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The line of numbers in the Objectivation Theory for the mobilization of mathematical knowledge

La escala de los números en la Teoría de la Objetivación para la movilización de saberes matemáticos

L'échelle des nombres dans la Théorie de l'Objectivation pour la mobilisation des savoirs mathématiques.

A escala dos números na Teoria da Objetivação para a mobilização de saberes matemáticos

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Abstract

In the realm of research focusing on Mathematics Education, the emphasis is on didactic possibilities for teaching this discipline. An alternative presented in national studies is the construction of an interface between history and Mathematics education developed by Saito & Dias (2013). In this article, the number scale devised by Edmund Gunter (1581–1626), presented in the treatise *The description and vse of the Sector, the Crosse-staffe, and other instruments, for such as are studious of Mathematicall practise* published in 1623, was appropriated for the construction of an interface. Thus, the aim was to analyze the mathematical knowledge mobilized in an activity applied with Mathematics undergraduates through the handling of the number scale. As a methodology, the Theory of Objectification (TO) was used, and for data analysis, the semiotic analysis indicated by TO was employed. The results revolve around distinct mathematical knowledges mobilized and interconnected. It is concluded that there was

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a mobilization of knowledge such as logarithms, perpendicular bisectors, and geometric construction based on the designed activity.

Keywords: Interface between history and mathematics education, Theory of objectification, Number scale*.*

Resumen

En el ámbito de las investigaciones que se centran en la Educación Matemática, se enfoca en posibilidades didácticas para la enseñanza de esta disciplina. Una alternativa presentada en estudios nacionales es la construcción de una interfaz entre la historia y la enseñanza de las Matemáticas desarrollada por Saito & Dias (2013). En este artículo, se apropió de la escala de números elaborada por Edmund Gunter (1581 – 1626) presentada en el tratado *The description and vse of the Sector, the Crosse-staffe, and other instruments, for such as are studious of Mathematicall practise* publicado en 1623 para la construcción de una interfaz. Así, se tuvo como objetivo analizar los saberes matemáticos movilizados en una actividad aplicada con estudiantes de licenciatura en Matemáticas a partir del manejo de la escala de números. Como metodología, se utilizó la Teoría de la Objetivación (TO) y para el análisis de los datos se empleó el análisis semiótico indicado por la TO. Los resultados obtenidos giran en torno a saberes matemáticos distintos movilizados y articulados entre sí. Se concluye que ocurrió la movilización de saberes como el de logaritmo, mediatriz y construcción geométrica a partir de la actividad elaborada.

Palabras clave: Interfaz entre historia y enseñanza de matemáticas, Teoría de la objetivación, Escala de números.

Résumé

Dans le cadre des recherches axées sur l'éducation mathématique, l'accent est mis sur les possibilités didactiques pour l'enseignement de cette discipline. Une alternative présentée dans les études nationales est la construction d'une interface entre l'histoire et l'enseignement des mathématiques, développée par Saito & Dias (2013). Dans cet article, on s'est approprié de l'échelle des nombres élaborée par Edmund Gunter (1581 – 1626), présentée dans le traité *The description and vse of the Sector, the Crosse-staffe, and other instruments, for such as are studious of Mathematicall practise* publié en 1623, pour la construction d'une interface. Ainsi, l'objectif était d'analyser les savoirs mathématiques mobilisés dans une activité appliquée avec des étudiants en licence de mathématiques à partir de la manipulation de l'échelle des nombres. Comme méthodologie, on a utilisé la Théorie de l'Objectivation (TO), et pour l'analyse des données, on a eu recours à l'analyse sémiotique indiquée par la TO. Les résultats obtenus tournent autour de différents savoirs mathématiques mobilisés et articulés entre eux. On conclut qu'il y a eu une mobilisation de savoirs tels que ceux du logarithme, de la médiatrice et de la construction géométrique à partir de l'activité conçue.

Mots-clés : Interface entre histoire et enseignement des mathématiques, Théorie de l'objectivation, Échelle des nombres.

Resumo

No rol de pesquisas que se dedicam à Educação Matemática, focaliza-se possibilidades didáticas para o ensino dessa disciplina. Uma alternativa apresentada em estudos nacionais é a construção de uma interface entre história e ensino de Matemática desenvolvida por Saito & Dias (2013). Neste artigo, apropriou-se da escala dos números elaborada por Edmund Gunter (1581 – 1626) apresentada no tratado *The description and vse of the Sector, the Crosse-staffe, and other instruments, for such as are studious of Mathematicall practise* publicado em 1623 para a construção de uma interface. Assim, objetivou-se analisar os saberes matemáticos mobilizados em uma atividade aplicada com licenciandos em Matemática a partir do manuseio da escala dos números. Como metodologia, utilizou-se a Teoria da Objetivação (TO) e, para a análise dos dados, empregou-se a análise semiótica indicada pela TO. Os resultados obtidos indicam saberes matemáticos distintos mobilizados e articulados entre si. Conclui-se que ocorreu a mobilização de saberes, como o de logaritmo, mediatriz e construção geométrica, a partir da atividade confeccionada.

Palavras-chave: Interface entre história e ensino de matemática, Teoria da objetivação, Escala dos números.

The line of numbers in the Objectivation Theory

Introduction

Discussions about mathematics education are becoming more prominent in national research. Many studies focus on using historical didactic resources to promote mathematical learning. Examples include studies by Batista and Pereira (2017), Saito (2019), and Pereira and Saito (2019a), as well as Santos et al. (2020). These studies draw on history to propose an interface with mathematics teaching.

One popular approach in this line of research is the interface between history and mathematics teaching developed by Saito & Dias (2013, p. 92), which they describe as "a set of actions and productions that promote reflection on the historical process of mathematical knowledge construction to develop didactic activities that seek to articulate history and mathematics teaching."

According to the precepts of Saito & Dias (2013), there are several possibilities in history for building an interface. According to Saito (2015), one can select a photograph, treatise, instrument, or painting as a potentially didactic historical resource for teaching mathematics. If the chosen resource is an original text, certain selection criteria must be followed for the mathematical treatise to be effective.

Given these criteria for constructing an interface, The Description and Use of the Sector, the Crosse-Staffe, and Other Instruments for Such as Are Studious of Mathematicall Practice, written by Edmund Gunter in 1623, was chosen. Specifically, the section dealing with the scale of numbers (line of numbers) was selected. Thus, a didactic treatment of the excerpt was created, and an updated historiographical approach was adopted that "aims to contextualize ancient knowledge within the past itself without considering subsequent discoveries" (Silva & Pereira, 2021, p. 225).

When constructing the interface, an activity must be developed using historical resources. However, the interface perspective developed by Saito & Dias (2013) does not provide a methodology for this. Thus, this article uses the Theory of Objectification (TO) to guide the development of an activity involving the number scale and its application in the initial training of professors.

The objective is to analyze the mathematical knowledge mobilized in an activity with mathematics teacher trainees based on handling the number scale, given the proposal to construct an interface using the number scale and the theoretical-methodological support of Objectification Theory.

Silva & Pereira (2021) established criteria for choosing a mathematical treatise. These criteria are as follows: what material will be used; how it will be used; what is the objective in choosing such a text; the school level to be applied; whether it is necessary to carry out a didactic treatment of the treatise; when to use the treatise; and what historiographical perspective to use.

Given these criteria for constructing an interface, The Description and Use of the Sector, the Crosse-Staffe, and Other Instruments for Such as Are Studious of Mathematicall Practice, written by Edmund Gunter in 1623, was chosen. Specifically, the section dealing with the scale of numbers (line of numbers) was selected. Thus, a didactic treatment of the excerpt was created, and an updated historiographical approach was adopted that "aims to contextualize ancient knowledge within the past itself without considering subsequent discoveries" (Silva & Pereira, 2021, p. 225).

When constructing the interface, an activity must be developed using historical resources. However, the interface perspective developed by Saito & Dias (2013) does not provide a methodology for this. Thus, this article uses the Theory of Objectification (TO) to guide the development of an activity involving the number scale and its application in the initial training of professors.

The objective is to analyze the mathematical knowledge mobilized in an activity with mathematics teacher trainees based on handling the number scale, given the proposal to construct an interface using the number scale and the theoretical-methodological support of Objectification Theory.

Thus, the article is divided into four sections. The first part presents the theoretical and methodological principles of research. The second part highlights the particularities of the activity developed according to OT principles. The third part discusses manipulating the number scale to find a proportional mean given two numbers. Finally, the fourth topic of the study presents the data collected during the training of mathematics teacher candidates and the analysis of this data.

Theoretical and methodological contributions

Regarding the theory adopted in this article, Radford (2018a, pp. 229–230) clarifies that

> The Theory of Objectification (TO) is an educational theory that focuses on the problems of teaching and learning and is based on the philosophy of Hegel (1830 [1991]) and the subsequent dialectical materialism developed by philosophers such as Marx (1932 [1998]) and Ilyenkov (1977). The dialectical philosophical support of objectification theory means, among other things, that OT is part of a

line of thinking in which human beings cannot be conceived as separate from the world and their cultures.

Thus, TO is characterized as a modern sociocultural educational theory based on the idea that learning is a process of knowing and becoming for those involved in an activity, including professors (Radford, 2017). Therefore, in addition to providing constructs for developing activities, this theory also guides professors' attitudes toward activities. This theory

> [...] Individuals activate and set knowledge in motion, giving concrete form to certain actions and reflections. When activated, knowledge transforms. As it transforms, it abandons its imperceptible form and reveals itself as something tangible and concrete. This process of materialization is referred to as "knowledge" in TO (Morey, 2020, pp. 56–57).

Considering the activity as a moment when knowledge comes into motion, the main element of TO is joint work, understood as the cooperation between professors and students with different roles. In this case, the professor must pay attention to selecting problems for the activity and organizing the topics to be addressed. In this regard, learning focuses on actively, creatively perceiving, and giving meaning to (potential) knowledge through actions and reflections (Radford, 2015, 2021; Paiva, 2019).

To advocate structuring the activity so that it includes an object and an objective and arranging the problems of the tasks in increasing order of difficulty, respecting the knowledge the student has already acquired. Thus, when designing the activity's tasks, three elements must be considered: general considerations, relative considerations, and considerations regarding the form of collaboration.

> (1) General considerations include:
> a. considering what students already know; and
> b. involving, as far as possible, the use of artifacts (concrete, technological, etc.).
> (2) Considerations regarding mathematical problems indicate that they should:
> c. be interesting from the students' perspective;
> d. offer students opportunities to engage with mathematical knowledge at deep levels of conceptualization;
> e. be organized according to a conceptual and contextual unity; and
> f. have increasing conceptual complexity.
> (3) Considerations regarding specific forms of human collaboration include organizing the classroom in a way that:
> g. encourages critical reflection; and
> h. fosters strong interaction among students, and between the teacher and students (Radford, 2021, p. 133, our translation)

In addition to these three elements, tasks must also comply with three levels of conceptualization:

> – The first level is associated with a concrete sensory experience, that is, experimentation and reflection with concrete materials [...].
> – The second level of conceptualization involves theoretical reflection based on the use of concrete objects that could highlight possible emerging connections that give meaning to mathematical objects.
> – The third level of conceptualization appears with the manipulation of mathematical symbols with which students elevate the previous experience (sensory, concrete experience) to another level of consciousness (Radford, 2021, p. 133-134, our translation) [1]. [1] The original text reads: "– A first level is associated with a concrete sensual experience; that is, with an experimentation and reflection through the use of concrete materials […].– A second level of conceptualization involves a theoretical reflection based on the use of concrete objects that could open up possible emerging links to the meaning of mathematical objects.– A third level of conceptualization appears with the manipulation of mathematical symbols with which the students raise the previous experience (e.g., concrete, sensual experience) to another level of consciousness" (Radford, 2021, pp. 133–134).

After following the precepts that govern the construction of the activity, the next step is applying it. At this point, the aforementioned joint effort comes into play. To analyze the collected data, the semiotic analysis method is used from the perspective of Objectification Theory (OT), as indicated by Radford (2021). This method of analysis is used in several OT studies and is recommended by the theory because it is based on Vygotsky's work. For this article, problem one of task five of the OT-based activity was analyzed using the number scale as a didactic resource.

The activity considering the Theory of Objectification

For this article, we considered a training course held in 2021 with eight mathematics students from the Universidade Estadual do Ceará (UECE). These eight students were divided into two groups, each with four members. For the data analysis, one group was selected to discuss the mathematical knowledge mobilized from an activity with the number scale, an activity developed based on the precepts of Objectification Theory[3].

It should be noted that this training was submitted to the ethics committee for review and approved for implementation in teacher training programs[4]. Due to the

[3] Excerpt from Santos' master's thesis (2022).
[4] Approved by the ethics committee under opinion number 4,391,528 and CAAE: 39612320.5.0000.5589.

pandemic, the training took place remotely via Zoom Meetings. Two rooms were created to divide the groups, with four students from different semesters in each room.

The training began on March 25, 2021, and ended on July 29, 2021. There were two 2-hour meetings per week, for a total of 40 hours. During this period, participants explored a seven-task activity based on Objectification Theory. Table 1 shows the training program in relation to the developed activity.

Table 1

Training program (Santos, 2022)

CONTENTS	HL
UNIT 1: Setting in 17th-century London	
1.1. The context in which the treatise The Description and use of the Sector. *The Crosse-staffe and other instruments*... 1.2. The mathematics are perceived on the frontispiece of the document.	4h/a
UNIT 2: The Cross-Staff Instrument and the number scale	
2.1. The Cross-staff instrument developed by Edmund Gunter; 2.2. Initial study on the scale of numbers; 2.3. The knowledge incorporated into the number scale.	4h/a
UNIT 3: The use of continuous proportion	
3.1. Initial study on the continuous proportion; 3.2. Mathematical knowledge mobilized in the use of continuous proportion based on the manipulation of the number scale.	8h/a
UNIT 4: Application of the continuous ratio	
4.1. Application of continuous proportion in a practical problem;; 4.2. Outline of the mathematical knowledge used in solving practical problems.	4h/a
UNIT 5: The use of proportional averages	
5.1. Initial study on proportional averages; 5.2. Mathematical knowledge used to obtain proportional averages by manipulating the number scale	6h/a
UNIT 6: Application of the proportional mean	
6.1. Application of proportional averages in a practical problem;; 6.2. Outline of the mathematical knowledge used in solving practical problems	10h/a
UNIT 7: Formalization of mathematical ideas	
7.1 Systematization of knowledge mobilized when manipulating the scale of numbers in the use of continuous proportion; 7.2 Systematization of knowledge mobilized when manipulating the scale of numbers in the use of proportional mean;	4h/a

The research activity consisted of seven tasks intertwined with a didactic resource derived from history: the number scale developed by Edmund Gunter (1581–1626). The tasks have different objectives that target the activity's overall goal.

The activity developed aimed to develop ideas about the geometric mean and mobilize mathematical knowledge incorporated in the number scale, from its manipulation to continuous proportion and proportional mean.

According to Objectification Theory, the problems in the activity should increase in conceptual difficulty throughout.

The tasks started with the historical context in which the number scale was developed. They delved into the instrument itself and the knowledge incorporated in it. Finally, they culminated in two manipulations of the scale to find the third, fourth, fifth, etc., in continuous proportion given two numbers. The tasks also aimed to obtain the proportional mean of two numbers. Table 2 shows the topics covered in each task of the activity.

Table 2

Topics related to tasks

Tasks	Tema
Task 1	Familiarization with the background to the drafting of the Treaty of Gunter (1623)
Task 2	Initial study of the Cross-staff instrument and the number scale
Task 3	Understanding how to obtain one-third, one-quarter, one-fifth, etc. in continuous proportion given two numbers on the number scale
Task 4	Application of the use of number scales in relation to continuous proportion in a practical context
Task 5	Understanding how to manipulate the scale of numbers to find the proportional average given two numbers
Task 6	Application of the use of the number scale referring to the proportional average in a practical situation
Task 7	Formalization of mathematical ideas used in the activity

For this article, which is based on an excerpt from the activity, we chose to explore the first problem in Task 5. This problem required the number scale, a compass, and Resource Card 5, which contains Gunter's (1623) explanation of how to use the scale to find the proportional mean given two numbers. The first problem consisted of understanding how to use the number scale to find the proportional mean of two numbers by reading resource card five.

Specifically, task five provides a conceptual and contextual unit, which refers to the background against which the problems will be posed (Radford, 2021), in accordance with the precepts of TO and the problem addressed in this article. Thus, the following scenario is used as a conceptual and contextual unit: After understanding how to manipulate the number scale to use the continuous proportion and complete the first mission on surveying instrument production in London, artisan Elias Allen asks the group to use the number scale to find the proportional average. From reading Resource Card Five:

1) Try to understand how to manipulate the number scale to find the proportional mean.

The next topic addresses how to manipulate the scale of numbers to find the proportional mean of two numbers. This topic is based on an excerpt from Edmund Gunter's treatise, The Description and Use of the Sector, the Crosse-Staffe, and Other Instruments, for Such as Are Studious of Mathematicall Practice.

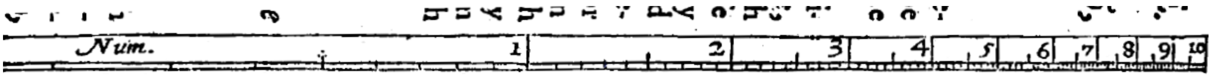
The scale of numbers and their manipulation to find the proportional average given two numbers

The scale of numbers (Figure 1), which Gunter (1623) called the line of numbers, was developed while he was a professor at Gresham College. According to Van Poelje (2004, p. 13, our translation), Gunter "[...] must have received many practical comments from ship captains and other navigators [...]" during his lectures. The influence of practitioners of mathematics can be seen in the development of scales for navigation, including the scale of numbers, which was also intended for navigation along with the other scales of proportions[5].

[5] On the scales of proportions, see Santos and Pereira (2021).

Figure 1

Scale of numbers (Gunter 1623, p. 31)



The scale of numbers has various uses, including arithmetic and astronomical calculations. It can also be combined with other proportional scales for additional applications. In his treatise The Description and Use of the Sector, the Crosse-staffe, and Other Instruments, for Such as Are Studious of Mathematicall Practice, Gunter (1623) presents the general uses of the scale of numbers, particularly in chapter six on the cross-staff instrument.

In this section, Gunter shows ten general uses of the scale of numbers with a compass.

1. given two numbers, to find one third in continuous proportion, one fourth, one fifth, and so on;

2. given two extreme numbers, to find a proportional mean between them;

3. to find the square root of any given number;
4. given two extreme numbers, to find two proportional means between them;
5. to find the cube root of a given number;
6. to multiply one number by another;
7. to divide one number by another;
8. three numbers being given, to find a proportional fourth;
9. three numbers being given, to find a fourth in a doubled proportion;

three numbers being given, to find a fourth in a tripled proportion.

The proportional mean is the second manipulation of the scale of numbers presented by Gunter (1623, p. 19; our translation), as previously highlighted, in which he states, 'Given two extreme numbers, find a proportional mean between them.' It is necessary to clarify what the 'extreme numbers' referred to by the author are. These numbers are the markings on which the compass points will be placed.

After presenting this manipulation, Gunter (1623) briefly explains how it works: 'Divide the space between the extreme numbers into two equal parts, and the foot of the compass will remain at the proportional mean' (Gunter, 1623, p. 19, our translation). Thus, the compass records the distance between the two given numbers and finding the middle of that distance verifies the proportional mean of those numbers on the scale.

The author states that the space collected by the compass should be divided into two equal parts but does not explain how to do this. Research into the context in which this scale was developed in the 17thcentury shows that Euclid's Elements had already been translated into Latin and English. Therefore, it is highly likely that Gunter had an in-depth knowledge of this work, given that he cites Proposition 6 in Book 1 of the Sester instrument. Therefore, the procedure for dividing a given space into two equal parts can be derived from this geometric study.

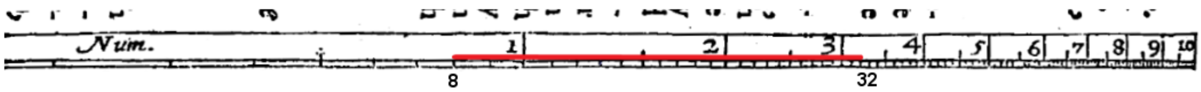
Gunter (1623) later provides an example of how to find a proportional mean given two numbers, stating, 'Given the extreme numbers 8 and 32, the mean between them is 16, as can be proven by the old proposition where it was shown that, as 8 is to 16, so also is 16 to 32' (Gunter, 1623, p. 19, our translation).

In the above passage, the author refers to the manipulation explained earlier in his treatise. By referring to the ancient proposition, Gunter returns to continuous proportion; that is, it is possible to find a sequence of numbers that are proportional to each other.

Thus, considering the information provided by Gunter, taking 8 and 32 as an example, the feet must be extended with a compass from 8 to 32 on the number scale to obtain the distance corresponding to the red segment in Figure 2.

Figure 2

Distance between numbers 8 and 32 (Adapted from Gunter, 1623, p. 31).



Given that, the distance between 8 and 32 has been measured using a compass, it is necessary to divide this space into two equal parts. As previously mentioned, the author was familiar with the constructions presented in Euclid's 'Elements', so it can be assumed that he knew how to divide a given space into two equal parts. In particular, he would have been aware of propositions I.1, I.9 and I.10.

Thus, given Proposition I.1 on constructing an equilateral triangle on a given limited line — the distance recorded by the compass when its points are positioned at marks 8 and 32 on the scale — we have [...]

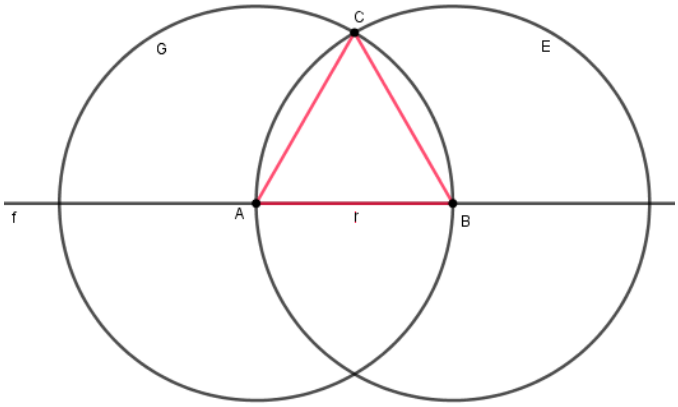
> Let AB be the given line segment. Then, construct an equilateral triangle on line segment AB.

> First, let circle BCD be described with center A and radius AB. Then, let circle ACE be described with center B and radius BA. Finally, connect lines CA and CB from point C, where the circles intersect, to points A and B.
> Since point A is the center of circle CDB, AC is equal to AB. Similarly, since point B is the center of circle CAE, BC is equal to BA. Since CA was also proven equal to AB, each of CA and CB must be equal to AB. Things that are equal to the same thing are also equal to each other. Therefore, CA is equal to CB. Thus, CA, AB, and BC are equal to each other.
> Thus, triangle ABC is equilateral and was constructed on the given finite line AB. Therefore, an equilateral triangle was constructed on the given finite line, which was necessary (Euclid, 2009, p. 99).

Figure 3 shows the construction of the equilateral triangle represented in red and the distance between 8 and 32 represented by.

Figure 3

Representation of the construction of an equilateral triangle



Once the equilateral triangle has been constructed, proposition I.9 is considered to bisect a given right angle[6], and we have:

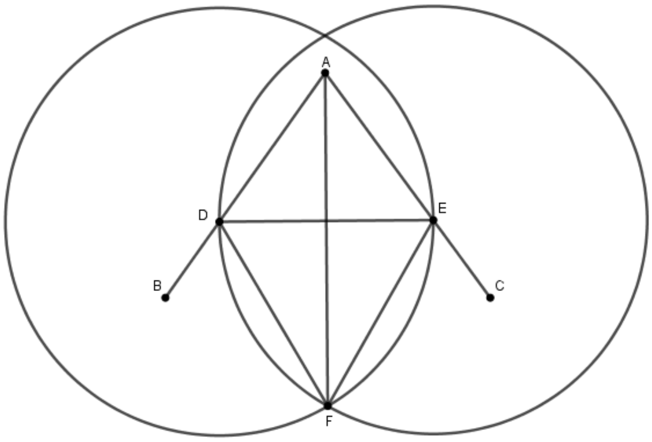
> Let the given right angle be BAC; it must then be cut in two.
> Take point D, found at random, on AB, subtract AE, equal to AD, from AC, connect it to DE, construct the equilateral triangle DEF on DE, and connect it to AF; I say that the angle under BAC was cut in two by the line AF.
> For, since AD is equal to AE, and AF is common, then the two DA, AF are equal to the two EA, AF, each to each. Furthermore, the base DF is equal to the base EF; therefore, the angle under DAF is equal to the angle under EAF.
> Therefore, the given straight angle, the angle under BAC, was cut in two by the line AF; which was necessary to do (Euclid, 2009, p. 105).

Figure 4 shows the geometric reconstruction of proposition I.9.

[6] A straight angle is defined as "[...] when the lines containing the angle are straight, the angle is called straight" (Euclid, 2009, p. 97).

Figure 4

Representation of the construction for dividing an angle into two equal parts



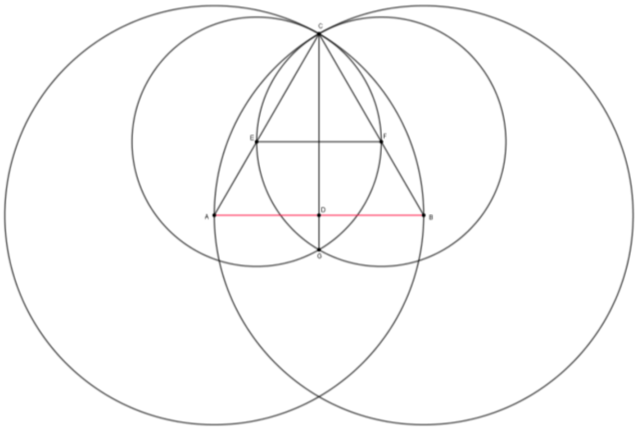
With the procedure of dividing a right angle into two parts, proposition I.10 follows: to divide a segment into two equal parts, the following applies:

> Let the given line segment be AB; it is then necessary to cut the line segment AB in two.
> Let the equilateral triangle ABC be constructed on it, and let the angle under ACB be cut in two by the line CD; I say that the line AB was cut in two at point D.
> Since AC is equal to CB, and CD is common, then the two AC, CD are equal to the two BC, CD, each to each; and the angle under ACD is equal to the angle under BCD; therefore, the base AD is equal to the base BD.
> Therefore, the given limited line AB was cut in two, D; which was necessary to do (Euclid, 2009, p. 106).

It is clear that to understand proposition I.10 in its construction links, it is necessary to know propositions I.1 and I.9. Figure 5 shows the geometric representation of proposition I.10. The segment referring to the distance between 8 and 32 is shown in red and point D marks the midpoint of this line segment.

Figure 5

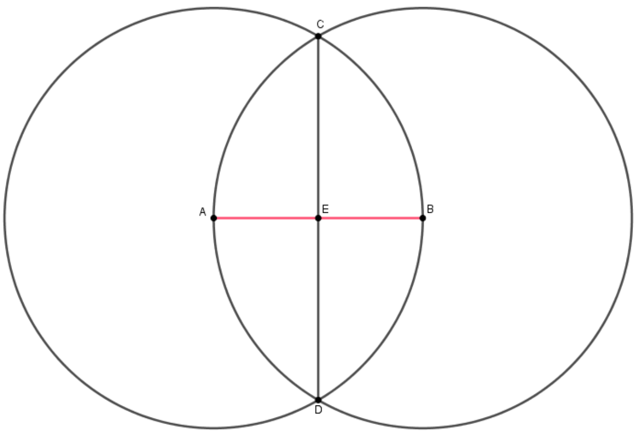
Representation of the division of a segment into two equal parts



Note that the geometric construction procedures described above can be performed with only a compass and an unmarked ruler. If you know how to divide a line segment into two equal parts, you can omit some steps in the construction, as shown in Figure 6. In this figure, constructions that are not relevant to finding the midpoint of a segment have been omitted.

Figure 6

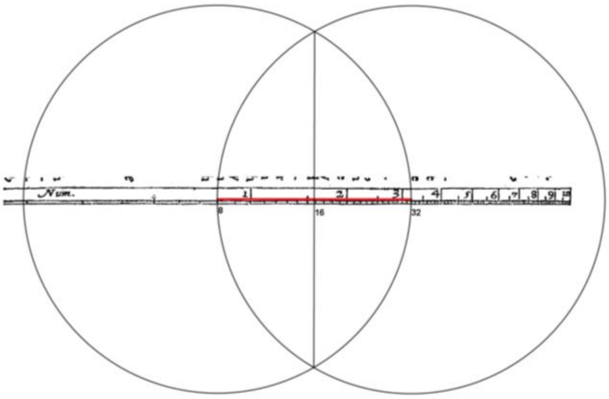
Simplified construction



Thus, by following the simplified steps, it is possible to divide the found segment, dividing the distance between 8 and 32 on the number scale into two equal parts. Starting at the end where the compass touches 8, the other end corresponds to 32, as shown in Figure 7. This results in 16 as the proportional average.

Figure 7

Proportional average of 8 and 32



Thus, the process of finding a proportional mean was explained using the instructions provided by Gunter (1623). Based on this, task five, specifically problem one, was proposed to demonstrate how to find the proportional mean of two given numbers, as presented in the following topic.

Discussions about mathematical links mobilized in training with mathematics undergraduates

Resource Card 5 of the proposed activity for mathematics students includes an excerpt from Gunter's 1623 treatise on manipulating the scale of numbers to find a proportional average given two numbers. This concept is illustrated in Figure 8 on Resource Card 5.

Figure 8

Resource card 5

Using the number scale to manipulate

proportional averages

RESOURCE CARD 05

The use of the line of numbers

2. Given two extreme numbers, to find a proportional mean between them.

Divide the space between the extreme numbers into two equal parts, and the foot of the compass will remain at the proportional mean. Therefore, the given extreme numbers being 8 and 32, the mean between them will be 16, which can be proven by the old proposition, in which it was shown that as 8 to 16, so also 16 to 32.

Extracted from Gunter (1623, p. 19).

After reading Resource Card 5, the group chosen for analysis began working on the first problem of this task: understanding how to use the number scale to find the proportional mean. It was proposed that the students manipulate the number scale with a compass, following Gunter's (1623) instructions to find the proportional mean of two numbers, as outlined in Resource Card 5. During this part of the activity, the students discussed the manipulation of the scale and attempted to grasp how it is performed.

To start, they considered the example given by Gunter (1623, p. 19, our translation) in the following excerpt: "[...] the extreme numbers given being 8 and 32; the mean between them will be 16[...]". Using this passage, the students manipulated the scale to find 16, the proportional mean of 8 and 32. This was their first experience using the number scale for this purpose.

The teacher trainees realized that Gunter (1623) does not explain how to divide the line segment corresponding to the distance between 8 and 32 on the scale. Therefore, the students needed to apply their knowledge of dividing a line segment into two equal parts. This is related to the median or midpoint of a segment, which is a potentiality of this scale and will be highlighted later.

At this point in the analysis, Student B's statement at the beginning of the manipulation process (Table 3) is important to highlight.

Table 3

Recording the distance between the given numbers

Speaker	Speech	Context
Student B	He gives 8 and 32; we have to take it at the beginning of the scale because there is no 32 after that.	Following Gunter's instructions (1623), the group attempts to manipulate the scale to find the distance that needs to be divided into two equal parts.

According to Student B (2021), the first step in performing this manipulation is "[...] to take the beginning of the scale [the number 8] because there is no 32 after it." He refers to finding the 8 at the beginning of the scale because, at the other end, 32 falls outside of it. Therefore, an 8 is found at the beginning, and the entire scale is multiplied by 10. The marking of 3.2 is equivalent to 32.

At this point in the task, students are objectifying—based on the precepts of TO—to gain a more profound understanding of the mathematical knowledge incorporated in it, specifically the concept of logarithms. Another well-established process in the group is multiplying the numbers on the scale by 10, 100, 1000, etc., according to the problem's needs and proposed calculations. Student B's brief statement reveals two mathematical concepts: the incorporation of logarithms into the scale and the use of multiplication by powers of ten to perform various calculations.

Thus, knowing where to position the compass, the students used their printed scales and compasses to follow Gunter's (1623) instructions and obtain the proportional mean of 8 and 32. However, the students noted that the author does not reveal how to divide the distance between the given numbers into two equal parts. This led to discussions about dividing a segment in half, as evidenced in Table 4.

Table 4

Discussions on how to divide a segment into two equal parts

Nº	Speaker	Speech	Context
1	Student B	There is one thing that works, which is using a compass, but I don't think it makes much sense; I don't know if they used that. You can divide it in the middle like this, using the **median line.** (Tone of voice expressing doubt)	
2	Student B	If you draw the circle there with the dry tip at 8 and the graphite tip at 32, and then draw the circle in reverse with the dry tip at 32 and the graphite tip at 8, you will get two different circles, right? (expresses uncertainty) They will intersect at two points. Drawing a straight line between these two points, it passes exactly through 16, which is the midpoint. (The student looks down, because while speaking he is manipulating the scale)	
3	Student C	Is the number 32 you're talking about the small number on the scale?	
4	Student B	Yes, yes. The **second mark after the 3**. [The student applied multiplication by 10]	
5	Pesquisadora	Try to mobilize this procedure.	
6	Student B	It's difficult to do with 8 and 32 because the circle goes beyond the sheet.	
7	Student B	Name two numbers whose **product gives an integer root.**	
8	Student A	**Do it with small numbers, 4 and 9, right?**	
9	Student B	But what is 4 and 9?	

10	Student A	36		Once numbers 4 and 9 have been chosen, they manipulate the scale with the compass, following the procedure explained in line 2.

In the dialogue presented in Table 4, the teacher trainees applied at least three of the mathematical concepts highlighted in the table. The first concept is the median, which divides a line segment into two equal parts. Another concept is implicitly considering the mark 3.2 as 32 due to multiplication by 10. Finally, the trainees recognized that a root is involved in finding a proportional average of two numbers.

When asked how to divide the segment found by placing the compass at marks 8 and 32, Student B (2021) suggests in a doubtful tone that 'You can divide it in the middle like this, using the **median,'** as explained in line 1 of Table 4. The other students in the group agree with them and help them remember how to perform this geometric construction. This association of knowledge helps the students to understand the process. At this stage of the task, links emerge between geometric knowledge (the median), arithmetic knowledge (the numbers on the scale in relation to the proportional mean), and the proportional mean of 8 and 32, since dividing the line segment into two equal parts results in a number on the scale that corresponds to this mean.

In line 2 of Table 4, Student B (2021) demonstrates to the other teacher trainees how to divide the line segment:

> If you draw the circle there with the dry tip at 8 and the graphite tip at 32, and then draw the circle in reverse with the dry tip at 32 and the graphite tip at 8, you will get two different circles, right? They will intersect at two points. Drawing a straight line between these two points, it passes exactly through 16, which is the midpoint.

In this explanation, Student B used the aforementioned propositions in a more objective and simplified manner to determine the median of the compass record of the distance between 8 and 32. At this point, the students had not yet manipulated the scale. When attempting to do so, Student B realized that the circles would have large radii and would not fit on the available A4 sheets. Student B (2021) then asked, "Name two numbers **whose product gives an integer root** [...]".

By referring to the product of two numbers with an integer root, Student B is associating the proportional mean with root extraction. However, this knowledge is not yet explicit or concrete in the manipulation, although there are indications of it.

Student A (2021) then suggests: "**Do it with small numbers, 4 and 9, right?**" By proposing this, Student A relates the distance between 4 and 9 to their magnitudes, offering another way of understanding mathematical concepts.

With numbers 4 and 9 established for the manipulation, the group follows Student B's instructions on how to divide the distance between 4 and 9 into two equal parts, using semiotic means of speech, gestures, and movements with the compass and the number scale, based on the same explanation transcribed in line 2 of Table 4. Everyone arrives at the same result, which is that the proportional average of 4 and 9 is equal to 6, a number found by drawing a straight line between the intersections of the circles centered at 4 and 9, respectively, and intersecting the number scale. There is no visual record of this moment, since the training took place remotely, so the camera did not record the movements with the scale. Thus, they did not transfer the distance of the compass to a straight-line segment in another location to perform the procedure; they only considered the distance of the compass between the numbers 4 and 9 on the scale and carried out the process.

It should be noted that the knowledge of rooting arose at this point in the task and initiates the process of objectifying the proportional mean, so the dialogue shown in Table 4 and the manipulation of the number scale illustrate a semiotic node.

The group also expressed in the problem report how they performed the manipulation geometrically, as follows.

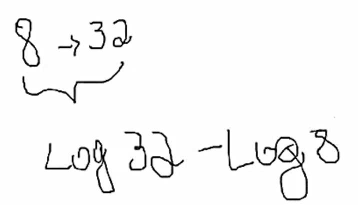
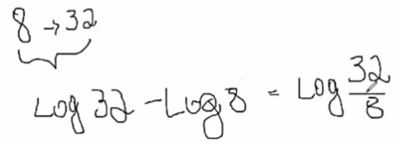
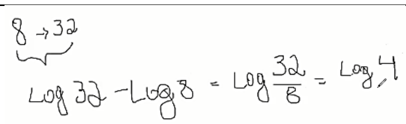
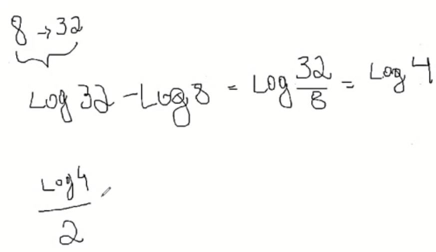
> To understand how Gunter calculated the proportional average using only a compass and a number scale, we drew two intersecting circles with a median line that intersects the proportional average between the two values at its center.
> For example, using points 4 and 9, we placed the dry tip of the compass on 4 and the graphite tip on 9. Then, we drew a circle. Next, we placed the dry tip on 9 and the graphite tip on 4 and drew another circle. Thus, two points were intercepted between these two circles. We drew a median line that intersected these two points and the number 6 on the number scale. This is the proportional average between 4 and 9 (Group 1, 2021).

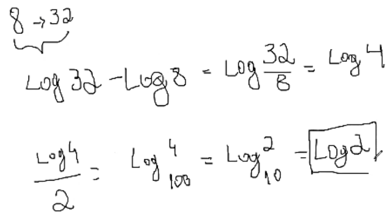
This would be enough to understand geometrically how to use the scale to find the proportional mean of two numbers. However, as the students had studied in previous tasks, particularly in task two, about the knowledge incorporated in the scale, logarithms, they delved deeper into understanding logarithmically how this manipulation works.

Since it is a logarithmic scale, an aspect seen and studied in task two of the activity, the group decides to analyze this manipulation based on the knowledge incorporated in the scale. Therefore, they mobilized their concepts about logarithms and their properties to find the proportional mean of 8 and 32 (Table 5).

Table 5

The proportional mean in relation to logarithms

Nº	Speaker	Speech	Representation
1	Student B	The example he gives is from 8 to 32, and what he asks you to do is divide that distance in half. [example given on resource card 5]	
2		**To divide into two equal parts, as this is a distance, we need to know what that distance is. This distance, expressed as a logarithm, is log 32 minus log 8. The distance is always the greater minus the lesser.**	
3		**However, by property, this is a log of 32 over 8.** [referring to the property of dividing logarithms with the same base]	
4		Which is 32 over 8, 4, so it's the log of 4.	
5		But he asks us to divide this distance into two equal parts, so dividing this distance is the same as dividing the log of 4, **because this distance is the log of 4.** Into 2 equal parts.	

6	Then the log of 4 divided by 2, I did that whole process to say that it's the same thing as log 4 to the base 100. Because of that property, right, it will take... except that log of 4 to the base 100 is the same thing as log of 2 to the base 10. Which is the same thing as just log of 2, you don't need the base because it's all to the base 10. **So, dividing that distance from 8 to 32 in the middle is dividing by log 2.**	

Source: Prepared by the authors.

Table 5 shows the different types of mathematical knowledge that are used when considering logarithms, which form the basis for constructing the number scale. At this point, Student B used the semiotic means of speech, the group's shared ethics, and the mathematical language that characterized a higher level of awareness of the mathematical knowledge involved in this process.

The semiotic means used by Student B complement each other since the implicit mathematical process occurring in the scale during this manipulation becomes visual. According to Student B (2021), the first step is to find the distance between the given numbers.

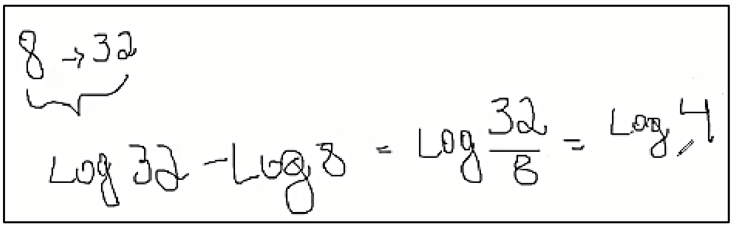
> To divide something into two equal parts, we first need to know the distance. This distance, expressed as a logarithm, is log$_{32}$ - log$_{8}$. The distance is always the larger number minus the smaller number. However, according to the property, it is log 32/8. That is 4, so it is log 4.

This statement makes it clear that Student B no longer views the distance between 8 and 32 as a line segment in a geometric sense, but rather as a logarithm. Specifically, the distance between 8 and 32 on the scale corresponds to the logarithm of 4.

When explaining the passage, Student B represents it mathematically, as shown in Table 5's representation column. First, he considers the numbers 8 and 32. Then, he represents the distance between them as the subtraction of their logarithms. Finally, he bases his reasoning on the property of logarithm division that states, "In any base a (0 < a < 1), the logarithm of the quotient of two positive real numbers is equal to the difference between the logarithm of the dividend and the logarithm of the divisor" (Iezzi et al., 2013, p. 63). Thus, he arrives at the distance between 8 and 32 on the scale, log 4, as shown in Figure 9.

Figure 9

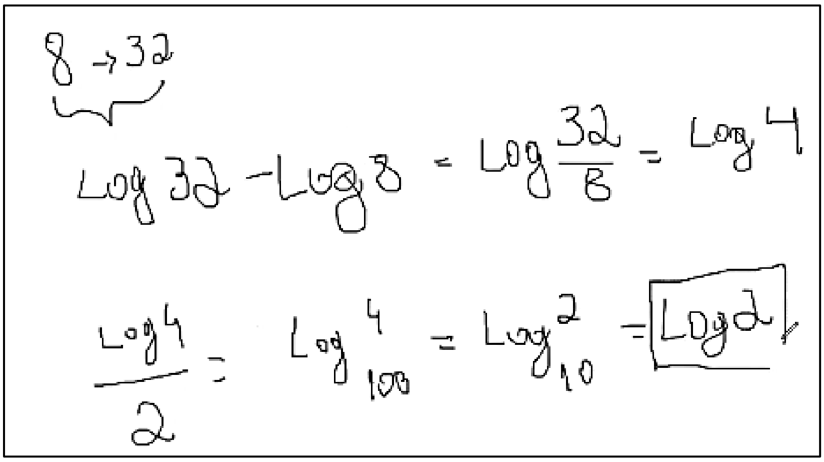
Logarithmic distance between 8 and 32 on the number scale (Survey data)



However, since we are looking for the proportional average between 8 and 32, we need to divide the distance between these numbers—that is, the log of 4—into two equal parts. In other words, we need to divide the log of 4 by 2, as shown in Student B's mathematical representation in Figure 10.

Figure 10

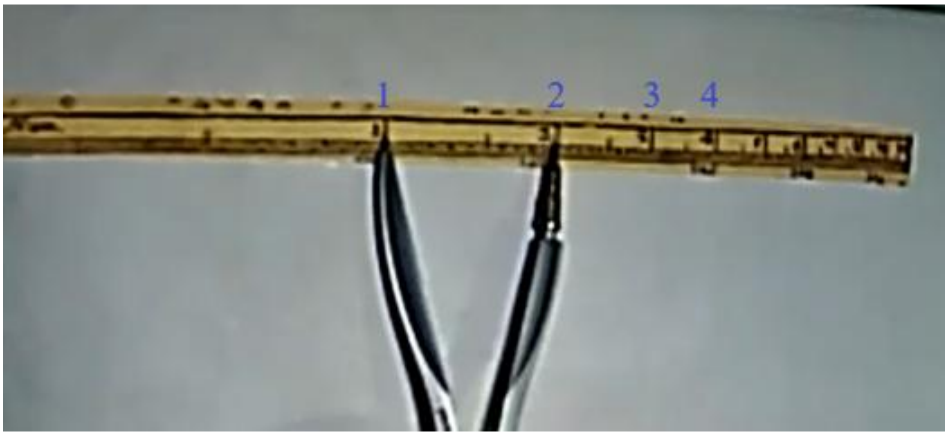
Divide log 4 into two equal parts on the number scale (Research data)



Following the event recorded in Table 5, the group provides further justification for these logarithmic concepts by considering the proportional averages of 8 and 32 on the scale. Student B (2021) reports: 'We found through calculations that half of this distance is precisely the log of 2,' and Student B then presents this distance on the compass, showing it on camera for the recording to be presented to the other students in the group, as shown in Figure 11.

Figure 11

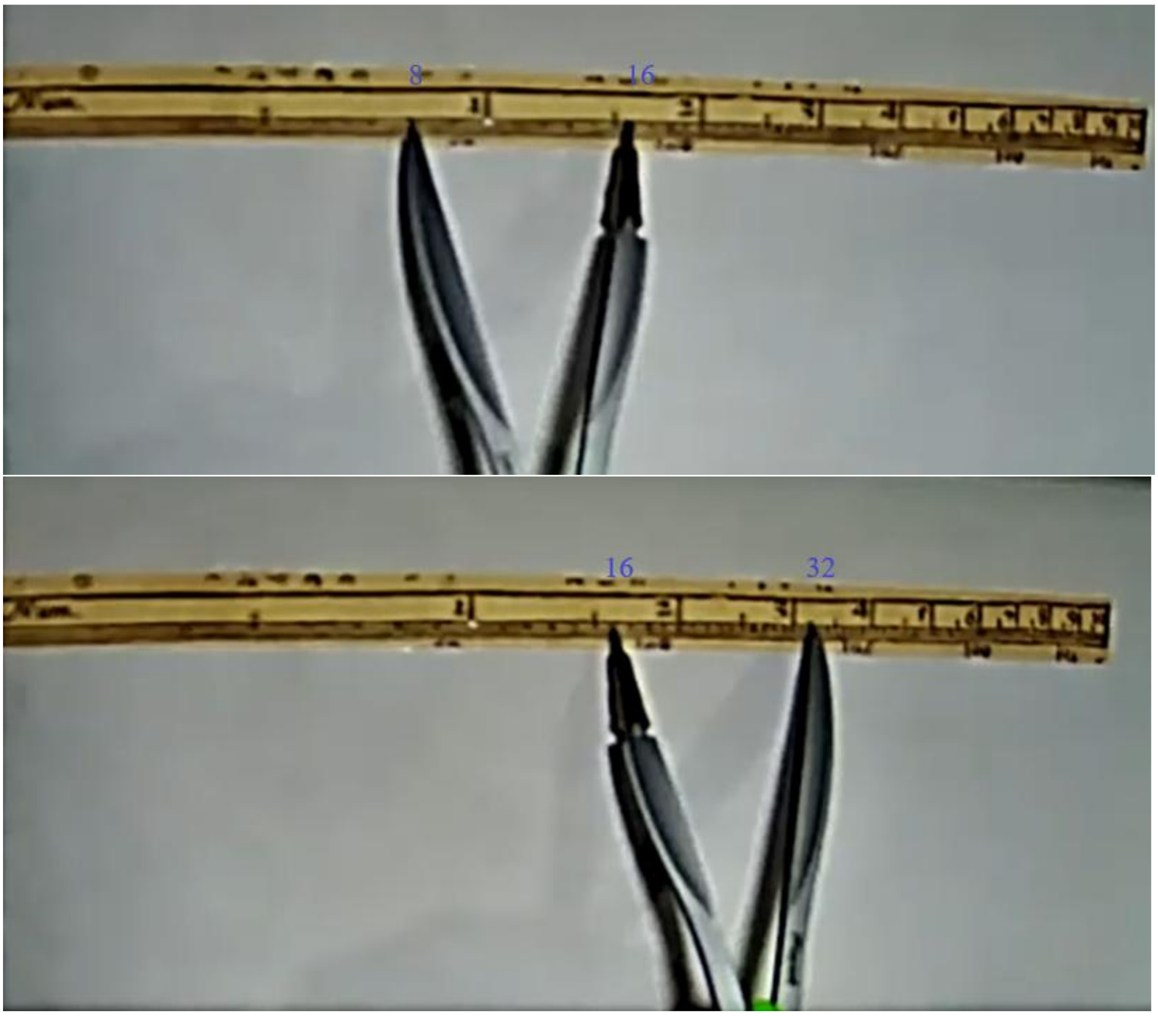
Recording the logarithm of 2 with a compass (Research data)



After recording log 2 on the compass, Student B (2021) says, 'So this [shows the compass with the recorded log 2 spacing] is half the distance between 8 and 32. To find the average of these numbers, place one end of the compass on 8 or 32.” He then places the dry tip on 8, with the other tip falling on 16 from left to right. When one tip is positioned on 32, the other ends to 16 from right to left, as shown in Figure 12.

Figure 12

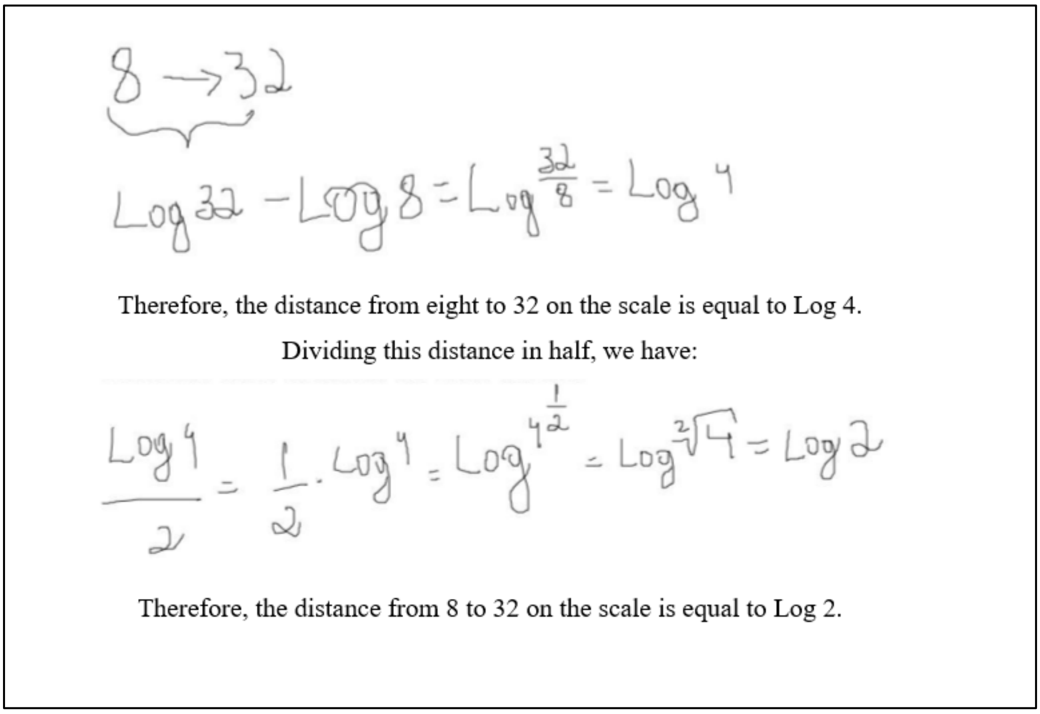
Sequence of movements based on logarithmic ideas (Research data)



The group further formalizes this way of mathematically interpreting the proportional mean in the final report on problem one. In it, they state: “As we know that Gunter's number scale is based on logarithms, we follow the use of logarithm properties:” (Group 1, 2021). They then add Figure 13 to the report to emphasize and formalize the discussions they had while working on the problem.

Figure 13

The mathematical procedures used by students when considering logarithms (Research data)



In the report, the group presents an alternative method for finding half of the log of 4, using the power property of logarithms: "In any base a (0 < a < 1), the logarithm of the nth root of a positive real number is equal to the product of the inverse of the root index and the logarithm of the radicand" (Iezzi et al., 2013, p. 67).

This alternative interpretation of the proportional mean is also characterized as a semiotic node based on logarithms. It reveals a deeper process of objectifying the proportional mean since it is once again associated with a root. At this point, the teacher trainees apply other knowledge, such as that of distance and the properties of division and power of logarithms.

Problem one of task five involved manipulating the number scale to obtain a proportional mean of two numbers based on the Theory of Objectification. This problem enabled the students to mobilize mathematical knowledge, such as logarithms, which were initially incorporated into the scale in a superficial way with basic notions of multiplication by 10, 100, 1000, etc. When referring to different calculations with the scale, they learned about the median and its geometric construction using an unmarked ruler and compass. They also learned to relate the geometric mean to a root and articulate different fields of mathematics, such as geometry and arithmetic. They mobilized knowledge about distance in relation to the magnitudes of numbers and more elaborate knowledge of logarithms in developing the problem. This included learning the properties of division and the power of logarithms. Thus, the activity based on Objectification Theory assumptions, which used the number scale as an artifact, enabled mathematics teacher trainees to mobilize mathematical knowledge often seen separately at different levels of schooling.

Final considerations

The construction of an interface based on Objectification Theory allows for the creation of an activity with a potentially didactic historical resource, which enhances students' ability to mobilize mathematical knowledge commonly seen separately at different levels of education.

Thus, an activity based on OT assumptions was developed, using the number scale developed by Edmund Gunter as a resource. This activity was applied in a training session with mathematics education students, enabling them to apply some of their mathematical knowledge.

Regarding training, this article aimed to analyze the mathematical knowledge applied by mathematics students when handling the number scale.

When analyzing problem one of task five, it was found that students applied their knowledge of logarithms and their properties, medians, geometric constructions, geometric means, and multiplication by ten. Based on the knowledge that emerged, it can be inferred that the students applied mathematical concepts not typically associated with basic education, such as geometric constructions and logarithms.

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