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On teachers' specific mathematics

Sobre las matemáticas específicas de los maestros

Sur les mathématiques spécifiques aux enseignants

Sobre a matemática específica dos professores

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Abstract

In this theoretical essay, I explore the specificity of teachers' mathematics, discussing the limitations of the models *Mathematical Knowledge for Teaching* (MKT), *Mathematics Teacher's Specialised Knowledge* (MTSK), and *Mathematics for Teaching* (MfT) in capturing the situated and regulated nature of teaching doing. I propose a distinction between Mathematics in Teaching (whose acronym in Portuguese is MnE), which manifests through pedagogical interaction with students, and Mathematics for Teaching (whose acronym in Portuguese is MpE), which encompasses MnE representations. I argue that these two dimensions are articulated recursively. The essay further supports the argument that both MpE and MnE are relational to the practice of school mathematics, considered evocative, and to the pedagogical context in which they are enacted. Based on this theoretical framework, I suggest that future studies investigate how public policies, curricular guidelines, and other socio-institutional dimensions shape the specificity of teachers' mathematics and deepen the understanding of how MpE and MnE interact and influence each other.

Keywords: Teacher education, Mathematics, School mathematics, Mathematics for teaching, Mathematics in teaching.

Resumen

En este ensayo teórico, exploro la especificidad de la matemática de los profesores, discutiendo las limitaciones de los modelos *Mathematical Knowledge for Teaching* (MKT), *Mathematics Teacher's Specialised Knowledge* (MTSK) y *Mathematics for Teaching* (MfT) para captar la naturaleza situada y regulada de la práctica docente. Propongo una distinción entre la

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Matemática en la Enseñanza (cuyo acrónimo en portugués es MnE), que se manifiesta a través de la interacción pedagógica, y la Matemática para la Enseñanza (cuyo acrónimo en portugués es MpE), que abarca las representaciones de la MnE. Sostengo que estas dos dimensiones se articulan de manera recursiva. El ensayo también apoya el argumento de que tanto la MpE como la MnE son relacionales a la práctica de la matemática escolar, considerada evocativa, y al contexto pedagógico en el que se llevan a cabo. Basado en este marco teórico, sugiero que futuros estudios investiguen cómo las políticas públicas, las directrices curriculares y otras dimensiones socio-institucionales moldean la especificidad de la matemática de los profesores y profundicen en la comprensión de cómo la MpE y la MnE interactúan e influyen mutuamente.

Palabras clave: Formación de Profesores, Matemáticas, Matemática escolar, Matemática para la enseñanza, Matemática en la enseñanza.

Résumé

Dans cet essai théorique, j'explore la spécificité des mathématiques des enseignants, en discutant les limites des modèles *Mathematical Knowledge for Teaching* (MKT), *Mathematics Teacher's Specialised Knowledge* (MTSK) et *Mathematics for Teaching* (MfT) pour saisir la nature située et régulée de la pratique enseignante. Je propose une distinction entre les Mathématiques dans l'Enseignement (dont l'acronyme en portugais est MnE), qui se manifestent à travers l'interaction pédagogique, et les Mathématiques pour l'Enseignement (dont l'acronyme en portugais est MpE), qui englobent les représentations de la MnE. Je soutiens que ces deux dimensions s'articulent de manière récursive. Cet essai appuie également l'argument selon lequel les MpE et MnE sont relationnelles à la pratique des mathématiques scolaires, considérées comme évocatrices, et au contexte pédagogique dans lequel elles se réalisent. Sur la base de ce cadre théorique, je suggère que les recherches futures explorent comment les politiques publiques, les prescriptions curriculaires et d'autres dimensions socio-institutionnelles façonnent la spécificité des mathématiques des enseignants et approfondissent la compréhension de l'interaction et de l'influence réciproque entre les MpE et les MnE.

Mots-clés : Formation des enseignants, Mathématiques, Mathématiques scolaires, Mathématiques pour l'enseignement, Mathématiques dans l'enseignement.

Resumo

Neste ensaio teórico, exploro a especificidade da matemática dos professores, discutindo as limitações dos modelos *Mathematical Knowledge for Teaching* (MKT), *Mathematics Teacher's Specialised Knowledge* (MTSK) e *Mathematics for Teaching* (MfT) em capturar a natureza situada e controlada do fazer docente. Proponho a distinção entre Matemática no Ensino (MnE), que se manifesta na interação pedagógica com os estudantes, e Matemática para Ensinar (MpE), que abrange as representações da MnE. Argumento que ambas se articulam de forma recursiva. Também sustento o argumento que tanto a MpE quanto a MnE são relacionais à prática da matemática escolar, considerada como evocativa, e ao contexto pedagógico onde se realiza. Com base nessa elaboração teórica, sugiro que estudos futuros investiguem como políticas públicas, prescrições curriculares e outras dimensões sócio-institucionais dão forma à especificidade da matemática dos professores e aprofundem o entendimento sobre como a MpE e a MnE se deslocam mutuamente.

Palavras-chave: Formação de professores, Matemática, Matemática escolar, Matemática para ensinar, Matemática no ensino.

On teachers' specific mathematics

Let us consider two classroom episodes, which have been adapted for this essay. I will use them to illustrate and articulate the arguments that will be outlined in this theoretical essay.

In the first episode (Episode I), the teacher presents the task in Figure 1 to their first-year high school students. This is a task suitable for exploring the concepts of even and odd functions.

Exploring the behavior of functions

Identify the function graphs that are symmetrical with respect to:

(i) the Oy axis;

(ii) the origin.

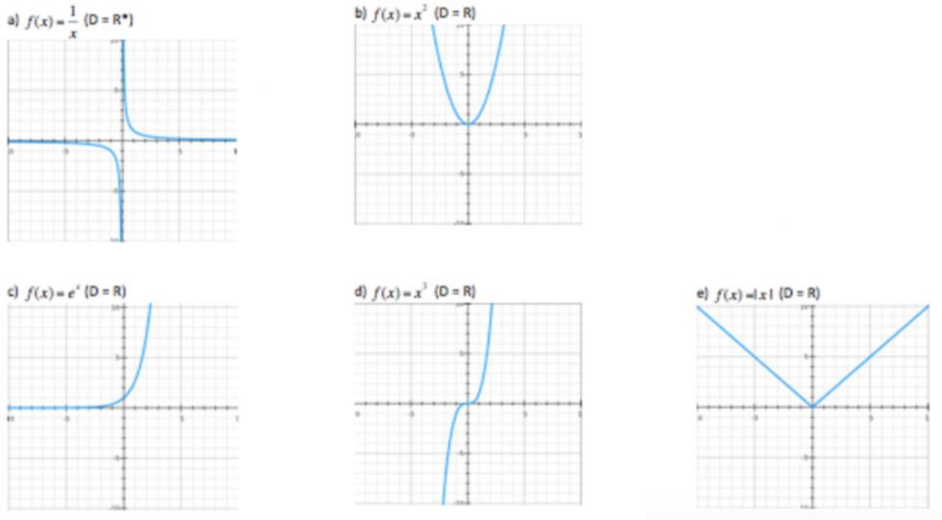

Figure 1.

Task used by the teacher

Students, working in groups, are expected to observe that some functions have graphs that are symmetric with respect to the y-axis, others with respect to the origin for every value in the function's domain, as well as the graph of a function that does not fit either of these patterns. Based on these observations, the teacher is equipped to define even and odd functions.

Let us examine an excerpt from the dialogue between the teacher and a group of students:

Teacher: Why did you choose graph (b) as symmetric with respect to the y-axis?
Student 1: Look, if you take 2, it gives 4, and if you take -2, it also gives 4.
Teacher: Only 2 and -2?
Student 1: No…
Student 2: The same happens with 1 and -1, 3 and -3…

Teacher: In other words, opposite numbers… But will this hold for all real opposite numbers?
(*Silence... the students observe the graph.*)
Student 1: Yes, isn't any number squared always positive?
Teacher: Yes, but is that the only reason?
Student 2: Yes, no matter which pair you take, if you take a number and its opposite, the y-value will be the same.
Teacher: Exactly, we are talking about opposite abscissas.

In the second episode (Episode II), the teacher stands in front of the class and announces the topic of the day: the parity of a function. The lesson begins with the presentation of the definitions of even and odd functions, and the teacher sketches on the board the representations transcribed in Figure 2.

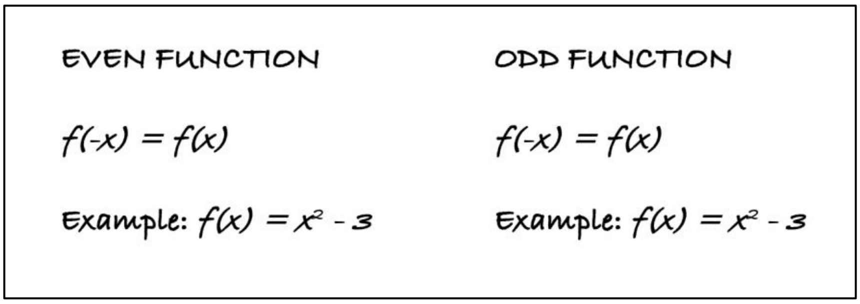

Figure 2.

Illustration reproducing the teacher's entry on the blackboard

Next, the teacher provides examples of an even function and an odd function, as recorded on the board (Figure 2). The students remain attentive, taking notes in their notebooks.

At this point, the teacher opens the floor for students' questions:

Student 1: Teacher, is every function either even or odd?
Teacher: Great question! Not every function will be either even or odd. This is something we will explore further later on.
The teacher then continues:
Teacher: Now, there is an important property you need to know.
(*He sketches the graphs of the real functions defined by* $f(x)=x^2$ *and* $f(x)=x^3$ *on the board.*)
Teacher: An even function is symmetric with respect to the y-axis, meaning it is mirrored along this axis. An odd function is also symmetric, but with respect to the origin. Imagine everything reflected around this point. (*He points to the origin.*)

As I will discuss later, these two episodes can be interpreted through theoretical perspectives such as **Mathematical Knowledge for Teaching** (MKT; Ball et al., 2008);

Mathematics Teacher's Specialized Knowledge (MTSK; Carrillo-Yañez et al., 2018); and **Mathematics for Teaching** (MfT; Davis & Renert, 2014). These perspectives are part of what Grilo et al. (2021) refer to as the **discourse on the specificity of teachers' mathematics**, which argues that teachers possess knowledge or practices inherent to the teaching profession.

In this paper, I will critically address the literature on the specificity of teachers' mathematics. I argue that theoretical models such as MKT, MTSK, and MfT fail to fully capture the situated and controlled nature of teachers' mathematics and, therefore, fall short in accounting for the variability in its realization. In light of these concerns, this study engages with these theoretical models to broaden theorizations about teachers' specific mathematics by integrating its situated and controlled dimensions. I acknowledge the existence of other theoretical models in the literature, such as the **Quartet Knowledge** (Rowland, 2014), which also offer relevant contributions, but for the sake of a focused discussion, I will limit my analysis to the aforementioned models.

Thus, this paper constitutes a theoretical essay. As defined by Barbosa (2018), this type of study is grounded in critical and in-depth reflection on a specific topic, fostering dialogue between different theoretical perspectives to formulate new interpretations. This format does not involve the collection of empirical data, although references to empirical events may be made to enrich the argument, as I am doing in this study. The results are, therefore, based on reflective and in-depth analysis, guided by theoretical frameworks drawn from the relevant literature.

I begin with a critical discussion of the theoretical models based on MKT, MTSK, and MfT. Next, I conceptualize the nature of school mathematics, leading me to distinguish between Mathematics in Teaching (translation of *Matemática no Ensino* and whose acronym I will keep here in Portuguese as MnT) and Mathematics for Teaching (translation of *Mathematics para Ensinar* and whose acronym I will keep here in Portuguese as MpT) as situated forms of teachers' specific mathematics. I then explore the relationship between these two forms and, finally, outline the implications for research in Mathematics Education. Episodes I and II will be revisited throughout the paper to illustrate the arguments developed.

What kind of math is that?

In both episodes, the teacher is teaching the parity of functions, which implies that specific task were organized for students to recognize the parity of a function. I do not intend to delve into the discussion on mathematical tasks (which I sometimes refer to simply as tasks), but for the sake of clarity, I adopt the definition by Smith and Stein (1998): a work proposal for students, consisting of one or more situations aimed at developing a specific mathematical idea.

In Episode I, the teacher proposed an exploratory task, expecting students to recognize the symmetry in the graphs so that he could define even and odd functions based on these observations. In Episode II, the teacher introduced the definitions of both types of functions, provided two corresponding examples, and emphasized their implications for the symmetry of the graphs. In both cases, the way mathematics is presented serves different pedagogical purposes, meaning that communication with the students was structured so that, from the teacher's perspective, they would be able to identify an even or odd function. This characterization aligns with what Grilo et al. (2021) broadly refer to as the specificity of teachers' mathematics.

However, it is not just about how pedagogical communication with students is organized. In Episode I, given how the class was structured, even and odd functions are linked to the symmetry of function graphs with respect to the y-axis and the origin, respectively. In Episode II, these functions are initially defined algebraically, with graphical properties presented as a consequence. This difference suggests variations not only in *how* to teach but also in *what* to teach. In Episode I, function parity is closely associated with graphical patterns, whereas in Episode II, it is tied to an algebraic regularity, as defined by the teacher. Therefore, I argue that the specificity of teachers' mathematics encompasses both *what* and *how* to teach mathematics.

According to Ball et al. (2008), this specificity of teachers' mathematics is understood as Mathematical Knowledge for Teaching (MKT), defined as "the subject matter knowledge required by teachers for specific teaching tasks" (p. 402). The authors developed a theoretical model that characterizes MKT across six domains of knowledge, which I will only mention here due to space limitations: common content knowledge, specialized content knowledge, horizon content knowledge, knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum. This framework has been particularly useful in shedding light on the specificity of teachers' mathematics, as demonstrated by studies such as Ribeiro et al. (2020).

Thus, Episodes I and II could be analyzed through the lens of MKT. In both cases, in different ways, it is assumed that teaching practice requires teachers to make decisions regarding how to position function parity within the school curriculum, how to relate it to other topics (e.g., symmetry of graphs), what to say about the topic, and how to sequence the content to ensure students' understanding. Although it is not my focus here, we could identify different MKT domains that were activated by the teacher in these episodes.

MKT is relational to the demands of classroom practice, which implies variability, as different learning situations call for different manifestations of MKT. However, the authors themselves acknowledge the challenges in linking mathematical knowledge for teaching with practice: "While this orientation aims to increase the likelihood that identified knowledge will be relevant to practice, it also brings some of the disorder and variability inherent in teaching and learning" (Ball et al., 2008, p. 403). They further clarify that MKT domains are not tied to any specific approach: "We do not view the knowledge we identify as closely linked to a particular reform agenda or teaching approach" (p. 404). Thus, while the MKT domains define *what* teachers need to know for teaching, they do not capture the variability in *how* to teach.

Another way to analyze these episodes is through the theoretical lens of the model developed by Carrillo-Yañez et al. (2018), called Mathematics Teacher's Specialized Knowledge (MTSK). Unlike Ball et al. (2008), who conceptualize MKT in terms of the mathematical knowledge needed for teaching—linked to the demands of practice—Carrillo-Yañez et al. (2018) develop MTSK in terms of "the knowledge that teachers may use to carry out any type of task as mathematics teachers" (p. 3). They assume that teachers' mathematical knowledge is specialized and mapped across six domains, which I will not detail here for the same reasons mentioned earlier.

The authors argue that these domains refer to teachers' specialized knowledge and do not cover other types of knowledge shared by other professionals. Thus, in the MTSK framework, common content knowledge is not included. Additionally, MTSK assumes that teachers' beliefs about mathematics and teaching permeate all knowledge domains. As the authors put it, there is a "reciprocity between beliefs and knowledge domains" (p. 5).

In Episodes I and II, we could apply the MTSK lens to identify the specialized knowledge that the teacher uses to conduct teaching tasks, such as what the teacher knows about function parity and its relationship to other curricular topics, how to define even and odd functions in class, and how students think and build knowledge about the topic. The variation in how this specialized knowledge manifests can be explained by the teacher's beliefs about mathematics and teaching. In Episode I, from the MTSK perspective, we could associate the teacher's specialized knowledge with the belief that mathematics is a discipline rooted in discovery, problem-solving, and exploratory learning. In Episode II, the teacher's knowledge reflects a belief that mathematics consists of facts and procedures to be followed, with teaching focused on teacher-led explanations. Thus, MTSK provides theoretical tools to capture the variability in teachers' knowledge, explained through their beliefs about mathematics and teaching.

However, the differences between MKT and MTSK extend beyond their heuristic capacity to their conceptualization of teachers' mathematical knowledge. While the former sees it as a response to the demands of practice, the latter views it as pre-existing, enabling practice. Furthermore, both models implicitly assume that knowledge is a tacit concept. Based on the way the term is used, knowledge appears to be understood as the cognitive processes and content that guide teachers' actions in their professional practice. This aligns MKT and MTSK with what is conventionally referred to as the **teacher thinking paradigm** (Barbosa, 2018). Within this tradition, teachers' actions are seen as stemming from their internal thought processes. In MKT, knowledge is activated by practice, while in MTSK, knowledge enables practice. Both models focus on the individual—the teacher, their thoughts, and actions—represented through theoretical models. Thus, we label these perspectives on teachers' mathematical specificity as **cognitive-representational** (Grilo et al., 2020).

Beyond the models discussed so far, Davis and Renert (2014) shift the focus from the individual to the teacher community. They describe teachers' specific mathematics as Mathematics for Teaching (MfT), which they view as an open disposition within the teaching community. According to the authors, "individual and collective knowledge cannot be dichotomized; collective possibilities reside within individual understandings and unfold them… it is simultaneously an individual and collective phenomenon” (p. 33). This view shifts the focus from individual knowledge to the participatory nature of professional learning.

Through Davis and Renert's (2014) lens, Episodes I and II do not merely reflect individual teachers' mathematical knowledge but represent ways of engaging with mathematics as part of a socially shared disposition. Thousands of other teachers could be organizing and conducting similar lessons. This is not just an expression of individual thought but a way of engaging with mathematics shaped by teachers' participation in a professional community, including their teaching career and earlier experiences such as teacher education and prior schooling.

In this perspective, MfT is seen as emergent (Davis & Renert, 2014), meaning that it depends on the interaction between the teacher and students, whether in the classroom or with other teachers in training situations. The social context establishes the conditions for certain ways of communicating mathematical concepts to emerge. Here, MfT shares common ground with MKT: both are responses to practice. However, unlike Ball et al. (2008), who view teachers' mathematics as individual knowledge that can be universally represented through theoretical abstraction, Davis and Renert (2014) see it as an open disposition for communicating mathematical concepts shared by teachers collectively.

From Davis and Renert’s perspective, Episodes I and II illustrate different ways of communicating the parity of functions. While the first episode focuses on graphical analysis, with mathematical definitions emerging as a consequence, the second emphasizes definitions, with graphical analysis following from them. Therefore, we categorize Davis and Renert's approach to teachers' mathematical specificity as **socio-discursive** (Grilo et al., 2020). This perspective helps us understand how teachers' mathematics unfolds through interaction with students. Episodes I and II can be seen as instances of teachers engaging with mathematics in different pedagogical situations. Similarly, Barwell (2013) argues that the unit of analysis is not what teachers know but how their mathematics is constituted discursively through classroom interaction. From this viewpoint, teachers' mathematics is not solely the property of the individual teacher but is relational to communication with students and other actors sharing the pedagogical context.

At this point, I inform you, the reader, that Episodes I and II describe lessons conducted by the same teacher at two different schools where they work. This may not be surprising, as many of us have encountered teachers who exhibit different specific mathematics in different contexts. Why, and how, does this happen? As Davis and Renert (2014) suggest, the “individual, social, institutional, and cultural dimensions in the generation of mathematical meaning” form an amalgam, making it difficult to separate these aspects. However, their conceptualization of MfT emphasizes the cultural dimension of teachers' mathematics, highlighting how the communication of mathematical concepts aligns with the practices and relationships within their professional community.

Although Davis and Renert (2014) provide a sociocultural understanding of teachers' mathematics, emphasizing the importance of collective participation, a gap remains in the literature regarding how these dimensions in teachers' work contexts influence, shape, and give form to their mathematics.

School mathematics is evocative

Mathematics teachers carry out their profession in contexts formally established for the purpose of teaching and learning mathematics, most often in schools. It is not new to conceive of school subjects as autonomous productions, with their own organization, internal economy, and efficacy, shaped by history (Chervel, 1990). This leads to the understanding that practices developed in schools are cultural in nature (Moreira & David, 2005; Valente, 2020). Thus, we can refer to this cultural practice developed within the school institution as **school mathematics** (Sfard, 2008).

It encapsulates practices that have historically taken shape (Valente, 2020), involving teachers and students, as well as other social actors whose actions affect its dynamics, such as curriculum developers, textbook authors, and policymakers. For this reason, Moreira and David (2005) emphasize the multiple factors influencing the historical and cultural constitution of school mathematics.

According to Sfard (2008), school mathematics constitutes a specific type of discourse, a form of communication defined by its vocabulary, visual mediators, routines, and narratives, all of which are understood as historically constituted. Consequently, the way mathematics teachers communicate with students (and with other actors such as colleagues and supervisors) is related to this practice in which they participate. Through the process of socialization into school mathematics—occurring largely during initial teacher education, prior schooling, and professional practice—teachers learn to recognize what is legitimate and how to act (Vilas Boas & Barbosa, 2016).

This does not imply a homogenization of this cultural practice. The Episodes I and II described above illustrate the variability that exists under the label of school mathematics. However, certain markers define this practice, such as the use of terms like even function and odd function (vocabulary), the graphical representation of functions (visual mediators), the identification of the axis of symmetry in a graph (routines), and the verbal explanation of why a function is even or odd (narratives). Thus, we can identify singularities in this cultural practice or, as Sfard (2008) prefers, discourse, while also acknowledging its variability.

When mathematics teachers participate in this social practice known as school mathematics, they can generate, evoke, or bring forth meanings, emotions, memories, and associations among participants (including students). According to Bernstein (2000), every pedagogical context, including school mathematics, has rules of recognition and realization that enable participants to identify and enact legitimate communication for that context. As such, mathematics teachers, in their professional practice, are positioned within school mathematics, which demands both *what* to teach and *how* to teach. In Episodes I and II, possibly because the topic of even and odd functions has a long tradition in mathematics curricula—featuring in textbooks and entrance exams—teachers feel compelled to address it. Both approaches to teaching even and odd functions are considered legitimate within school mathematics, even though the triad of exposition, examples, and exercises remains dominant in mathematics classrooms.

Thus, we can say that **school mathematics is evocative**, as Bernstein (2000) describes any pedagogical context: it evokes specific modes of communication, implying particular ways of selecting, organizing, and communicating mathematics, which are regarded as more or less legitimate. From the perspective of situated theoretical frameworks (e.g., Bernstein, 2000), **the**

pedagogical context not only influences what teachers do but also constitutes their professional actions, as teachers respond to the demands evoked by school mathematics. This response to school mathematics may exhibit great variability, as seen in Episodes I and II, but it also involves a certain degree of specialization, reflected in the mathematical discourse markers discussed by Sfard (2008).

From this perspective, we can only understand what mathematics teachers do in relation to the pedagogical practice in which they participate—that is, school mathematics within a specific context. Teachers' professional practice, school mathematics, and the context in which it occurs are analytically inseparable, as they are ontologically intertwined. Thus, theoretical models based on the teacher cognition paradigm are limited in understanding professional practice, as they focus on the relationship between thought and action, treating the context as merely an intervening factor (Barbosa, 2018).

Despite the specialization of school mathematics, as I suggested earlier, this social practice is neither homogeneous nor isolated from other social practices and determinants. As Moreira and David (2005) point out, school mathematics has a history that has led to its specialization but also reflects multiple local and macrostructural factors. Surely, readers recognize that the content and methods of teaching mathematics in an urban school differ significantly from those in a rural school; similar variations occur across other contexts, reflecting the socio-economic environments in which schools are embedded. Each mathematics classroom is thus unique, shaped by its participants—students, teachers, supervisors, parents, and others. However, there is also similarity in the content to be communicated, uniting these classrooms within the broader practice known globally as school mathematics or the discourse of school mathematics.

Macrostructural dimensions also influence school mathematics, shaping new demands on teachers' work, such as public policies. In Lira and Barbosa (2023), for example, we analyzed an intervention program developed by the Municipal Department of Education in Teresina in partnership with the Alga e Beto Institute (IAB). We identified how this public policy shaped mathematics lessons by emphasizing student performance in assessments, requiring the use of IAB-designed materials, organizing teacher training sessions on these materials, conducting pedagogical planning meetings, observing teachers' lessons, and administering regular student assessments. This policy context, focused on performativity, highlights the situated nature of school mathematics and how teachers participate in it. Although I will not elaborate on other examples, the Teresina policy illustrates how macro-level factors—whether municipal, regional, national, or international—shape school mathematics, evoking different forms of participation from teachers and students.

Therefore, school mathematics is a social practice with a long-standing tradition of *what* and *how* to communicate. At the same time, it is shaped by local configurations and macrostructural determinants. In their professional practice, teachers recognize the demands placed upon them by the pedagogical context, and their actions are responses to these demands, either aligning with or resisting them to varying degrees. Thus, my argument is that we cannot fully understand what teachers do through theoretical lenses that focus solely on their cognition. Instead, we must acknowledge the inseparability between teachers' actions, school mathematics, and the context in which their profession is practiced.

Mathematics in Teaching

Considering that school mathematics is a social practice with its own characteristics, understood as evocative in the previous section, the teacher is a participant in this practice. This can be seen from situated perspectives (Vilas Boas & Barbosa, 2016), where the position of the teacher is already socially instituted by school mathematics, setting boundaries for how and what the teacher participates in within this practice.

In Episodes I and II, both teachers understand that it is their responsibility to organize the pedagogical work in class, conduct the lesson, and address the topic of even and odd functions. They know that when covering this topic, they must teach students the definitions of even and odd functions and how to recognize them. However, teachers participate in school mathematics with varying qualities. In Episode I, the teacher presents an exploratory task in which students are asked to observe graphical patterns to formalize the definitions of even and odd functions. In Episode II, the teacher begins with the definitions and then provides examples. The differences between the two episodes extend to how each teacher manages the pedagogical work: in the first episode, the teacher organizes students into groups for exploration, while in the second, the teacher keeps students seated in rows to listen. We could continue illustrating the varying qualities of teachers' participation in school mathematics (e.g., the communication pattern, the type of tasks used, etc.).

Both episodes represent qualities (characteristics) of how teachers participate in school mathematics, which I define as **Mathematics in Teaching** (MnT). With this concept, I emphasize what and how teachers communicate with students in their interactions. Inspired by Bernstein (2000), I see teachers' participation in terms of their communicative interactions with students, occurring in pedagogically organized situations such as lessons. This includes not only what teachers say, write, and gesture but also how they interact with artifacts during their participation.

We can identify certain patterns in how teachers participate in school mathematics (Vilas Boas & Barbosa, 2016). For example, it is a pattern to structure a lesson by first providing an explanation, then assigning exercises for students, followed by corrections. Another common pattern is always answering students' questions immediately. On the other hand, a different pattern involves structuring the lesson by introducing a problem situation, having students work in groups, and then formalizing the knowledge. These patterns represent the qualities of teachers' participation in school mathematics—how they engage in this social practice.

The contracted preposition ***in*** within the expression ***Mathematics in Teaching*** emphasizes that this refers to the mathematics communicated by teachers during the act of teaching. It occurs neither before nor after but precisely at the moment teachers interact with students. Therefore, Episodes I and II describe lessons that took place, capturing the teachers' participation patterns in school mathematics—forms of MnT.

The concept of MnT is not detached from school mathematics. As presented here, the former refers to how mathematics teachers participate in the latter. This participation is not arbitrary, as discussed in the previous section; school mathematics is evocative, meaning it prompts specific forms of involvement from teachers. Thus, MnT can only be understood in relation to school mathematics. This does not mean that school mathematics determines MnT; instead, MnT reflects a greater or lesser alignment and resistance to its demands.

MnT is a response to the evocation of school mathematics—what it demands from teachers' participation. These demands originate from various sources, which we can group into four categories:

- Previous experiences: Through their participation in school mathematics (as teachers and in their own experiences as students), teachers recognize what is considered valid to communicate in this social role.

- Professional development: Experiences in continuing education programs may prompt teachers to participate in school mathematics in specific ways.

- Curricular documents: Curricular guidelines and textbooks demand certain qualities from teachers' MnT.

- large-scale assessments and public policies: through the educational policies implemented by the state, teachers are required to teach mathematics in a certain way, with emphasis on the inductive role of large-scale assessments;

- Other school actors: Other social actors within the school, such as colleagues, supervisors, principals, and parents, may also influence how teachers participate in school mathematics.

None of these sources alone determines teachers' MnT. However, teachers draw from them to shape their MnT. When I say that MnT responds to the evocation of school mathematics, I reiterate that this is not a determination. MnT can align with, oppose, or adapt to these demands, but it is always a response to the evocation of school mathematics.

In their communicative interactions within the school culture, teachers respond to these demands through MnT. Colleagues, supervisors, parents, and students can all influence teachers' participation. Imagine, for instance, in Episode I, that students resist engaging in the exploratory task, which asks them to observe that some functions have graphs symmetric with respect to the y-axis, others with respect to the origin, and one function that does not fit either pattern. In Episode I, the teacher's MnT requires students to engage in exploration to later consolidate new knowledge based on the observed regularities. However, as often happens in lessons where students are unfamiliar with such pedagogical approaches, they may remain passive, waiting for the teacher to solve the problem on the board, or even request a direct explanation (e.g., “Just explain it, teacher!”). In this situation, the evocative nature of school mathematics is expressed through students' participation.

Episodes like the one described above demand a response from the teacher—an approach to dealing with students' resistance. The teacher might reinforce the exploratory nature of the task, encouraging student engagement by asking questions such as, “What is the task asking you to do?” or “What do you observe?” Alternatively, the teacher might go to the board and provide a brief explanation to support the students' exploration. We don’t know in advance—the teacher will decide in the moment. However, the teacher remains aware of their participation in school mathematics, which will influence their decision in addressing students’ resistance.

Davis and Renert (2014) argue that Mathematics for Teaching (MfT) is emergent, involving continuous reinterpretation and recontextualization of mathematical content according to students' needs and characteristics. They suggest that mathematics takes shape and transforms through pedagogical practices. Inspired by this idea, I also argue that MnT is emergent. Although school mathematics in a given context demands specific forms of teacher participation, the way it materializes depends on communicative interaction with students.

Even in a lesson like the one illustrated in Episode II—characterized by exposition, where the teacher can anticipate its structure—a student’s unexpected question can shift the course of the lesson. Imagine a student asking, “Can a function be even on one interval and odd on another?” The teacher might not have an immediate answer, as they would need to reflect on the definition of function parity. This could be unsettling, as the question was not anticipated. The teacher will need to decide in the moment how to respond.

We can recognize the emergent nature of MnT in any interaction with students. Teachers do not communicate with students without considering their profile, mathematical experiences, and potential difficulties. Lins (1999) reminds us that communication is shaped by the speaker's interpretation of the listener. This is evident in how a teacher might communicate differently with classes of varying profiles or tailor individual interactions within the same class. This communicative relationship between speaker and listener occurs within a context conceptualized here as school mathematics.

Therefore, I emphasize that school mathematics and MnT are inseparable, but MnT is not determined by the former. While school mathematics establishes boundaries for what and how to teach, MnT encompasses a range of possibilities and remains emergent, shaped by pedagogical interactions with students.

Mathematics for Teaching

School mathematics not only shapes MnT but also what happens beyond and before the interaction between mathematics teachers and students. Teachers decide in advance what they will teach in class, what materials they will use, how they will conduct the lesson, etc. Teacher education programs and curriculum guidelines provide indications of how MnT should or could take place. Large-scale assessments demand results, expecting students to achieve specific outcomes, signaling to teachers what and how to teach. In Mathematics Education research, we researchers describe what happens in mathematics classes, pointing out implications for teaching practice. These are just a few examples of social situations that refer to and represent the pedagogical interactions between those who teach and those who learn mathematics in the school environment. However, these situations are not to be confused with the actual interactions with students as they occur. They are representations of how MnT takes place or could take place but are not MnT itself, as it is instantaneous and emergent, as discussed in the previous section.

All these representations of how MnT happens or could happen are what I refer to here as **Mathematics for Teaching** (MfT). The expression **for Teaching** indicates purpose or intent. Thus, all the situations listed in the previous paragraph aim at or demand certain ways for teachers to teach mathematics. However, as discussed earlier, the way MnT materializes will depend on interaction with students. The situations listed above are not part of MnT but occur outside the interactive context of the classroom, even though they refer to it; therefore, they are MfT.

As the reader may note, I use the term *Mathematics for Teaching* more narrowly than Davis and Renert (2014). While those authors use the term to describe ways of communicating mathematical concepts shared by the collective of mathematics teachers, activated according to the situation at hand, I use the term to indicate any re-presentation of *Mathematics in Teaching*. The term *re-presentation* emphasizes that this is a new presentation of MnT—it is not MnT itself but a shift that takes place outside the interaction between teachers and students.

Let us return to Episodes I and II, where teachers teach even and odd functions in different ways. These episodes describe the interaction between teachers and students, showing us the teachers' participation—MnT. However, beyond this interaction, we can find several re-presentations of what and how to teach even functions. Teachers may have based their approach on how the topic is presented in textbooks and curricular materials. Perhaps they relied on training programs they attended or drew on their experiences teaching or observing others teach the topic. All these situations simplify how interaction with students actually occurs or could occur. None of them, for instance, could predict the dialogue in Episode I, but they can suggest certain ways of participating.

Therefore, there is a clear distinction between MfT and MnT. While MnT refers to teachers' actual participation in school mathematics through interaction with students, MfT refers to any representation of MnT, without including real-time interaction with students, even though it can represent it. I believe this distinction is essential for describing teachers' specific mathematics because, although lesson plans, curricular documents, teacher training programs, past experiences, and curricular materials may suggest ways for teachers to participate in mathematics lessons, their participation is realized through interaction with students. MfT refers to moments before interaction with students; MnT refers to the interaction itself.

MnT draws upon MfT, as teachers enter the classroom with a sense of what and how to teach, informed by sources that re-present MnT, as previously listed. In other words, MnT shifts from MfT, with teachers modifying their participation based on how students engage, a topic I will explore in more detail in the next section.

In summary, Mathematics for Teaching (MfT) refers to any representation of MnT, including those found in curricular documents, textbooks, teacher education programs, large-scale assessments, or even Mathematics Education research, as well as how teachers interpret their past experiences with mathematics teaching.

MfT has three properties: it is representational, imaginary, and demanding. By **representational**, I mean that MfT offers a simplified view of what and how mathematics can be taught. Any description considered MfT cannot capture the richness of the real-time interaction between teachers and students but only certain aspects. For example, a textbook might present a sequence for teaching a topic, such as a problem situation, its resolution, the formalization of a concept, and examples. The teacher's version of the textbook may include notes about potential student difficulties or tips for addressing specific issues, but it can never predict exactly how the interaction will unfold. Thus, a textbook provides only a representation of how MnT might or should occur. The same reasoning applies to other descriptions that fall under MfT.

By **imaginary**, I mean that MfT refers to a generalized classroom and student. As sociocultural and communicative perspectives emphasize, every student is different (Lins, 1999). No representation of MnT can encompass all possible forms of student participation, and therefore, all possible forms of MnT. Perhaps the most emblematic example of MfT's imaginary nature is found in curricular documents. For instance, Brazil's BNCC curriculum framework prescribes the development of the same mathematical skills for all students nationwide, overlooking regional, economic, social, and cultural differences. This reflects an idealized student generalized across the country. Even MfT that acknowledges variations in student profiles cannot fully capture the diversity of MnT. For example, Mathematics Education research that identifies common student errors in a given topic (Cury, 2007) may provide insights into various student responses, requiring teachers to adopt different approaches. However, classroom situations often surprise us, with students presenting new difficulties that require teachers to devise solutions on the spot.

By **demanding**, I mean that MfT establishes expectations and norms about what and how teachers should teach. It provides direction for teaching practice through curricula, textbooks, or large-scale assessments that assume an idealized model of teaching and learning. Thus, MfT not only offers a simplified view of MnT but also demands certain forms of participation in school mathematics—certain varieties of MnT. However, even as a demanding force, MfT does not exhaust the possibilities of real-time interaction with students.

In summary, **Mathematics for Teaching** (MfT) and **Mathematics in Teaching** (MnT) are intrinsically connected, as MfT represents, imagines, and demands MnT. However, as discussed, MnT is not a simple execution of MfT's directions. Instead, the dynamic interaction between teachers and students in the classroom introduces new nuances and modifications that cannot be entirely predicted or represented by MfT. This relationship between MfT and MnT will be explored further in the next section.

How do Mathematics for Teaching and Mathematics in Teaching work together?

The relationship between Mathematics for Teaching (MfT) and Mathematics in Teaching (MnT) can be understood as a process of pedagogical recontextualization, as developed by Bernstein (2000). Bernstein argues that knowledge is not simply transmitted from one context to another but undergoes transformations in its meaning and organization as it shifts between social spheres. In the case of MfT and MnT, the transitions are neither linear nor direct; it is a dynamic movement in which the former informs but does not determine the latter, and vice versa. I will explain this further.

When teachers enter the classroom, MfT—which includes anticipations such as lesson plans, curricular documents, textbooks, and training—gets translated into teachers' mathematical communication with their students (MnT). This process involves instantaneous and adaptive decisions: the teacher must transform what is articulated in MfT into a form of communication they consider suitable for their students within a particular pedagogical context. This is a shift from representations of how mathematics should or could be taught to real-time communication with students, where MnT emerges as situated.

For example, a lesson plan may outline an expository sequence followed by exercises and corrections. However, during the lesson, the teacher may notice that students are experiencing unexpected difficulties or that a question raised requires an alternative explanation. This deviation from the original plan is a manifestation of emergent MnT, which takes shape in the moment of interaction and requires the teacher to interpret the context and the signals conveyed by the students. Events like these demonstrate that the transition from MfT to MnT is adaptive to the pedagogical context.

The transition from MfT to MnT is not merely the execution of a pre-established plan but involves flexibility and transformation. The teacher decides, based on the students' needs and the characteristics of the pedagogical context, what to prioritize and how to adjust the plan. MfT offers an anticipation of the pedagogical work, but it is within MnT that this work is updated, reconfigured, and realized.

The shift also occurs in the opposite direction: from MnT to MfT. To conceptualize this transition, I draw on the notion of reverse recontextualization, as the concept of pedagogical recontextualization in Bernstein (2000) was originally more suited to the transition from MfT to MnT. In Barbosa (2013), I called this process reverse recontextualization, which occurs when teachers transform their concerns and classroom experiences into their lesson plans, discussions with colleagues, written reflections, etc. In other words, it happens when teachers represent MnT within MfT.

This occurs when teachers' practical experiences in the classroom extend beyond the immediate pedagogical interaction with students, informing future decisions and planning. When reflecting on their lessons, teachers may reconfigure their practices for future lessons, modify teaching materials, or share experiences with colleagues during pedagogical meetings.

This retroactive process ensures that MfT is constantly renewed, incorporating insights and adaptations derived from past practices. For example, if a teacher realizes that students struggled to understand the concept of even and odd functions through an expository approach, they might explore an alternative approach based on exploratory tasks for a future lesson or adjust the way they deliver the exposition. By reflecting and planning, the teacher recontextualizes their practice, transforming MnT experiences into new resources for MfT.

MfT and MnT are articulated in a non-deterministic manner. While MfT may set expectations about what and how to teach, classroom practice often escapes the control of curricular prescriptions. As Bernstein (2000) argues, recontextualization implies that every pedagogical context is unique, and the way a teacher responds to MfT’s expectations depends on the specific conditions of their interaction with students. Thus, we can say that MfT and MnT are articulated recursively, meaning the process is continuous and cyclical.

Final remarks

In this essay, I began by presenting two classroom episodes that illustrate the specificity of teachers' mathematics. I used them to argue that theoretical models based on the teacher cognition paradigm, such as Mathematical Knowledge for Teaching (MKT) and Mathematics Teacher's Specialized Knowledge (MTSK), are limited in capturing the situated nature of teachers' specific mathematics. I also contended that perspectives developed to address this limitation, such as Mathematics for Teaching (MfT) proposed by Davis and Renert (2014), fail to encompass the controlled nature of teachers' specific mathematics.

While acknowledging the significant contributions these perspectives have made to the field of Mathematics Education, I sought in this essay to advance the thesis that teachers' specific mathematics cannot be understood solely as a set of cognitive knowledge or practices. I further argue that, although social approaches are relevant, they have not adequately advanced the understanding of the socio-institutional constraints that shape and permeate teachers' work. To deepen this understanding, I emphasized the need to recognize that teachers' specific mathematics is intrinsically relational to school mathematics and the pedagogical context in which it takes place. Thus, I highlighted the evocative nature of these practices.

I argued that it is essential to distinguish between *Mathematics in Teaching* (MnT) and *Mathematics for Teaching* (MfT). MnT refers to teachers' participation in school mathematics during the moment of pedagogical interaction with students, while MfT represents the anticipations and guidelines about how this participation may or should occur. The articulation between MnT and MfT is recursive and non-deterministic, meaning that while MfT guides teachers' mathematical communication with students, classroom situations often evolve as teachers respond to emerging needs and dynamics.

In light of the discussions developed, future studies need to investigate how different institutional and social dimensions shape teachers' specific mathematics. In particular, it is necessary to advance research that examines the influence of public policies, curricular documents, and large-scale assessments on the configuration of both MnT and MfT. It is equally important to deepen the analysis of the emergent nature of MnT, exploring how teachers respond to students' resistance, improvise, and adjust MfT according to students' profiles and needs—in other words, how the shift from MfT to MnT occurs.

Another promising area of study is the exploration of reverse recontextualization, referring to the process by which teachers' classroom experiences are transformed into future planning, pedagogical discussions, and professional development. These investigations can provide insights into how teaching practices are continuously renewed and how MnT contributes to updating MfT.

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