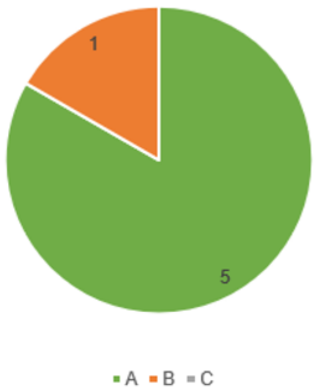


Figure 7.

Answers to question 15 regarding KPM about the definition of equivalent fractions (Authors)

In this case, the preservice initial years teachers should comprehend the definition of equivalent fractions, attesting to KPM when selecting alternative A. Figure 7 shows that an undergraduate student chose the conceptually wrong answer (P2). Nonetheless, there were positive signs of KPM as the five other prospective educators answered correctly about the concept they will be teaching in the classroom.

Moriel Junior and Camacho (2022) affirm that this subdomain relates to the teacher's understanding of the contents' characterizations, thus improving their presentation and explanation. According to the authors, teachers who display the KPM offer a more rigorous mathematical learning process. An analysis of the data indicates that the pre-service teachers exhibit signs of Knowledge of Practices in Mathematics.

The following table presents the data derived from the analysis of the questionnaire responses.

Table 11.

Relationship between knowledge and signs from the data collected (Authors)

MTSK Subdomains	Do they show signs of knowledge about the subdomain?
Knowledge of Features of Learning Mathematics (KFLM)	Yes
Knowledge of Mathematics Teaching (KMT)	Partially
Knowledge of Mathematics Learning Standards (KMLS)	No
Knowledge of Topics (KoT)	No
Knowledge of the Structure of Mathematics (KSM)	Yes
Knowledge of Practices in Mathematics (KPM)	Yes

Table 11 shows an overview of the possible gaps in the Mathematics Teacher's Specialized Knowledge in pre-service initial years teachers when referring to KFLM, KSM, and KPM.

Final Considerations

This study aimed to further the analysis of the Mathematics Teacher's Specialized Knowledge (MTSK) of prospective initial years teachers of *Ensino Fundamental*. The research question was: which subdomains of MTSK have pre-service initial years teachers, who have previously studied Mathematics disciplines, demonstrated signs of understanding?

Due to the thorough analysis of MTSK of undergraduate students of the Pedagogy program, the complexity and diversity of the involved facets are evident. The results revealed a challenging scenario, with varied signs in the different subdomains. The respondents demonstrated signs of KFLM, KSM, and KPM. Furthermore, they exhibited partial signs of KMT but not of the KMLS or KoT subdomains.

The main results suggest that the research participants depict signs of the Knowledge of Features of Learning Mathematics (KFLM). Most of them could identify the possibilities and difficulties of the division process; especially when pointing out that teaching through examples about sharing is easier.

Regarding the Knowledge of Mathematics Teaching (KMT), the pre-service initial years teachers efficiently indicated how to use the didactic resources to enhance Mathematics teaching. However, they did not know how to explain the materials' limitations. It is noteworthy that the participants demonstrated a lack of comprehension about the shortcomings of Geoboard in teaching plane geometry. Furthermore, P2, P3, and P6 did not know how to provide metaphors or everyday situations to teach the subtraction operation.

Concerning the Knowledge of Mathematics Learning Standards (KMLS), only one-third of the respondents could link the mathematical skill EF03MA11, concerning equality for addition and subtraction operations of two natural numbers, to its thematic unit. They did not show a clear understanding of aligning the content for the first years of *Ensino Fundamental* with the appropriate grade level. This fact became more evident when the participants did not know when the division would be revisited during the last years of *Ensino Fundamental*. Therefore, it can be concluded that they do not have reasonable signs of KMLS.

The prospective teachers did not exhibit signs of Knowledge of Topics (KoT) since they could not explain divisibility concepts, nor could they remark on the possibility of using the decomposition method in division operations. Concerning the Knowledge of the Structure of Mathematics (KSM), the subjects demonstrated signs about how to simplify fraction representation. Nonetheless, only P5 suggested how to improve the complexity of the topic. Lastly, the undergraduate students indicated signs of Knowledge of Practices in Mathematics (KPM) by understanding the equivalence between two fractions.

It is also worth mentioning that the purpose of the research was not to attribute value judgment to the knowledge of the study participants, classifying them as "good" or "bad" aspiring teachers based on the data collected from the questionnaire. This research aimed to document how pre-service teachers present signs of these types of knowledge. The findings here point to the need to improve Mathematics study in the training of initial years teachers, particularly regarding the understanding of limitations of didactic resources, didactic sequencing, knowledge of procedures, mathematical rules, etc.

In line with these observations, Oliveira et al. (2021) showed that pedagogy undergraduates exhibited deficiencies in mathematics teaching, using methods reflective of their own education, which were based on a technical rationality that requires to be revised. According to Lacerda (2023), this situation can be overcome by re-evaluating the curricula and instructional hours dedicated to mathematics teaching in Pedagogy courses, as an attempt to enhance undergraduate students' access to specialized content to mitigate potential classroom challenges.

Although few participants responded to the questionnaire, meaningful information was gathered about the prospective teachers' knowledge and training in Mathematics. Hence, this study can contribute to future research on the knowledge of initial years teachers who will work in Mathematics teaching. Furthermore, it indicates the need for training aimed at building specialized knowledge for teaching mathematics and, consequently, improving teaching-learning in elementary school. This improvement would promote the development of more fruitful pedagogical practices, in line with educational demands. Another limitation of the present research was addressing the signs within the same subdomain in a fragmented manner, which could be approached differently in future studies.

Finally, this research does not end here. Following the demonstration of potential MTSK deficits among pre-service initial years teachers, further investigation is needed to explore the specific knowledge areas where gaps were observed. Therefore, this fact suggests avenues for additional research on the subject.

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