EMP Educação Matemática Pesquisa
ISSN 1983-3156
Qualis A1

http://dx.doi.org/10.23925/1983-3156.2025v27i5p057-080

Articulating combinatorics and probability: modeling a middle school curriculum

Articulando combinatória e probabilidade: modelando o currículo dos anos finais do ensino fundamental

Articulando combinatoria y probabilidad: modelando el currículo del liceo

Articuler la combinatoire et la probabilité: modéliser le curriculum du collège

Ewellen Tenório de Lima[1]
Universidade Federal de Pernambuco
Doutora em Educação Matemática e Tecnológica
https://orcid.org/0000-0002-3654-0370

Rute Elizabete de Souza Rosa Borba[2]
Universidade Federal de Pernambuco
Doutora em Psicologia Cognitiva
https://orcid.org/0000-0002-5098-4461

Abstract

This work presents an excerpt from a doctoral thesis study that addressed the following research question: How are combinatorics and probability approached in various middle school curricula, and what can be done to articulate these topics? In this regard, and based on the theory of conceptual fields and classifications of different combinatorial and probabilistic situations, the research aimed to investigate the interplay between combinatorics, probability, and their articulation within both prescribed and presented middle school curricula. To this end, official curriculum documents and textbooks were analyzed. Based on the findings from these stages of the research, a teacher-oriented resource was developed. This paper presents material consisting of eight sets of tasks adapted from problems found in the analyzed textbooks – materials already accessible to teachers and their students. The proposed adaptations aim to expand the contexts addressed, allowing for the exploration of different cognitive demands of probability through combinatorial problems. Moreover, the questions also cover a range of difficulty levels, related to the number of possibilities involved and the symbolic representations presented or required from the students. It is important to emphasize that this material, just as any other educational resource a teacher may use, should not be viewed as

[1] ewellen.lima@ufpe.br
[2] resrborba@gmail.com

definitive or complete. Rather, it is intended as a starting point —one that can and should be adapted by teachers to suit the specific needs and goals of their own classrooms.

Keywords: Combinatorics, Probability, Articulations, Proposal, Middle school.

Resumen

Este trabajo consiste en un recorte de un estudio de tesis doctoral que tuvo el siguiente problema de investigación: ¿Cómo se abordan la combinatoria y la probabilidad en diferentes instancias curriculares del liceo y qué se puede hacer para articular estas temáticas? En este sentido, a la luz de la teoría de los campos conceptuales y de clasificaciones de diferentes situaciones combinatorias y probabilísticas, la investigación en cuestión tuvo como objetivo investigar la combinatoria, la probabilidad y sus articulaciones en los currículos prescriptos y presentados a la Escuela Intermedia. Para ello, se analizaron documentos curriculares oficiales y libros de texto, y a partir de los hallazgos de estas etapas de la investigación, se construyó un material dirigido al profesorado. En el presente texto se presenta dicho material, el cual está compuesto por ocho bloques de cuestiones que consisten en adaptaciones de problemas presentes en los libros de texto analizados (material al cual el profesor, y sus estudiantes, ya tienen acceso). Las adaptaciones propuestas tuvieron como objetivo ampliar los contextos abordados, permitiendo que, a partir de problemas combinatorios, se exploren las diferentes demandas cognitivas de la probabilidad. Además, se contemplaron distintos niveles de dificultad, en función del número de posibilidades involucradas y de las representaciones simbólicas presentadas o solicitadas a los estudiantes. Cabe destacar que dicho material, al igual que otros a los que el profesor pueda tener acceso, no debe considerarse como definitivo y acabado, sino más bien como un punto de partida: este puede y debe ser modelado (por el profesor) a las diferentes necesidades y objetivos de su aula.

Palabras clave: Combinatoria, Probabilidad, Articulaciones, Propuesta, Liceo.

Résumé

Ce travail constitue un extrait d'une étude de thèse de doctorat qui avait pour problématique: Comment la Combinatoire et la Probabilité sont-elles abordées dans différentes instances curriculaires du Collège, et que peut-on faire pour articuler ces thématiques? Dans cette perspective, à la lumière de la Théorie des Champs Conceptuels et de classifications de différentes situations combinatoires et probabilistes, la recherche avait pour objectif d'étudier la Combinatoire, la Probabilité et leurs articulations dans les curriculums prescrits et mis en œuvre au Collège. Pour ce faire, des documents curriculaires officiels ainsi que des manuels scolaires ont été analysés. À partir des résultats obtenus dans ces étapes de la recherche, un matériel à destination des enseignants a été élaboré. Le présent article présente ce matériel, composé de huit blocs de questions, constitués d'adaptations de problèmes issus des manuels scolaires analysés (matériel auquel les enseignants et leurs élèves ont déjà accès). Les adaptations proposées visaient à élargir les contextes abordés, permettant d'explorer, à partir de problèmes combinatoires, les différentes exigences cognitives de la Probabilité. En outre, différents niveaux de difficulté ont été pris en compte, en fonction du nombre de possibilités impliquées et des représentations symboliques présentées ou demandées aux élèves. Il convient de souligner que ce matériel, comme tout autre auquel l'enseignant peut avoir accès, ne doit pas être considéré comme un produit fini et définitif, mais plutôt comme un point de départ: il peut et doit être adapté (par l'enseignant) aux besoins et objectifs spécifiques de sa classe.

Mots-clés: Combinatoire, Probabilité, Articulations, Proposition, Collège.

Resumo

Este trabalho consiste em um recorte de um estudo de tese de doutorado que teve o seguinte problema de pesquisa: Como a combinatória e a probabilidade são abordadas em diferentes instâncias curriculares dos anos finais e o que se pode fazer para articular essas temáticas? Nesse sentido, à luz da teoria dos campos conceituais e de classificações de diferentes situações combinatórias e probabilísticas, a pesquisa em questão teve por objetivo investigar combinatória, probabilidade e suas articulações em currículos prescritos e apresentados aos anos finais do ensino fundamental. Para tal, foram analisados documentos curriculares oficiais e livros didáticos e, a partir dos achados de tais etapas da pesquisa, foi construído um material direcionado ao professor. No presente texto, esse material é apresentado, sendo o mesmo composto por oito blocos de questões que consistem em adaptações de problemas presentes nos livros didáticos analisados (material ao qual o professor, e seus estudantes, já têm acesso). As adaptações propostas visaram ampliar os contextos abordados, permitindo que a partir de problemas combinatórios sejam exploradas as diferentes demandas cognitivas da probabilidade. Ainda, foram contemplados diferentes níveis de dificuldade, em função do número de possibilidades envolvidas e das representações simbólicas apresentadas ou solicitadas aos estudantes. Destaca-se que tal material, assim como os demais aos quais o professor pode vir a ter acesso, não deve ser encarado como pronto e acabado, mas, sim, como um ponto de partida: o mesmo pode e deve ser moldado (pelo professor) às diferentes necessidades e objetivos de sua sala de aula.

Palavras-chave: Combinatória, Probabilidade, Articulações, Proposta, Anos finais.

Articulating combinatorics and probability: Modeling a middle school curriculum

Combinatorics is the area of mathematics that develops tools to “enumerate all the possible ways in which a given number of objects can be combined in such a way that one is sure that none of the possibilities has been omitted” (Batanero et al., 1996, p. 17, free translation). In turn, Morgado et al. (1991) affirm that probability “creates, develops, and generally researches models that can be used to study random experiments or phenomena” (p. 119). Such models become essential for analyzing situations where we are not sure of the results, as, “between the certain or the safe (what will necessarily occur or what is true without any doubt) and the impossible (what can never occur) is the probable” (Godino et al., 1991, p. 19, free translation).

In this way, combinatorics and probability enable the understanding of random events, providing us with mathematical tools that we can employ to determine the number of results and/or the possible results in a given context, as well as predict the probability of a specific event.

These are relatively recent themes in the basic education mathematics curriculum, having appeared for the first time in the block of information treatment of the National Curriculum Parameters (Brasil, 1998). In this sense, teaching, learning, and the evolution of these themes in the curriculum have been the focus of research in mathematics education.

The research reported here defines curriculum following Sacristán (2000), i.e., we understand that the notion of curriculum extends far beyond an ordered selection of content. We believe the curriculum undergoes several transformations, and we also believe it is influenced by and influences various agents until it materializes in teachers’ practice and students’ learning. Thus, the author considers “six moments, levels or phases in the development process, which discover peculiar fields of defense, which help us to understand connections between such levels and which make manifest how [...] these other phases exist” (Sacristán, 2000, p. 104), namely: prescribed curriculum, presented curriculum, curriculum shaped (or modeled) by the teacher, curriculum in action, implemented curriculum, and assessed curriculum.

This article presents an excerpt from a doctoral thesis that encompassed three studies (Figure 1), directly related to the first three curriculum instances mentioned above, and with the following specific objectives: 1. Investigate the curriculum prescriptions for the study of combinatorics and probability in middle school; 2. Analyze how such themes are addressed in textbooks at this stage of schooling, defending the articulation between both themes in view of the broad development of combinatorial and probabilistic reasoning, 3. Develop a proposal outlining the articulated problems to be explored in the classroom.

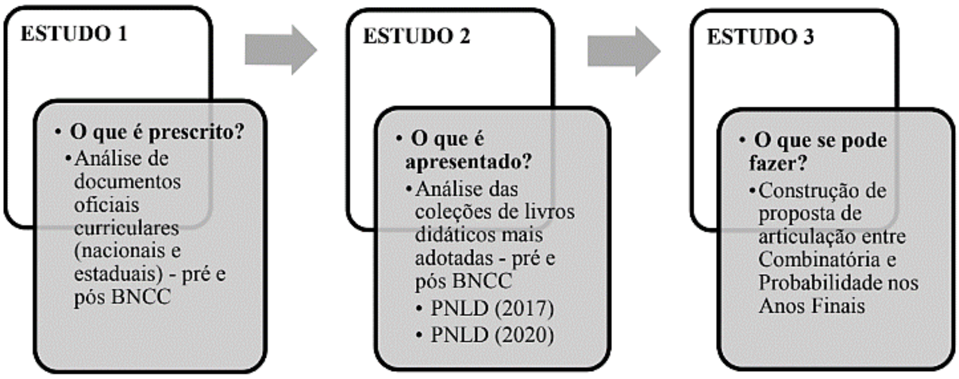


Figure 1.

Research structure (Lima, 2022)

The product of this last objective (Study 3), a set of articulated questions, is the main focus of this text. This type of material represents a curriculum that was modeled (by the researchers), but that is presented (to the teacher), since the instance referring to the *shaped curriculum* is essentially characterized by the teacher as an agent of curriculum transformation, as this professional:

> [...] is a very decisive active agent in the realization of the contents and meanings of the curricula, shaping any proposal made to it based on its professional culture, whether through administrative prescription or the curriculum developed through materials, guides, textbooks, etc. [...] is a 'translator' who intervenes in the configuration of the meanings of the curriculum proposals. The plan that teachers make for teaching, or what we understand as programming, is a moment of special significance in this translation. (Sacristán, 2000, p. 105)

The main theoretical contributions used in the thesis research in question, and which, therefore, support the discussions developed in this work, are presented in the following section.

Theoretical contributions

The theory of conceptual fields (Vergnaud, 1996) enables the analysis of teaching and learning processes, starting from the perspective of concept development. According to the author, a concept is formed by a tripod comprising sets of *situations* that give meaning to such a concept, *invariants* —immutable properties that characterize each type of situation —and *symbolic representations* used to represent them. Vergnaud (1996) also highlights that "to study the functioning and development of a concept, it is necessary to consider these three plans at the same time" (p. 166).

This author defines a conceptual field as "a set of situations, the mastery of which requires a range of concepts, procedures, and symbolic representations in close connection" (Vergnaud, 1986, p. 10). It is worth highlighting that the themes focused on in this work, combinatorics and probability, are inserted in the conceptual field of multiplicative structures, since this conceptual field is composed of the "set of situations that require multiplication, division, or a combination of these two operations" (Vergnaud, 1996, p. 167).

In a complementary manner, to specifically characterize combinatorial situations, we adopted Borba's (2010) categorization. The author unified previous classifications and organized such situations based on their invariants of exhaustion (common to all situations), of order, and choice. Thus, in this research, we consider the combinatorial situations of the product of measurements, arrangement, combination, and permutation.

Problems addressing the situation of product of measurements are those in which two or more sets of elements are involved, from which one element of each must be chosen to form the possibilities. In this type of situation, the order of the elements does not determine distinct possibilities. Example: In how many ways can a person dress on a trip if they have three pairs of pants, four t-shirts, and two shoes in their suitcase? (24 ways).

In arrangement problems, only one set is involved, and the choice consists of some elements from that set. In this case, the order determines several possibilities. Example: How many distinct two-digit passwords can Marlon form using letters from his name? (30 passwords).

In turn, in permutation problems, only one set is involved, but all elements must be used simultaneously. Thus, the order of these elements will determine the various possibilities. Example: In how many ways can three people queue up? (six ways).

Finally, combination problems also involve elements from a single set, from which some elements must be chosen. However, in this type of situation, unlike in an arrangement, the order of the elements does not determine distinct possibilities. Example: How many different trios can be formed from a group of five friends? (ten trios).

Regarding the characterization of different probabilistic situations, the present research was based on Bryant and Nunes (2012). These authors state that a broad understanding of probability is linked to four cognitive demands, which are related to: randomness, sample space, comparison and quantification of probabilities, and correlation.

Understanding randomness refers to the understanding that the results of random experiments cannot be predicted with certainty, as well as the understanding of the independence of events in random sequences. For example, when flipping a coin, we are not sure of the result we will get (it could be either heads or tails). Furthermore, when flipping this coin 50 times, it is essential to realize that each flip is independent of the previous results.

Related to the second cognitive demand pointed out by the authors, the sample space consists of the set of all possible outcomes of a given random experiment, having a close relationship with combinatorial thinking. By mastering this cognitive demand, students must be able to understand and construct simple and compound sample spaces, which will support the analysis of probabilities, including the ability to compare and/or quantify them.

It is noteworthy that the authors present the comparison and quantification of probabilities as a single demand, related to proportional thinking. To compare probabilities, it is necessary to consider the sample spaces involved, rather than just the absolute quantity of elements in each case. Furthermore, to quantify probabilities, the ratio between favorable cases and all possible existing cases (sample space) in a given context is used.

Finally, correlation is related to understanding associations or influence between events. This demand is not included in middle school curricula; therefore, we removed it from our proposal in this article. It is worth noting, however, that understanding correlations plays a crucial role in more advanced stages of schooling, as it enables the analysis of probabilistic risks and the application of knowledge of probability to inform decisions related to real-world situations.

In the following sections, we will discuss symbolic representations commonly used when working with combinatorics and probability. These representations are essential in problem-solving and must be gradually learned so that both types of reasoning can be fully developed. We argue that different types of combinatorial and probabilistic situations should be addressed in the classroom (also making connections), so that their invariants can be explored and the use of varied symbolic representations appropriate to each stage of schooling can be encouraged.

Below, we review the main results of Studies 1 and 2, which were previously presented in other works. These results will serve as a basis for the presentation and discussion of the proposal, which consists of a set of articulated problems aimed at exploring different aspects of concepts and situations covering combinatorics and probability in a middle school classroom.

Combinatorics and Probability in Middle School

The analyses of prescribed and presented curricula for middle school, stages of the thesis research focused on here, were addressed in detail in Lima and Borba (2021; 2022a).

In summary, Study 1 consisted of the analysis of the curricula prescribed for middle school. The objective of this study was to investigate how work with combinatorics and probability is prescribed and whether and how relationships are established between knowledge related to these themes in official documents aimed at the final years of elementary school (middle school). The analysis of this curriculum instance was conducted in light of its various roles, including serving as a means of controlling teaching practices, organizing knowledge within schooling, and laying the groundwork for a common culture and equality of opportunities in official schooling (Sacristán, 2000). This type of document influences the selection of what will actually be taught in the classroom and, therefore, affects other curriculum instances.

In this sense, in light of the theoretical references previously presented, we analyzed national and state documents, namely: the National Curriculum Parameters [Parâmetros Curriculares Nacionais] (Brasil, 1998), the Common National Curriculum Base [Base Nacional Comum Curricular] (Brasil, 2018), the Parameters for Basic Education in the State of Pernambuco [Parâmetros para a Educação Básica do Estado de Pernambuco] (Pernambuco, 2012), and the Pernambuco Curriculum [Currículo de Pernambuco] (Pernambuco, 2019).

In this initial study of combinatorics, the results primarily indicated that insufficient attention has been devoted to this topic. The analyzed curriculum documents do not detail work with the several types of combinatorial situations at this stage, with the product of measures being the only combinatorial situation explicitly mentioned and/or exemplified in them. Probability, on the other hand, gains more space: work with different cognitive demands is prescribed for all years of elementary school, highlighting the importance of teaching that provides contact with varied situations, thereby fostering the deepening of probabilistic knowledge (Lima & Borba, 2021).

In Study 2, the focus is on textbooks because this resource "usually translates the meaning and content of the prescribed curriculum for teachers, providing an interpretation of it. Prescriptions are usually very generic and, to that extent, are not sufficient to guide educational activity in classes" (Sacristán, 2000, pp. 104-105). In this sense, from a comparative perspective pre and post BNCC, we analyzed 24 volumes, distributed among three collections aimed at middle school, most distributed by the National Book and Teaching Material Program 2017 [Programa Nacional do Livro e do Material Didático (PNLD)], whose authors have collections also approved by PNLD 2020. The collections were: Desire to Know / Reality & Technology (Collection A), Understanding and Practice (Collection B), and Teláris Project / Teláris (Collection C) [*Vontade de Saber / Realidade & Tecnologia* (Coleção A), *Compreensão* *e Prática* (Coleção B) e *Projeto Teláris / Teláris* (Coleção C)].

In the selected collections, we surveyed chapters that addressed combinatorics and/or probability. Then, reading these chapters in full helped us identify the problems to be analyzed in light of the theoretical contributions adopted (Vergnaud, 1986, 1996; Borba, 2010; Bryant & Nunes, 2012).

One hundred and eighty-nine combinatorics activities were identified (53% addressing the product of measures situation, 28% of arrangement, 13% of permutation, and only 5% of combination) (Lima & Borba, 2022a). Combination problems were present in only one collection, further highlighting an unbalanced distribution. The present study argues that failure to explore different situations can limit reflections on the invariants that differentiate them and also restrict the use of varied symbolic representations when exploring them in the classroom, hindering the broad development of combinatorial reasoning. A greater emphasis on probability was also observed in textbooks (Lima & Borba, 2022a): in total, 995 activities were identified in the analyzed collections. The activities address various cognitive demands to foster a comprehensive understanding of probability. However, quantification of probabilities was much focused (70.4% of the probabilistic problems identified). We argue that not addressing the intrinsic aspects of probability (understanding randomness, sample spaces, and proportionality) in middle school, sometimes because teachers rely on the fact that it has already been covered in earlier stages of schooling, can generate gaps in the development of probabilistic reasoning and lead to the practice of merely carrying out probability calculations without attributing meaning to them.

Finally, about the connections between these themes, in the prescribed curricula, the mention of the use of symbolic representations in common between combinatorics and probability with emphasis on the fundamental counting principle (FCP) highlights the development of combinatorial knowledge as a tool for calculating probabilities (Lima & Borba, 2021). In turn, in textbooks, we identified a potential for articulation between combinatorics and probability not only arising from common symbolic representations, but also from unique random contexts: activities of a random nature that allow both combinatorial and probabilistic concepts to be explored to understand the proposed problems. Among all the identified combinatorial and probabilistic activities (1,184), we perceived potential for articulation in approximately 25% (Lima & Borba, 2021). Some of these activities served as the basis for the construction of the articulation proposal that comprised Study 3, presented below. This study was based on the premise of expanding combinatorial contexts, also to address the different cognitive demands of probability.

Articulation Proposal

Study 3 of the thesis, the focus of this article, aimed to develop a proposal for articulation between combinatorics and probability, targeting teachers to promote teaching that fosters the development of combinatorial and probabilistic reasoning among middle school students. In this sense, this study involved constructing a set of problem situations to explore various interconnected combinatorial and probabilistic situations.

This articulation proposal takes into account the theoretical contributions adopted, as indicated by previous studies, and, in particular, the results obtained in Studies 1 and 2, mentioned in the previous section.

Thus, the proposal aims to support teachers in incorporating randomness into the classroom, exploring its diverse facets, and encouraging the development of combinatorial and probabilistic reasoning. It is therefore based on the desire to fill gaps observed in the prescriptions and textbooks analyzed with regard to work involving combinatorics and probability – specifically, the articulation between them.

The set of problem situations was built mainly from some of the problems with potential for articulation identified in Study 2 (What is displayed? – analysis of textbooks). In this way, they could be adapted to accommodate the objective of exploring a variety of combinatorial situations (Borba, 2010) and probabilistic demands (Bryant & Nunes, 2012) in the proposed activities. In this sense, Study 3 involved the development of eight blocks of articulated problems, as outlined in Table 1.

Table 1.

Structure of the articulation proposal (Lima, 2022)

	Combinatorial situation	Probabilistic Demands			
Block 1	Product of measurements	EA 1	AL 1	COMP 1	QUANT 1
Block 2	Product of measurements	EA 2	AL 2	COMP 2	QUANT 2
Block 3	Arrangement	EA 1	AL 1	COMP 1	QUANT 1
Block 4	Arrangement	EA 2	AL 2	COMP 2	QUANT 2
Block 5	Permutation	EA 1	AL 1	COMP 1	QUANT 1
Block 6	Permutation	EA 2	AL 2	COMP 2	QUANT 2
Block 7	Combination	EA 1	AL 1	COMP 1	QUANT 1

Block 8	Combination	EA 2	AL 2	COMP 2	QUANT 2

EA1: sample space with up to 24 possibilities; EA2: sample space with more than 24 possibilities; AL1: randomness – identification of random experiment; AL2: randomness – random sequence; COMP1: comparison of probabilities in the same sample space; COMP2: comparison of probabilities in different sample spaces; QUANT1: quantification of probabilities (ratio); QUANT2: quantification of probabilities (percentage).

In this sense, we sought to complement the problems to which the teacher already has access (present in textbooks, as exemplified in Figure 2), taking advantage of their statements and some images. The adaptations (based on the development of different items related to the same problem and guidance on the use of specific symbolic representations) aimed to expand the potential for articulation between combinatorics and probability, starting from the same context, this being a possibility for the development of both reasonings in question.

4 Uma indústria de brinquedos fabrica a mesma boneca com algumas variações de roupas e tons de cabelo. Veja a seguir as variações.

a) Conforme as opções acima, de quantas maneiras diferentes essa boneca pode ser vendida? 18 opções

b) A mãe de Mariana comprou uma dessas bonecas. Qual é a probabilidade de ela ter comprado uma boneca de cabelo preto, vestido amarelo e sapato preto? $\frac{1}{18}$

c) Mariana pediu uma boneca que tivesse cabelo preto. Se sua mãe comprou a boneca conforme o pedido de Mariana, qual é a probabilidade de ela ter vestido amarelo e sapato preto? $\frac{1}{6}$

Figure 2.

Combinatorial situation (product of measurements) with probabilistic deepening (quantification of probabilities), Understanding and Practice Collection, 8th grade (Silveira, 2018, p. 121)

The presence of two blocks related to each combinatorial situation (Table 1) is related to the variation in the difficulty level of the problems, especially regarding the number of choice steps involved and the order of magnitude of the sample space (and the understanding of when, or not, it is feasible to explain all possibilities one by one in a random context); the identification of randomness in a direct context or associated with experimentation in a random sequence (independence of events); the comparison of probabilities (within the same sample space or between different sample spaces); and the representation used to communicate results obtained from the quantification of probabilities.

Thus, regarding the variation in the level of difficulty of the proposed problem blocks, while odd blocks have a lower level of complexity, even blocks have a higher level of difficulty. This variable is linked to: the number of choice steps, the number of possibilities in the problem and the type of approach to the cognitive demands of probability related to randomness (in even blocks it is linked to the understanding of the independence of events in random sequences), comparison of probabilities (in even blocks, the comparison occurs between different sample spaces, requiring consideration of the proportional nature of the quantities in question) and conversion of the representation used to communicate the results in terms of probabilities quantification (in even blocks, the use of percentages is requested). These blocks are presented below (Figures 3-10) and have response expectations described for each item.

The proposed adaptation of Block 1 was designed to use the context of the original problem (see Figure 2); that is, the assembly of a doll considering three characteristics (hair; clothes; shoes) as a basis for working with the combinatorial situation implicit in the problem (product of measurements), but also with different cognitive demands of probability.

A toy company manufactures the same doll with some variations in clothing and hair colors, as illustrated in the image.

a) According to the options above, in how many different ways can this doll be sold? How did you achieve that answer? (Product of Measurements – Expected answer: 18 ways)

b) If you haven't already done so in item a, use your preferred method to indicate each of the possible characterizations of this doll model that this toy industry manufactures.. (Sample Space – Expected response: Explanation of the 18 possibilities via listing, drawing, diagram, possibilities table or possibilities tree, among other possible representations)

c) This toy company decided to sell this doll in a special edition: 'Surprise Doll,' in which the doll's features are only discovered after purchase and opening the packaging. If Mariana's mother buys a doll from this edition, what color hair will the doll have? What will the doll's outfit be? What about the shoes? Justify your answer. (Randomness – Expected answer: We have no way of knowing the characteristics before opening the package, it is possible to buy any of the 18 dolls)

d) If Mariana's mother buys a 'Surprise Doll,' is it more likely to have a single-colored outfit or a varied-colored outfit? Explain how you achieved your answer. (Comparison of probabilities in the same sample space – Expected answer: It is more likely that the doll has clothes with different colors, as two of the options in the image fit this description. Only the yellow dress is a single-colored outfit)

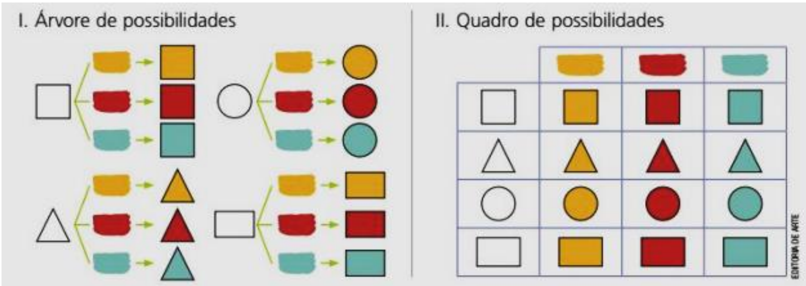
e) Mariana's mother bought one of these dolls. What is the probability that she bought a doll with black hair, a yellow dress, and black shoes? Indicate your answer as a ratio/fraction. (Quantification of Probabilities – Expected answer: 1/18)

Figure 3.

Problem proposal with articulation potential, Block 1 (product of measurements) (Lima, 2022, p. 152) - adapted from the Understanding and Practice Collection, 8th grade (Silveira, 2018, p. 121)

This type of adaptation allows for expanding the possibilities of analyzing a context addressed in a problem, which enables the teacher to broadly explore the probability linked to the initial combinatorial context in the classroom. The other blocks are presented in a similar manner.

A group of students will make a game and need to choose a figure and a color to prepare each card. Check out two ways to identify the different possible chips with 4 shapes (square, triangle, circle, and rectangle) and 3 colors (yellow, red, and blue) as options.



a) The group decided to add new shape options (pentagon and hexagon) and new color options (green, purple, pink, and black). Now, how many different chips can be created? How did you achieve this result? (Product of Measurements – Expected answer: 42 ways)

b) Given the new number of shapes and colors available, is it feasible to use the possibility tree or the possibility chart to know how many different chips can be created? Why? Also explain which strategy you chose in item a and why. (Sample Space – Expected answer: When the number of possibilities and/or choice stages is greater, it is more viable to use strategies that do not indicate the possibilities one by one. For example, the fundamental counting principle)

c) After building all types of pieces according to item a, Juliana, a student in this group, took one piece of each type and put it in a box. She is playing by drawing one piece at a time, returning the piece after each drawing. Juliana has already drawn five pieces and got the following results: red triangle; black triangle; blue triangle; green triangle, and pink triangle. If she draws a new chip, which piece will she get? Justify your answer. (Randomness – Expected answer: We cannot know the characteristics of the piece. Even though in the previous draws Juliana drew triangles, in the sixth draw any of the 42 pieces can be drawn)

d) With some extra pieces, Juliana assembled two boxes: in the first, there are 5 circular chips (3 green, 1 blue, and 1 purple); in the second box, there are 8 square chips (4 green, 2 red, and 2 black). When drawing without looking at just one piece from each urn, is it more likely to get a circular or square green piece? Explain how you achieved your answer. (Comparison of Probabilities in different sample spaces – Expected answer: It is more likely to draw a green piece in box 1 (circular pieces), as there are, proportionally, more green pieces in this box. $3/5 > 4/8$)

e) Using all valid pieces in the game, what is the probability of drawing a piece whose shape has four sides? Indicate your answer as a percentage. (Quantification of Probabilities – Expected answer: 14/42, i.e., 33%)

Figure 4.

Problem proposal with potential for articulation, Block 2 (product of measurements) (Lima, 2022, p. 155) – adapted from the Reality & Technology Collection, 6th grade (Souza, 2018, p. 52)

It is worth highlighting that in this second block of questions, priority was given to presenting different symbolic representations in a simpler problem (as in the original problem, from which the illustration was taken), which would serve as a basis for confronting a problem in which the number of possibilities is greater, in which case it would not be feasible to explain each of the possibilities.

Raquel put a chain with a padlock with a three-digit password on her bike. When she tried to open the lock, she realized she had forgotten the password. She just remembered that the password included numbers that were part of her birthday date and that there were no repeating digits. Knowing that Raquel was born on 6/12, answer:

a) How many different three-digit passwords can be created using the digits from Raquel's birthday date? How did you achieve that answer? (Arrangement – Expected answer: 24 ways)
b) If you haven't already done so in item a, use your preferred method to indicate each of the passwords considered. (Sample Space – Expected answer: Explanation of the 24 possibilities via listing, drawing, diagram, possibilities chart, or possibilities tree, among other representations)
c) If Raquel draws three of the four digits that make up her birthday date, which number will she get? Justify your answer. (Randomness – Expected answer: We have no way of knowing which number will be formed, it is possible that Raquel's draw will generate any of the 24 possibilities)
d) Is it more likely that Raquel's password will be an even or odd number? Explain how you achieved your answer. (Comparison of probabilities in the same sample space – Expected answer: It is more likely that the password ends in an even digit, as there are three even digits – 0, 2, and 6 – and only one odd digit – 1)
e) What is the probability of Raquel getting the password right in a single try? Indicate your answer as a ratio/fraction. (Quantification of Probabilities – Expected answer: 1/24)

Figure 5.

Problem proposal with potential for articulation, Block 3 (arrangement) (Lima, 2022, p. 157) – adapted from Teláris Project Collection, 9th grade (Dante, 2015, p. 296)

The original problem presented a padlock password context with 500 possibilities, restricting the number of elements (3) and requiring that the last digit should be even and that the quantification of the probability should be to get the password right on the first try. In the proposed adaptation, as it involves an odd block of work with a combinatorial arrangement (a block with a lower level of difficulty), the number of possibilities in the problem was reduced to 24. Proposing an arrangement of four elements, taken three by three, also aimed to make it possible to explain the passwords one by one, as requested in item b.

Among the three medalists in a swimming event, we can calculate the various possible results for first and second place using the fundamental counting principle, as follows:

POSSIBILITIES FOR 1ST PLACE: 3
POSSIBILITIES FOR 2ND PLACE: 2
(note that the second position cannot be occupied by the athlete who occupies the first place)
Therefore, we have: 3 x 2 = 6 possibilities

a) How many possible outcomes are there for the top three finishers in an Olympic swimming final contested by eight athletes? How did you achieve this result? (Arrangement – Expected answer: 336 results)
b) In your opinion, is it feasible to use the list to indicate all possible podiums relating to item a? Why? Also explain which strategy you chose in item a and justify your choice. (Sample Space – Expected answer: When the number of possibilities and/or choice stages is greater, it is more viable to use strategies that do not indicate the possibilities one by one. For example, the FCP)
c) The lane is the position occupied by each swimmer in a competition. In this final, each of the eight competitors will occupy a position defined by a draw. Three competitors have already drawn their positions and have been assigned lanes 1, 2 and 3, respectively. When continuing with the draw, which lane will the fourth swimmer occupy? Justify your answer. (Randomness – Expected answer: We cannot know which lane the next swimmer will occupy. Even if in previous draws the numbers drawn formed a sequence, any of the other lines may be drawn)
d) Is it more likely to draw a lane numbered with an odd number from lanes 1, 2, and 3 or from the other lanes (4, 5, 6, 7, and 8)? (Comparison of Probabilities in different sample spaces – Expected answer: It is more likely to draw an odd-numbered lane in the first group (lanes 1, 2, and 3), as there are, proportionally, more odd numbers in this group. $2/3>2/5$)
e) What is the probability of someone correctly guessing who will win the gold medal in the final? Indicate your answer as a percentage. (Quantification of Probabilities – Expected answer: 1/8, that is, 12.5%)

Figure 6.

Problem proposal with potential for articulation, Block 3 (arrangement) (Lima, 2022, p. 159) – adapted from Teláris Project Collection, 9th grade (Dante, 2015, p. 281)

In the adaptation proposed in Block 4, a simpler version of the problem is initially proposed, in which the choice stages and the number of choices in each stage are explained when using the FCP. The example presented in the proposed statement concerns an arrangement of three elements taken two at a time. Only then, in items a and b, is the problem proposed in its order of magnitude present in the original problem: eight elements taken three by three. It is also worth noting that the original problem only requested, without explicitly providing symbolic representations, the calculation of the arrangement relating to the podium in a swimming final in which eight athletes participate. Thus, the adaptation used the context to further discuss each of the cognitive demands of probability.

We call the different positions of the letters in a word anagrams. To create an anagram, we use all the letters of a word, each one only once.

a) How many anagrams can be formed using the letters of the word AMOR [LOVE]? How did you achieve that answer? (Permutation – Expected answer: 24 ways)

b) If you haven't already done so in item a, use your preferred method to indicate each of the anagrams considered. (Sample Space – Expected answer: Explanation of the 24 possibilities via listing, drawing, diagram, possibilities chart, or possibilities tree, among other representations)

c) Sabrina wrote down all the possible anagrams, cut out each word, and put all the papers in a box. If Sabrina randomly draws a single piece of paper, what word will she get? Justify your answer. (Randomness – Expected answer: We have no way of knowing which word will be drawn, it is possible that Sabrina will draw any of the 24 papers)

d) In the drawing for item c, is it more likely that Sabrina draws a word starting with the letter A or starting with a consonant? Explain how you achieved your answer. (Comparison of Probabilities in the same sample space – Expected answer: It is more likely that the word begins with a consonant, as there are 2 options: M and R)

e) If you draw one of these anagrams at random, what is the probability that it will end in a vowel? Indicate your answer as a ratio/fraction. (Quantification of Probabilities – Expected answer: 12/24 or 1/2)

Figure **7**.

Problem proposal with potential for articulation, Block 5 (permutation) (Lima, 2022, p. 161) – adapted from Teláris Project Collection, 7th grade (Dante, 2015, p. 268)

In the original problem, the author of the textbook presents some examples of anagrams that constitute the possibilities regarding the permutation of the letters of the word ‘amor’, highlighting the invariants of order and choice of this type of combinatorial situation: all the elements must be used (four letters), only once each and the change of order between them is what will determine the different possibilities. Furthermore, it requests the quantification of the probabilities of drawing a word ending in a vowel and a word beginning and ending in consonants. In Block 5, it was decided not to present such examples, but rather to clarify the text of the statement to make such invariants explicit by explaining what an anagram is. The student is asked to list all possibilities (all existing anagrams) in item b, and from there, the other items in the block in question can be easily solved.

Claudio wrote all three-digit numbers using the digits 4, 5, and 6. Look at the numbers he wrote:

456, 465, 546, 564, 645, 654

a) If, in addition to 4, 5, and 6, Claudio also uses the digits 1 and 3, how many different five-digit numbers can he form? How did you achieve this result? (Permutation – Expected answer: 120 ways)

b) Given the new number of numbers that Claudio can form, is it feasible to use the list to indicate all the various numbers that exist? Why? Also explain which strategy you chose in item a and justify your choice. (Sample Space – Expected answer: When the number of possibilities and/or choice stages is greater, it is more viable to use strategies that do not indicate the possibilities one by one. For example, the FCP)

c) With the help of some classmates, Claudio wrote down all the possible numbers that could be formed according to item a. Then he put all the numbers in a bag and asked his classmates to draw just one number, with replacement. Five colleagues have already drawn the lottery and they all got even numbers. If Claudio draws a new number, what result will he get? Justify your answer. (Randomness – Expected answer: We have no way of knowing the characteristics of the number that will be drawn. Even though in the previous draws only even numbers were drawn, in the sixth draw any of the 120 numbers can be drawn)

d) Using the same chosen digits, Cláudio wrote some two-digit numbers and separated them as follows, into two boxes: I - numbers 13, 14, 15, and 16 and II - numbers 41, 43, 45, 46, 51, 54, and 56. Which box is most likely to draw an even number? (Comparison of Probabilities in different sample spaces – Expected answer: It is more likely to draw an even number in box 1, as there are, proportionally, more even numbers in that box. 2/4>3/7)

e) Using all the numbers formed in item a, what is the probability of drawing a number less than 30000? Indicate your answer as a percentage. (Quantification of Probabilities – Expected answer: 24/120, that is, 20%)

Figure 8.

Problem proposal with articulation potential, Block 6 (permutation) (Lima, 2022, p. 162) – adapted from Teláris Collection, 6th grade (Dante, 2018, p. 286)

In this case, the listing presented in the original problem (permutation of three digits) was retained as an example in the statement. In turn, the proposed items a and b concern the permutation of five digits, thus increasing the number of possibilities. Furthermore, while the problem only required the quantification of the probability of drawing an even number among the possibilities considered, the proposed adaptation addresses the other cognitive demands of probability.

Finally, the two blocks that present a proposal for articulated work based on the combinatorial situation of combination, a situation less addressed in the curriculum presented, as highlighted previously, as well as indicated by previous studies (Pessoa, 2009; Lima, 2010;

Azevedo, 2013; Lima, 2018) as the one with which students have the most difficulty, were previously presented and published in detail at the last National Meeting on Mathematics Education [Encontro Nacional de Educação Matemática (XIV ENEM)] (Lima & Borba, 2022b).

A travel agency offers a travel plan to Northeast Brazil in which you can choose three of the five available capitals: Salvador (S), Recife (R), Maceió (M), Natal (N) and Aracaju (A)

a) Based on the options above, how many choices you have when purchasing a travel plan from this agency? How did you achieve that answer? (Combination – Expected answer: 10 ways)

b) If you haven't already done so in item a, use your preferred method to indicate each of the possible choices of the Northeastern capitals included in the plan offered by the travel agency. (Sample Space – Expected answer: Explanation of the ten possibilities via listing, drawing, diagram, possibilities chart, or possibilities tree, among other representations)

c) Mocking their indecision in choosing which capitals to visit, a couple decided to hold a draw using five pieces of paper of equal size on which they wrote the names of the five capitals. If the choice is made by drawing lots, which three cities will be chosen? Justify your answer. (Randomness – Expected answer: We cannot know which cities will be chosen, it is possible to draw any of the ten possible routes)

d) When drawing the first city, is it more likely that a capital whose name begins with a vowel or a consonant will be drawn? Explain how you achieved your answer. (Comparison of Probabilities in the same sample space – Expected answer: It is more likely that the city has a name beginning with a consonant, as this characteristic applies to four of the five available cities)

e) The couple really wants to visit Recife. Choosing the itinerary via draw, what is the probability of this city being included in the package? Indicate your answer as a ratio/fraction. (Quantification of Probabilities – Expected answer: 6/10)

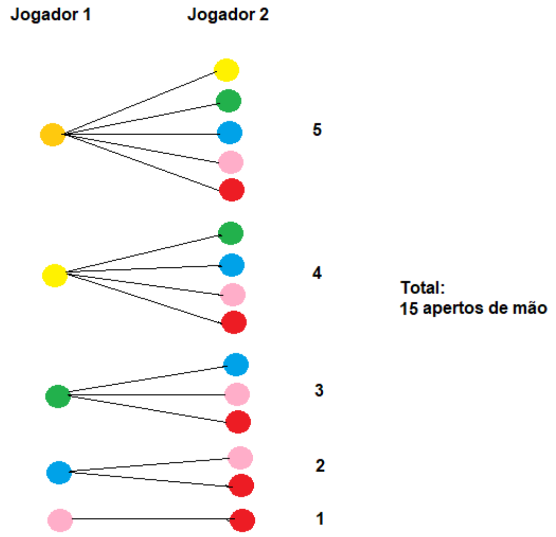
Figure 9.

Problem proposal with articulation potential, Block 7 (combination) (Lima, 2022, p. 165) – adapted from Teláris Collection, 8th grade (Dante, 2018, p. 13).

In this case, the original problem only asked for a list of possible choices of two capitals to visit from among four available options. The adaptation proposal considered increasing the number of possibilities, as well as leveraging the context (travel itinerary) to explore different cognitive demands related to probability.

A volleyball team is made up of six players. Before starting a game, each player greeted the others with a handshake.

Check out the tree of possibilities, which is a way to find out how many handshakes were given in total. Each of the six players was represented by a color.



Note that only one handshake was considered between each pair of players, therefore totaling 15 handshakes.

a) And if we consider a football team, with 11 players, how many handshakes would be given? How did you achieve this result? (Combination – Expected answer: 55 results)

b) In your opinion, is it feasible to use the possibility tree to indicate all possible handshakes related to item a? Why? Also explain which strategy you chose in item a and justify your choice. (Sample Space – Expected answer: When the number of possibilities and/or choice stages is greater, it is more viable to use strategies that do not indicate the possibilities one by one. For example, the FCP)

c) It is known that the order of entry of the 11 players was defined by drawing. The goalkeeper, number 1, has already shaken hands with four teammates, numbers 2, 4, 6, and 8. What is the uniform number of the next player who will shake hands with the goalkeeper? (Randomness – Expected answer: We cannot know which player will be the next to shake the goalkeeper's hand. Even though the previous handshakes followed a numerical sequence of the uniforms, the order of entry was drawn, so any of the other players on the team could be next)

d) A volleyball team has six players on the court, three of which are defense players. A football team has eleven players on the field and, in a given team, the following defense was adopted: goalkeeper, two defenders, and two full-backs. Considering such formations, when drawing a player at random, is it more likely to obtain a player who occupies a defensive position from a volleyball team or a football team? (Comparison of Probabilities in different sample spaces – Expected answer: It is more likely to draw a defense player in a volleyball team, as there are, proportionally, more players in that position. 3/6>5/11)

e) Taking any handshake from all the possible ones considered in item a, what is the probability that the team captain is involved? Indicate your answer as a percentage. (Quantification of Probabilities – Expected answer: 10/55, that is, 18.2%)

Figure 10.

Problem proposal with potential for articulation, Block 8 (combination) (Lima, 2022, p. 167) – adapted from Teláris Collection, 8th grade (Dante, 2018, p. 13).

In turn, the original problem adapted in Block 8 only required the calculation of the combination referring to the number of handshakes by six players on a volleyball team. In the adaptation proposal, this initial context was used as an example (along with the representation of the possibility tree) to illustrate the structure of the combination and highlight its order invariant. Thus, in the proposed items, one could explore a combination involving a greater number of possibilities (considering 11 players) and address the cognitive demands of probability from the same context.

Given the above, Study 3 of the thesis research discussed here expanded on textbook contexts already commonly used in the classroom, highlighting the possibility of articulating combinatorics and probability through the exploration of randomness in a more in-depth manner in problems with simple and easy-to-understand statements for middle school students. It is worth emphasizing, however, that the proposal presented here does not claim to be a unique or correct way to work with combinatorics and probability in the classroom. Indeed, it was built to demonstrate an alternative approach to modeling the prescribed and presented curricula, thereby enhancing the development of students' combinatorial and probabilistic reasoning in the classroom. Teachers can and should also model such curricula themselves, based on their experiences and objectives, and according to the particular needs of their students.

Some considerations

This work presents a proposal that has as its main bias the adaptation of questions from textbooks, which were analyzed in a previous study, to deepen the articulation between combinatorial and probabilistic problems. We prioritized exploring various situations, symbolic representations, and other aspects related to the level of difficulty of the problems, such as the number of choice stages and the number of possibilities, resulting in eight blocks of questions.

The study proposes a method (but not the only one) for exploring the connections between combinatorics and probability in the classroom, thereby enhancing the development of the two reasonings linked to them. It is important to remember, however, that both themes are, independently, of great importance. They are essential to the understanding of randomness (very present in everyday life) and contribute to the development of formal reasoning. Thus, articulating combinatorics and probability is optional. The present research took this path because previous studies have shown the potential of exploring the relationships between these two themes to contribute to the development of combinatorial and probabilistic reasoning.

As material aimed at teachers, the set of questions serves as support for our adaptations (shaped curriculum), aiming to explore the random in a broader way in the classroom and taking advantage of contexts present in teaching materials that are easily accessible to teachers and students in middle school.

References

Azevedo, J. (2013). *Alunos de anos iniciais construindo árvores de possibilidades: é melhor no papel ou no computador?* [Dissertação de Mestrado em Educação Matemática e Tecnológica, Universidade Federal de Pernambuco]. https://repositorio.ufpe.br/handle/123456789/13237

Batanero, C., Godino, J. & Navarro-Pelayo, V. (1996). *Razonamiento combinatorio*. Síntesis.

Borba, R. (2010). O raciocínio combinatório na Educação Básica. *Anais do Encontro Nacional de Educação Matemática – X ENEM*. https://atelierdigitas.net/CDS/ENEM10/artigos/PA/Palestra15.pdf

Brasil. (1998). *Parâmetros Curriculares Nacionais: Matemática – 3º e 4º ciclos.* Ministério da Educação.

Brasil. (2018). *Base Nacional Comum Curricular*. Ministério da Educação.

Bryant, P. & Nunes, T. (2012). *Children's undertanding of probability: a literature review*. Nuffield Foundation.

Dante, L. R. (2015). *Projeto Teláris: Matemática – Anos Finais*. 4 volumes – 2. ed. Ática.

Dante, L. R. (2018). *Teláris: Matemática – Anos Finais*. 4 volumes – 3. ed. Ática.

Godino, J., Batanero, C. & Cañizares, M. J. (1991). *Azar y probabilidad.* Síntesis.

Lima, E. & Borba, R. (2021). Probabilidade nos Anos Finais: o currículo prescrito pré e pós BNCC. In: *Anais do 8º Seminário Internacional de Pesquisa em Educação Matemática – VIII SIPEM.* https://www.even3.com.br/anais/viiisipemvs2021/372722-probabilidade-nos-anos-finais--o-curriculo-prescrito-pre-e-pos-bncc

Lima, E. & Borba, R. (2022a). Combinatória, Probabilidade e suas articulações em livros didáticos de Matemática dos Anos Finais do Ensino Fundamental. *Boletim de Educação Matemática – Bolema,* 36(72). https://www.scielo.br/j/bolema/a/84X6mfyJHcxMQBdh4krsLbb/

Lima, E. & Borba, R. (2022b). Problemas de combinação nos Anos Finais: o que é prescrito, o que é apresentado e o que se pode fazer articulado à Probabilidade? *Anais do Encontro Nacional de Educação Matemática – XIV ENEM.* https://www.even3.com.br/anais/xivenem2022/477015/

Lima, E. (2018). *Raciocínios combinatório e probabilístico na EJA: investigando relações*. [Dissertação de Mestrado em Educação Matemática e Tecnológica, Universidade Federal de Pernambuco]. https://repositorio.ufpe.br/handle/123456789/29717

Lima, E. (2022). *Combinatória, Probabilidade e suas articulações no currículo dos Anos Finais do Ensino Fundamental: o que é prescrito, o que é apresentado e o que se pode fazer?* [Tese de Doutorado em Educação Matemática e Tecnológica, Universidade Federal de Pernambuco]. https://drive.google.com/file/d/1Rnkqwb5y6kdLziF28YLJ2qHzi2XS9UVr/view

Lima, R. (2010). *O raciocínio combinatório de alunos da educação de jovens e adultos: do início da escolarização até o Ensino Médio* [Dissertação de Mestrado em Educação

Matemática e Tecnológica, Universidade Federal de Pernambuco]. https://repositorio.ufpe.br/handle/123456789/3868

Morgado, A., Pitombeira, J., Pinto de Carvalho, P. & Fernandes, P. (1991). *Análise combinatória e probabilidade*. Graftex.

Pernambuco. (2012). *Parâmetros para a Educação Básica do Estado de Pernambuco – Parâmetros Curriculares de Matemática para o Ensino Fundamental e Médio*. Secretaria de Educação.

Pernambuco. (2019). *Currículo de Pernambuco: Ensino Fundamental – área de Matemática*. Secretaria de Educação e Esportes.

Pessoa, C. (2009). *Quem dança com quem: o desenvolvimento do raciocínio combinatório do 2º ano do ensino fundamental ao 3º ano do Ensino Médio* [Tese de Doutorado em Educação, Universidade Federal de Pernambuco]. https://repositorio.ufpe.br/handle/123456789/4189

Sacristán, J. G. (2000). *O currículo: Uma reflexão sobre a prática*. Artmed.

Silveira, E. (2018). *Matemática: Compreensão e Prática: Anos Finais.* 4 volumes – 5. ed. Moderna.

Souza, J. (2018). *Matemática Realidade & Tecnologia: Anos Finais*. 4 volumes – 1. ed. FTD.

Vergnaud, G. (1986). Psicologia do desenvolvimento cognitivo e didática das matemáticas, um exemplo: as estruturas aditivas. *Análise Psicológica*, *1*(1), 75-90.

Vergnaud, G. (1996). A Teoria dos Campos Conceituais. In Brum. (org.), *Didáctica das Matemáticas* (pp. 155-191) – Lisboa. Horizontes Pedagógicos.