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The didactic problem of the decimal number system in the light of didactic transposition

El problema didáctico del sistema numérico decimal a la luz de la transposición didáctica

Le problème didactique du système des nombres décimaux à la lumière de la transposition didactique

O problema didático do sistema de numeração decimal à luz da transposição didática

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Abstract

In this article, based on the theory of didactic transposition and, more broadly, the anthropological theory of the didactic, we conjecture the didactic problem of the naturalization of the decimal positional numbering system, which renders invisible, if not difficult, part of the knowledge and its relationships with each other, which give meaning and significance to its structuring. In this way, we aim to highlight the role of studying a situation in a potentially real context as an initial condition for tackling the naturalization of numerals in decimal groupings. In this theoretical-methodological context, the study and research activity is proposed as a

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didactic device involving non-decimal numerals to highlight and deal with this problem. In order to address the focus of the research, on decimal and non-decimal numerals, an empirical study was carried out with elementary school students through the study of a problem situation, as this includes the *raison d'être* or rationality that gives meaning to (re)signifying mathematical activities. The results found in the empirical study confirm our hypotheses about the important strategic role played by the problem situation, as a provider of indispensable conditions for building and expanding the quality of students' relationships with different objects of knowledge. In addition, they encourage future research into numerals, especially with the use of manipulable materials available to elementary school students, in order to give relevance to the knowledge of the social practice of quantification as the structuring genesis of various objects of mathematical knowledge.

Keywords: Didactic transposition; Decimal number system; Problem situation.

Resumen

En este artículo, a partir de la teoría de la transposición didáctica y, más ampliamente, de la teoría antropológica de lo didáctico, conjeturamos el problema didáctico de la naturalización del sistema de numeración posicional decimal, que vuelve invisible, cuando no difícil, parte del conocimiento y de sus relaciones entre sí, que dan sentido y significado a su estructuración. De esta forma, pretendemos destacar el papel del estudio de una situación en un contexto potencialmente real como condición inicial para abordar la naturalización de los numerales en las agrupaciones decimales. En este contexto teórico-metodológico, la actividad de estudio e investigación se propone como un dispositivo didáctico que involucra numerales no decimales para evidenciar y abordar esta problemática. Para abordar el foco de la investigación, sobre los numerales decimales y no decimales, se realizó un estudio empírico con alumnos de primaria mediante el estudio de una situación problema, ya que ésta incluye la razón de ser o fundamento que da sentido a las actividades matemáticas (re)significantes. Los resultados encontrados en el estudio empírico confirman nuestras hipótesis sobre el importante papel estratégico que desempeña la situación problema como proveedora de condiciones indispensables para construir y ampliar la calidad de las relaciones de los alumnos con los diferentes objetos de conocimiento. Además, animan a futuras investigaciones sobre los numerales, especialmente con el uso de materiales manipulables al alcance de los alumnos de primaria, para dar relevancia al conocimiento de la práctica social de la cuantificación como génesis estructurante de diversos objetos de conocimiento matemático.

Palabras clave: Transposición didáctica, Sistema numérico decimal, Situación problemática.

Résumé

Dans cet article, en nous appuyant sur la théorie de la transposition didactique et, plus largement, sur la théorie anthropologique du didactique, nous conjecturons le problème didactique de la naturalisation du système de numération positionnelle décimale, qui rend invisible, voire difficile, une partie des savoirs et de leurs relations entre eux, qui donnent du sens et de la signification à leur structuration. De cette façon, nous voulons mettre en évidence le rôle de l'étude d'une situation dans un contexte potentiellement réel comme condition initiale pour aborder la naturalisation des chiffres dans les groupements décimaux. Dans ce contexte théorico-méthodologique, l'activité d'étude et de recherche est proposée comme un dispositif didactique impliquant des nombres non décimaux pour mettre en évidence et traiter ce problème. Afin de répondre à l'objectif de la recherche sur les nombres décimaux et non décimaux, une étude empirique a été menée avec des élèves de l'école primaire en étudiant une situation-problème, puisque celle-ci comprend la raison d'être ou le raisonnement qui donne un sens aux activités mathématiques (re)signifiantes. Les résultats de l'étude empirique confirment nos hypothèses sur le rôle stratégique important joué par la situation-problème en tant que fournisseur de conditions indispensables à la construction et à l'expansion de la qualité des relations des élèves avec différents objets de connaissance. En outre, ils encouragent les recherches futures sur les chiffres, en particulier avec l'utilisation de matériel manipulable à la disposition des élèves de l'école primaire, afin de donner de la pertinence à la connaissance de la pratique sociale de la quantification en tant que genèse structurante de divers objets de la connaissance mathématique.

Mots-clés*:* Transposition didactique, Système de numération décimale, Situation-problème.

Resumo

Neste artigo conjecturamos, a partir da teoria da transposição didática e, mais amplamente, da teoria antropológica do didático, o problema didático da naturalização do sistema de numeração posicional decimal que invisibiliza, senão dificulta, parte dos saberes e de suas relações entre si, que dão sentido e significado em sua estruturação. Desse modo, objetivamos evidenciar o papel do estudo de uma situação em contexto potencialmente real como condição inicial provedora ao enfrentamento da naturalização dos numerais de agrupamentos decimais. Nesse contexto teórico-metodológico, é proposta a atividade de estudo e pesquisa como dispositivo didático envolvendo numerais não decimais para evidenciar e tratar dessa problemática. Para atender o enfoque da pesquisa sobre os numerais de agrupamentos decimais e não decimais, foi realizada uma empiria com alunos do ensino básico por meio do estudo de uma situação problema, por esta incluir a razão de ser ou a racionalidade que dá sentido em (re)significar as atividades matemáticas. Os resultados encontrados na empiria ratificam nossas hipóteses sobre o importante papel estratégico desempenhado pela situação problema, como provedora de condições indispensáveis para construção e ampliação da qualidade de relações dos alunos com diferentes objetos de saberes. Além disso, estimulam futuras pesquisas com os numerais, principalmente, com o uso de materiais manipuláveis ao alcance dos alunos do ensino básico, de modo a dar relevância aos saberes da prática social de quantificação como gênese estruturante de vários objetos do conhecimento matemático.

Palavras-chave: Transposição didática, Sistema de numeração decimal, Situação-problema.

The didactic problem of the decimal number system in light of didactic transposition

The teaching of objects of school mathematical knowledge has been configured as topics of interest for different researchers in the field of mathematics education, concerned with the learning of these objects, in particular, and of our interest, on the decimal number system (DNS), as demonstrated, for example, by Lerner and Sadovsky (1996), Sadovsky (2005), Terigi and Wolman (2007), Itzcovitch (2008), Lendínez, Garcia, and Sierra (2017), Sierra and Gascón (2018), Ferreira and Guerra (2020), and Ferreira (2020).

According to Ferreira (2020) and Ferreira and Guerra (2020), different civilizations, such as the Egyptians, Greeks, Romans, and Mayans, among others, contributed to the historical construction of the Hindu-Arabic number system, also known as DNS. Its importance in the ordering and organization of the use of different human activities, recognized by culture, can be understood as a *wise knowing*, in the sense proposed by Chevallard (2005), that is, as knowledge chosen by culture, and not necessarily by 'academics', bearing in mind that "the title of wise does not belong *intrinsically* to knowing. It is granted by culture and can be lost. In short, a knowing is not wise knowing because its producers are 'academics,' it is just the opposite[5] (Chevallard, 2005, p. 162, author's emphasis, our translation).

Maybe that is why Ferreira and Guerra (2020, p. 2) highlight, based on Ifrah's (1985) observations, that "this system would have been brought by the Arab civilization to Europe, in the middle of the 7th century, from where it was spread to other civilizations, becoming dominant in the world today." The social and cultural importance of using DNS to serve the organization of different social practices, including that of basic education, for example, places centrality on teaching it as an object of knowledge. Because of this,

> There seems to be no doubt that teaching in basic schools must include the decimal number system (DNS), as decimal numbers are shown within our society to be indispensable knowledge for dealing with different routine practices and specialists from different areas of knowledge of humanity and, among them, that of fundamental knowledge for the development of school mathematics (Ferreira, 2020, p. 26).

From this perspective, the DNS, by transversalizing the basic school curriculum, occupies, according to Terigi and Wolman (2007), a strategic place in the mathematics curriculum of several countries from the initial years of students' education, although they state that this object of knowledge is considered problematic in its teaching and, thus, responsible

[5] Fragments of the text: [...] que el título de sabio no pertenece jamás *intrínsecamente* a un saber. Es otorgado por la cultura y puede perderse. En resumen, un saber no es sabio porque sus productores sean "académicos": es exactamente lo inverso lo que escierto.

for low student performance in understanding it. In this sense: "School failure creates the need for an analysis of regular education that allows for a better understanding of the unfavorable factors that may be acting in teaching situations"[6] (Terigi & Wolman, 2007, p. 63, our translation).

Along this path, some researchers have highlighted problems regarding the DNS teaching, among them, Lerner and Sadovsky (1996), Sadovsky (2005), Terigi and Wolman (2007), Itzcovitch (2008), Lendínez, Garcia and Sierra (2017), Sierra and Gascón (2018), Ferreira and Guerra (2020), and Ferreira (2020), above all, due to a non-critical view, as Sierra and Gascón (2018) point out, on the different teaching objects dealt with in basic school, in general, without further questioning of knowings. "It is more difficult to find works that take the questioning of school mathematics as a starting point to address the problem of teacher education" (Sierra & Gascón, 2018, p. 81, our translation).

In the wake of this construction signaled by Sierra and Gascón (2018), who considered it necessary to question the objects of knowings, including DNS, Sadovsky (2005) highligts that the teaching-learning process is "conditioned not only by social factors, but also by a certain vision of how knowledge circulates within classes. Reviewing the mathematics that exists in school, questioning it, analyzing it, is essential to conceive other scenarios" (Sadovsky, 2005, p. 13) that allow, in some way, proposing challenges to the student through teaching situations.

Our interest in choosing the DNS aligns with what Terigi and Wolman (2007) present, as it is strategic in the process of mathematical education of students at school, which demands the creation or recreation of didactic-mathematical organizations for teaching the DNS, because:

> Limiting didactic work to a few numbers in the series, presenting them one by one without progressing until the child masters the name and correct arrangement of each one, or showing only one way – the conventional one – of solving operations, makes it impossible for children to access the world of relationships that numerical notation entails and puts at risk not only their learning about it, but also their future possibilities of appropriating new mathematical knowledge[7] (Terigi & Wolman, 2007, p. 65, our translation).

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[6] Fragments of the text: El fracaso escolar provoca la necesidad de un análisis de la enseñanza habitual que permita comprender mejor los factores productivos que puedan estar operando en las situaciones de enseñanza.

7 Fragments of the text: *Limitar el trabajo didáctico a unos pocos números de la serie presentándolos de uno en uno sin avanzar hasta que no se domine el nombre y el correcto trazado de cada uno, o mostrar un único modo –el convencional– de resolver las operaciones, hace imposible que los niños accedan al mundo de relaciones que supone la notación numérica, y pone en riesgo no sólo sus aprendizajes sobre el sino sus SN posibilidades futuras de apropiación de nuevos conocimientos matemáticos.*

According to the authors above, the notion of DNS as a teaching object does not aim merely to translate quantities into graphic forms. They, therefore, point out that its use as an instrument for representing physical quantities of the socially shared real world occurs, above all, due to the necessary denaturalization of the DNS to be taught in basic school.

This view does not seem to be present only in Terigi and Wolman's highlights (2007), as, in Itzcovich's (2008) approaches, we observe assumptions about the complexity involved[8] in teaching decimal numerals at school as an object-product of human culture used naturally or as a non-problematic object-product. Thus, "the fact that the number system is knowledge that we use constantly sometimes makes us lose sight of the complexity that its functioning entails and the difficulties that, consequently, those who are trying to learn this mathematical object may encounter"[9] (Itzcovich, 2008, p. 31, our translation).

In his analyses, Itzcovich (2008, p. 31, our translation) highlights that:

> Our number system is a cultural creation with its own characteristics, which differ from other systems belonging to other cultures. Like any object of cultural construction, it is a convention and, as such, arbitrary; therefore, the possibility of this system being learned by new generations depends on teaching[10].

The text extract from Itzcovich (2008) seems to suggest that the supposed success of learning numbers depends on teaching, that is, on the possibility of creating mathematical organizations aimed at meeting, even if only partially, the needs of teaching the DNS within the context of school mathematics activities.

In this regard, we believe that the creation of conditions for teaching the DNS may prove to be indispensable to school education, since "at the beginning of school mathematics education, when children enter elementary school, the DNS is the key element," taking into account the supposed role of these conditions, especially since these "[...] cannot be enumerated a priori: their discovery is progressive [...]" (Chevallard, 2009, p. 12, our translation). These conditions must also be understood as possibilities for dealing with the naturalization of decimal numerals, as recommended by Terigi and Wolman (2007) and Ripoll et al. (2016).

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[8] The expression *decimal numerals* is adopted here as being the non-negative integer numerals of decimal groupings or simply of numerical base ten.

[9] Fragment of the text: *El hecho de que el sistema de numeración sea un conocimiento que utilizamos permanentemente, a veces, nos hace perder de vista la complejidad que encierra su funcionamiento y las dificultades que, en consecuencia, pueden encontrar aquellos que están intentando aprender este objeto matemático.*

[10] Fragment of the text: *Nuestro sistema de numeración es una creación cultural con características propias, que difieren de las de otros sistemas pertenecientes a otras culturas. Como cualquier objeto de construcción cultural, es una convención y, como tal, arbitraria; por ol tanto, al posibilidad de que este sistema pueda ser aprendido por las nuevas generaciones depende de la enseñanza.*

In the wake of this construction, Ripoll et al. (2016) reveal the naturalization of numerals as one of the problems that "have often been avoided or suppressed in mathematics teaching, when mathematical concepts are artificially naturalized, that is, when their nature and necessity are assumed as given" (Ripoll et al. 2016, p. XX).

Perhaps for this reason, Itzcovich (2008) warns that some efforts have been made to minimize the complexity of teaching numerals: "In some of these attempts, we oscillate between a trivialization and a naturalization of the object. In other words, it is transformed by trivializing it as if it were not complex and, at the same time, it is treated as if its appropriation were natural or spontaneous"[11] (Itzcovich, 2008, p. 32, our translation).

According to Itzcovich (2008), the rules of the numbering system, far from being natural, are products of intentional human action, and, as such, the author draws attention to the great challenge of teaching this object, due to its proximity to the universe of knowledge of the real world of students, which may require or demand efforts from the teacher to create teaching situations that make it possible to denaturalize so-called "adult" knowledge.

Otherwise, "the didactic problem consists in finding the appropriate situations to make these rules explicit to children"[12] (Itzcovich, 2008, p. 34, our translation), which leads us to the problem of interest of the theory of didactic-institutional transposition (Chevallard, 2005; 2019), assumed here as a theoretical contribution, by highlighting the questioning of knowings through a straightforward yet comprehensive question. According to Chevallard (2019), its understanding can be paraphrased as follows: *What is this thing you call DNS?* In other words, for example, what is the "version" of knowledge of the DNS that lives in the school? This questioning of interest in the theory of didactic transposition is also strongly recommended by Ferreira (2020, p. 35), when expressing the need to:

> Questioning decimal numerals as a product of a didactic transposition activity, here understood as the work of transformations and adaptations of knowledge related to decimal numerals to be taught, taking into account their relations with other mathematical and non-mathematical knowledge, including non-disciplinary knowledge, which can contribute to a didactic organization that meets the needs of teaching and, consequently, learning.

The description of the DNS, based on the theory of didactic transposition, can be paraphrased in the following terms: What do we call DNS? Where does the DNS come from?

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11 Fragments of the text: *En algunos de esos intentos, se oscila entre una banalización y una naturalización del objeto. Es decir, se lo transforma banalizándolo como si no fuera complejo y, al mismo tiempo, se lo trata como si su apropiación fuera natural o espontánea.*

12 Fragments of the text: *El problema didáctico consiste en encontrar las situaciones adecuadas para explicitar estas reglas a los niños.*

How is the DNS epistemologically legitimized? Or must the DNS be rebuilt? These developments are also highlighted by Terigi and Wolman (2007), Itzcovich (2008), Ripoll et al. (2016), Ferreira and Guerra (2020), and Ferreira (2020), when reiterating the need to construct situations for teaching, in order to meet, even partially, the complex naturalization of this object.

From this perspective, the "theory of the didactic transposition postulates two fundamental principles. The first is that the version of school knowledge and academic knowledge are almost always different" (Chevallard, 2019, p. 73, our translation), which can lead to some distortions about the notions of numbers and numerals –as if one were the other and vice versa, in general, with the prevalence of the notion of number, according to guidelines from official documents of so-called "academic" mathematics– without observing, in our understanding, that the reference to the supposed idea of numbers is dominantly associated in school teaching in close relation to real-life objects.

In school institutions, there is a "specificity of didactic functioning, of its very nature, irreducible, without mediation, to the 'secular' functioning of the corresponding knowledge in the 'academic community'" (Chevallard, 2005, p. 76, author's emphasis, our translation). In other words, in the school institution, the knowings must be "constituted by a certain past/future contradiction" (Chevallard, 2005, p. 76, author's emphasis, our translation) in which the object of teaching must "appear as an object with two faces, contradictory to each other" (Chevallard, 2005 p. 77, author's emphasis, our translation).

We assume that the study of situations in potentially real contexts with non-decimal numerals, i.e., in other numerical bases distinct from the decimal grouping base, can provide minimum conditions for the denaturalization of the DNS, given the recommendations highlighted by Ferreira (2020, p. 8) for "the conception and construction of praxeological organizations on decimal numerals for teaching the initial years of elementary school, under the understanding developed here, for experimental validation in the classroom." Thus, we aim to highlight the role of studying a situation in a potentially real context as an initial condition, providing support for tackling the naturalization of decimal numerals.

Numerals as a "version" of a school knowing

Although the process of studying numerals in basic school, as a product of social practices of quantifying physical quantities, seems confined under the perspective of the notion of numbers, in the sense of 'academic' mathematics –and as recommended in official documents, such as the National Common Curricular Base (Base Nacional Comum Curricular

- BNCC) (Brasil, 2018)– can bring into play the historical-epistemological genesis, through the action of counting real objects for the construction of numerals, and, consequently, reveal part of the complexity of the structuring of decimal numerals.

From this perspective, which focuses on the didactic problem of naturalizing numerals in decimal groups, endowed with their complexities in their structuring, as recognized by Itzcovich (2008), Terigi and Wolman (2007), and Ripoll et al. (2016), it is essential to recognize the intricate relationships between mathematical and non-mathematical knowings, including the non-academic ones, as noted by Ferreira (2020).

In this path, the numbering system, "as a social instrument, implies that the analysis of this object that is necessary for the conception of its teaching is not limited to the knowledge of its mathematical aspects; it requires bringing into play other knowledge that is not that of the specialist in the mathematical field" 13 (Terigi & Wolman, 2007, p. 68, our translation).

This view seems to go towards the notion of numerals used in school, although this object is described in some cases as being numbers, as pointed out by Ferreira and Guerra (2020) and Ferreira (2020), specifically in the sense adopted by the institution of academic mathematics, which seems to reject the notion of numeral, whose path is also reiterated, in our understanding, by the BNCC (Brasil, 2018) when it highlights the thematic unit of numbers:

> The purpose of the Numbers thematic unit is to develop numerical thinking, which involves knowledge of ways to quantify attributes of objects and to judge and interpret arguments based on quantities. In the process of constructing the notion of number, students need to develop, among others, the ideas of approximation, proportionality, equivalence, and order, which are fundamental notions in mathematics. For this construction, it is important to propose, through significant situations, successive expansions of the numerical fields. In studying these number fields, records, uses, meanings, and operations must be emphasized (Brasil, 2018, p. 268).

The text extract makes it clear that, during the process of constructing the notion of numbers, students must develop mathematical ideas that, in our understanding, are interwoven with ideas based on the practices of academic mathematics, whose focus given by this thematic unit is on the emphasis on records, uses, meanings, and operations, dispelling, in our understanding, the non-visibility of the social practice of quantification as a genesis for the structuring of numerals and developing other knowings used in different social practices, such as school and academic mathematics.

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13 Fragments of the text: *Como instrumento social implica que el análisis de este objeto que se requiere para diseñar su enseñanza no se agota en el conocimiento de sus aspectos matemáticos; requiere poner en juego otros saberes que no son los del especialista en el campo matemático.*

This glance reinforces our understanding of the coexistence of institutional powers that "hover" over the old dichotomy between the notions of numbers and numerals, in some cases, where one notion is taken as the other and vice versa. It is necessary to understand that the notion of numbers conceived by the institution of "academic" mathematics is not at stake when resorting to the notion of counting that involves the use of real-world objects. In this sense, the notion of numbers, in addition to not being confused with that of numerals, assumed by school mathematics, is not necessarily dependent on relations with objects in the real world, because:

> It is essential to note that the teaching of school mathematics is frequently shaped by understandings of academic mathematics, and the concept of numbers is one of them. The last two understandings are present in different organizations for teaching numbers, with one of the main concerns being to differentiate numbers from numerals, as if this were a simple and easy task (Ferreira, 2020, p. 92).

It is worth noting that, although the notion of number is recommended by official documents of the basic education curriculum, such as the BNCC, and seems dominant in the discourse of school mathematics teaching practices, it is distinct from the notion of numerals available in different social practices. Thus, as such, it is dependent on other knowledge, including non-mathematical knowledge present in its structuring, as Itzcovich (2008) and Ferreira (2020) warn. Because of this,

> The notion of mathematical number is not at stake when we resort to counting and this has implications when we deal with concrete or sensitive objects –which can be understood as the objects that are counted, in this case a unit of magnitude, as well as the counting process and, in an indispensable way, sensitive objects that are manipulated according to quantities to produce new sensitive objects that can be manipulated according to new quantities (Ferreira, 2020, p. 97).

Perhaps this is why Ferreira (2020, p. 100) adds that "adjectival numerals are products of actions on physical quantities and are only achieved with a way of counting and a numbering system in line with this process," whose progress highlights the important role given to school mathematics "as a tool for reading the world" (Ferreira, 2020, p. 100), based on the ideas coordinated by the numeral, endowed with physical units for the quantification process.

In order to make part of this complex distinction explicit, we turn to Wittgenstein (1987), who highlights the propositions of mathematics practiced in academic institutions, since they are not to be confused with the mathematical propositions shared by institutions that deal with real-world problems or that strive to teach the use of mathematical objects as instruments for readings of the real world, as "this does not mean in any way that the propositions of mathematics fulfill the function of empirical propositions" (Wittgenstein, 1987, p. 322, our translation).

Wittgenstein's (1987) excerpt can be exemplified through an additive operation between two numerical entities: (2 + 3). From the point of view of the institution of "academic" mathematics, this additive operation between the two entities can admit another numerical response distinct from the numerical entity designated by 5 (five), including being capable of admitting any other response, such as 0 (zero), for example, depending on the type of operation in which it is defined.

On the other hand, from the point of view of social practices with real-world objects, another result of the operation between physical quantities, such as '2 pens + 3 pens = 5 pens' may be inconceivable or even cause social strangeness, if the accepted answer is different from the numeral that expresses the quantity 5 (five).

Furthermore, Wittgenstein's (1987) excerpt seeks to give visibility to mathematical propositions that independ on propositions shared in the real world, such as what often seems to occur in the classroom when the teacher, in their didactic action, points to a quantity of physical objects (number of desks, number of pencils, etc.) and then records a numerical data to express that quantification. Thus:

> It is necessary to insist that numerals are not works of mathematicians in their Platonic universe, even when understood as clusters of "ones," but rather human works to construct answers to questions that emerge in situations within social spaces, in human beings' relationships with the world, others, and themselves (Charlot, 2003). Numerals are tools developed by human beings to facilitate their various practices necessary for different activities (Ferreira, 2020, p. 95).

Thus, it seems useful, if not indispensable, to create conditions, in the sense of didactic-institutional transposition (Chevallard, 2005; 2019) and, more broadly, of didactic science, since "conditions are objects of study of didactics, they cannot be enumerated a priori: their discovery is progressive and the understanding of their role in the dissemination of a certain entity of knowledge are the permanent objectives of research in didactics" (Chevallard, 2009, p. 12, our translation) to make possible the denaturalization of decimal numerals, which leads us to:

> Questioning didactic organizations about decimal numbers means questioning as a product of a didactic transposition activity, here understood as the work of transformations and adaptations of knowledge related to decimal numbers to be taught, taking into account their relations with other mathematical and non-mathematical knowledge, including non-academic knowledge, which can contribute to a didactic

organization that meets the needs of teaching and, consequently, learning (Ferreira, 2020, p. 255).

To meet the objective of our investigation, we resorted to an empirical approach in relation to a group of elementary school students, when considering carrying out a study and research activity (Bosch & Gascón, 2010) as a methodological instrument of research, whose progress can lead the subject to eventual changes in *quality of relationships* (Chevallard, 2005) with the study of decimal numerals, and thus, denaturalize the DNS, even if partially, by creating the conditions for the study.

Theoretical-methodological elements of the research

The theoretical-methodological foundations of this research are based on highlighting and addressing, even if partially, a didactic problem recognized by different authors in the literature in the area (Sadovsky, 2005; Terigi & Wolman, 2007; Itzcovitch, 2008; Ripoll et al. 2016; Ferreira & Guerra, 2020; Ferreira, 2020; Guerra & Ferreira, 2022) of the naturalization of numerals of decimal groupings, through notions of the theory of didactic transposition and, more broadly, from notions of the anthropological theory of the didactic (ATD) (Chevallard, 1999), more precisely, when considering the notion of study and research activity (SRA) (Bosch & Gascón, 2010), assumed as a didactic device to deal with the problem of school teaching in the DNS. In this sense, the notion of SRA:

> [...] resumes a concern inherent to the theory of didactic situations in its proposal for the functional reconstruction of mathematical knowledge based on 'fundamental situations,' the objective of which is to place the 'reason for being' or the 'meaning' of this knowledge at the center of the study process (Bosch & Gascón, 2010, p. 77, our translation).

In the didactic of the SRA, it is necessary to explicitly highlight the generating question of the study process, whose developments are guided, in this investigation, by an initial condition from an unusual situation, in a potentially real context, with the use of non-decimal numerals, "recreated" to meet teaching intentions, from the perspective of the didactic transposition (Chevallard, 2005, 2019), based on the type of problem proposed by Ferreira (2020) and Ferreira and Guerra (2020):

> A "flying saucer" lands from a planet manned by human-like beings with I mouth, V eyes and Z limbs. These beings differ from humans in that they only have A fingers, that is, Z minus I, on each of their limbs, in addition to not having hair, that is, O hair on their entire body. On our planet, we cultivate grain and tubers like Earthlings and, in our last solar year AIOOO, which corresponds numerically to the Earth's solar year 2000, we obtained the following productions: On my planet, we only use the

representation registers V, A, Z, I and O to represent the quantities (Ferreira, 2020, p. 112).

Table 1.

Representation of grains or tubers (Text adapted from Ferreira (2020, p. 112)

PRODUCTS	PRODUCTIO
Beans	AZOIO
Rice	ZVAII
Cassava	ZZAAV

Based on the information described in the text of the problem situation, a group of twenty-five (25) students in the 6th grade of elementary school at a public school were asked to give greater emphasis to groupings with decimal numbers in their school education, which are generally culturally assumed to be an object of non-problematic knowledge and, therefore, naturalized at the heart of social practices, such as school mathematics.

The study process was guided by the following question, adapted from Ferreira (2020): ***Q_1*** *– What do these people and their flying saucer probably look like?*

The analyses presented below aim to confirm or refute the hypotheses outlined here based on the five groups of students described by ***$G_i = (G_1, G_2, G_3, G_4, G_5)$***, with the installation of auxiliary teaching systems to address the issue ***Q_1*.**

Analysis of results with 6th-grade elementary school students

The SRA conducted with groups of students follows the model provided by the ATD of the semi-developed Herbartian scheme given $[S_p\,(G_i, y, \mathrm{Q}_1) \Rightarrow M] \Rightarrow \mathrm{St}_i^{♥}$, with i = {1, 2,..,5}, in which St_i designates the *situational responses* constructed and defended by each auxiliary teaching system, symbolized here by $S_i\,(G_i, y, \mathrm{Q}_1)$ which, together, are integrated into the central didactic system $[S_p\,(G_i, y, \mathrm{Q}_1)] \Rightarrow \mathrm{St}_i^{♥}$.

Furthermore, *M* represents the *medium* equipped with resources or works studied by students to delimit the answers found under the class's appreciation during socializations and defenses; and (y) symbolizes the teacher or director of the investigation. Thus, one of the primary functions of the investigation director consisted not only of proposing the questioning ***Q_1***, but also of contributing to students in the study processes, encouraging them to take the lead in the actions undertaken.

The auxiliary didactic systems installed showed different contexts of doubts during the reading of the unusual problem, including a broad 'strangeness' of the class when trying to establish possible qualities of relationships (Chevallard, 2005), if they do not have them, with objects of knowings available in their *cognitive universe (CU)* (Chevallard, 2009).

Perhaps when the class feels strange when reading the problem statement, it is necessary to consider "that to understand a cultural production (literature, science, etc.) it is not enough to refer to the textual content of that production, nor the social context, contenting oneself with establishing a direct relationship between the text and the context" (Bourdieu, 2004, p. 20), due to the possible existence of other practical or theoretical knowings as conditioning factors, not always explicitly stated in the reading of a text such as the problem situation.

It is no coincidence that this 'strangeness' perceived during the reading of the problem in context and in the flow of the study process motivated students to ask other questions, such as:

$Q_{1.1}$ *- Which Hindu-Arabic numerals correspond to each letter of the human-like 'beings' described in the text?;* and ***$Q_{1.2}$*** *– Teacher [...] wouldn't it be good if these beings, who they say are similar to humans, went to school to study more [lol] since they only know how to count up to four quantities? What do you mean [...]?*

During the collective reading stage of the problem text, specifically in the part of the table that highlights the production of each product, the students would read the quantities as if they were a word in the Portuguese language, considering the direction from left to right. For example, the quantity of beans: AZOIO was initially referenced as if it were a word devoid of meaning regarding quantities expressed in physical units.

Along the way, teaching systems intervened with the following highlights: *[S_3 (G_3, y, Q_1)] St_3Mates, if the word AZOIO can be compared with our numbers, then in comparison it would be 34010, thirty-four thousand and ten for beans? Is that so?; [S_4 (G_4, y, Q_1)] St_4: what does it mean? But, if they only know how to count up to four quantities, lol [...] then it can't be thirty-four thousand and ten, because these beings from another planet don't use our numbers in tens. What a confusing thing, huh... it's because they are from another planet.*

These manifestations of didactic systems [S_3 (G_3, y, Q_1)] St_3 and [S_4 (G_4, y, Q_1)] ➡ St_4 seem to reveal the explicit naturalization of decimal numerals ratified by the literature in the area (Sadovsky, 2005; Terigi & Wolman, 2007; Itzcovitch, 2008; Ripoll et al. 2016; Ferreira, Guerra, 2020; Ferreira, 2020; Guerra & Ferreira, 2022), including, in a dominant way, due to the non-recognition of a quantity of units described by AZOIO, for example, when this register was interpreted as being a possible "word" of the Portuguese language, which, with the help of the Hindu-Arabic numerals in step 1 of Figure 1, recorded the numeral described by: 34010.

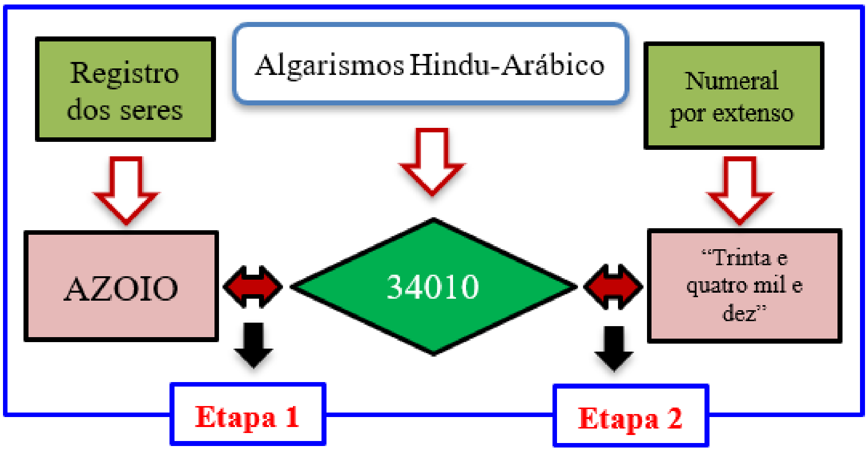

Figure 1.

Representation of different registers of numerals (Construction by the authors, 2024)

The registration of the number 34010, using Hindu-Arabic numerals, was questioned by the auxiliary didactic system [S_4 (G_4, y, Q_1)] St_4, (identified in step 2 of Figure 1) by this numeral, when recited by the didactic system [S_3 (G_3, y, Q_1)] St_3, by following the rules of the usual reading of decimal numerals in full: "thirty-four thousand and ten," without noting that beings similar to humans, as noted by the didactic system [S_4 (G_4, y, Q_1)] St_4, do not count units according to the practical logic of decimal numberals, that is, in groups of tens.

In this context of defending responses to the class, there is, in addition to a clear "confrontation" of practices for facing the complex desnaturalization of decimal numerals compared to non-decimal numerals, changes in the cognitive universe (Chevallard, 2009), and students' creations of relationships, even if partially, through their encounter with the practical logic of non-decimal numerals, which are devoid, for example, of numerical registers with specific names.

Based on these approaches, didactic systems focused on developing answers to question $\boldsymbol{Q_1}$. When considering the correspondence technique between the registers of human-like beings described in the unusual problem regarding supposedly human decimal numerals, the auxiliary didactic systems: [S_1 (G_1, y, Q_1)] ➡ St_1; [S_3 (G_3, y, Q_1)] St_3 and [S_4 (G_4, y, Q_1)] ➡St_4 highlighted qualities of relationships supported by Hindu-Arabic numerals, taking decimal numerals as a reference through the following relationships: O = 0; I = 1; V = 2; A = 3, and Z = 4, as highlighted in Figure 2.

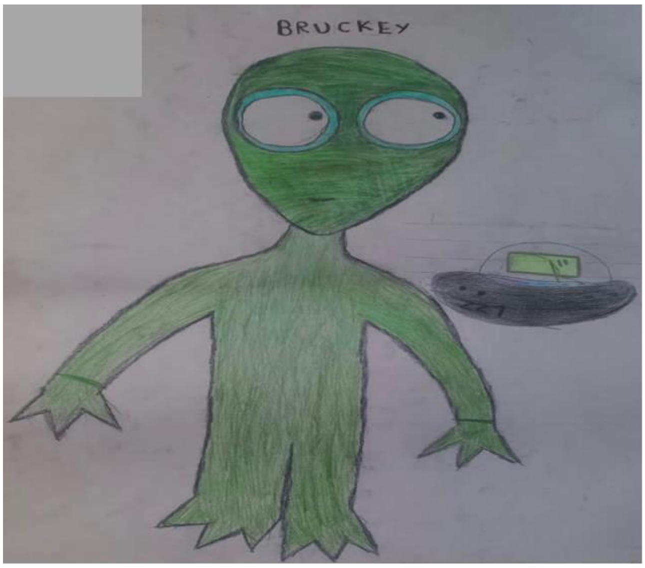

Figure 2.

Response from the auxiliary didactic system [S_1 (G_1, y, Q_1)] ➥ St_1 (Research collection, 2023)

The physical totalities in the structuring of the bodily constitution of the probable appearance of the ‘people’ make it clear that the auxiliary didactic system [S_1 (G_1, y, Q_1)] ➥St_1, as for the [S_3 (G_3, y, Q_1)] St_3, identified physical units with the decimal numeral unit by comparing one as if it were the other, and vice versa. In their defense, the class questioned the didactic systems about what inspired the elaboration of the situations (St_1 e St_3), whose answers were explained in the following terms: *[S_1 (G_1, y, Q_1)] ➥St_1: we were inspired by the cartoon character Hulk, and so we decided to paint it using the color green and named it “Bruckey”; [S_3 (G_3, y, Q_1)] ➥St_3: our drawing is inspired by the concept of “Area 51,” as scientists in that area conducted several experiments, developing and testing planes and possible spaceships for espionage, according to our research on Internet sites.*

Situations St_1 and St_3 forwarded by these auxiliary didactic systems reveal the use of decimal numerals in a naturalized way, taken, according to Ferreira and Guerra (2020), as unquestionable physical quantities; that is, objects of knowledge learned without programmed teaching, “but incorporated by use in social practices” (Ferreira & Guerra, 2020, p. 12).

On the other hand, auxiliary didactic systems [S_2 (G_2, y, Q_1)] St_2 and [S_5 (G_5, y, Q_1)] ➥ St_5 revealed the following responses to the question $\boldsymbol{Q_1}$:

Figure 3.

Responses from auxiliary teaching systems St_2 *and* St_5*, respectively* St_1 *(Research collection, 2023)*

In defense of situations St_2 and St_5 presented to the class, the didactic systems assumed in the study of the problem the symbol V of quantification of beings as one of the digits of the Roman numerals corresponding to the physical quantity of five units. In this journey, the didactic systems [S_2 (G_2, y, Q_1)] St_2 and [S_5 (G_5, y, Q_1)] St_5 also presented in their defense the main symbols used by the Romans to express quantities, after researching textbooks available at the school for study.

However, situations St_2 and St_5 were questioned by the class because, according to other didactic systems, beings similar to humans were 'similar' in several physical aspects. Therefore, the class did not accept that these beings have five eyes in their physical composition. These observations raised by the didactic systems [S_1 (G_1, y, Q_1)] St_1; [S_3 (G_3, y, Q_1)] St_3 and [S_4 (G_4, y, Q_1)] St_4 were based on descriptive observation of the problem text and the relationships between the quantities of beings, compared with Hindu-Arabic numerals (O = 0; I = 1; V = 2; A = 3, and Z = 4).

The dynamics of research from the perspective of the didactics of the SRA appear to preserve the *principle of methodical doubt* (Chevallard, 2009), suggesting that any statement presented by the didactic systems should not be accepted or rejected a priori, as it is necessary to treat these statements as conjectural.

Thus, studying the problem in a potentially real context led students to encounter different situations associated with it, emerging in each auxiliary didactic system, which were not always accepted by the class without observing the principle of methodical doubt. It is worth highlighting that the situation that may emerge from the problem is not confused with this, since "the notion of situation with mathematics that may emerge from it is the product of the subject's abstraction in the face of the type of problem considered and, with this, draws on his experiences accumulated in their personal and institutional life history" (Ferreira, 2023, p. 205).

Ultimately, the study process undertaken by these teaching systems revealed, in our understanding, potentialities of the situation in an unusual context for students, acting as a provider of conditions, in the sense of didactic transposition, which made possible different changes in the qualities of relationships (Chevallard, 2005) of these students in facing the naturalization of decimal numerals through non-decimal numerals.

Otherwise, "the mobilization of knowledge in quantification situations, with the use of unusual written records and with strangeness, undoubtedly contributed to the emergence of questions" (Ferreira, 2020, p. 270) of interest to the central didactic system $[S_p\,(G_i, y, Q_1)\,M]$ $St_i^{\heartsuit}$ that were necessary for the development of attitudes towards the pedagogy of questioning.

In this sense, the study of the problem situation in an unusual context allowed didactic systems to reach the minimum condition to leave the "comfort zone," as recommended by Ripoll et al. (2016, p. 32), because:

> Thinking about positional systems in other bases can help to "clear up" some of the properties and procedures that we are so used to dealing with in base 10, but about which we give little thought. In other words, carrying out these procedures on other bases takes us out of the "comfort zone" that base 10 provides, and forces us to consider the justifications for each step that we execute automatically, whose validity is often assumed to be certain.

Students' demonstrations made clear, in our understanding, the "confrontation" of naturalized practices with decimal numerals by contrast with non-decimal numerals, thus avoiding possible "confusion" of rules of the numerals themselves with rules of their representation.

Furthermore, the study of the situation created conditions that led students to encounter different objects of knowings, such as Roman numerals referenced by the didactic systems [S_2 (G_2, y, Q_1)] St_2 and [S_5 (G_5, y, Q_1)] St_5, derived from their past institutional relations and resumed in the investigative flow of the study of the situation. From this perspective, it is worth noting that "your personal relationship to an object *o* is formed by the integration, over time, of the influences exerted by different institutional relationships to which the person has been subjected"[14] (Chevallard, 2009, p. 3, our translation) in different institutional positions.

Final considerations and perspectives

In this research, the aim was to highlight the important role of studying an unusual problem situation, in a potentially real context, as a provider of minimum conditions, in the sense of didactic-institutional transposition (Chevallard, 2005, 2019). This was due to the need to address the naturalization of decimal numbers as part of a didactic problem recognized by different authors in the literature on teaching DNS (Sadovsky, 2005; Terigi & Wolman, 2007; Itzcovitch, 2008; Ripoll et al. 2016; Ferreira & Guerra, 2020; Ferreira, 2020; Guerra & Ferreira, 2022).

In order to address, even if partially, the didactic problem of the DNS, which consists, according to Itzcovitch (2008), in finding suitable situations to make the rules of the DNS explicit, we sought to observe Ferreira's (2020) recommendations for the (re)construction or use of mathematical organizations on decimal numerals that would allow validating or refuting hypotheses experimentally put forward in the classroom about the DNS.

This problem was addressed based on notions of the theory of the didactic transposition and, more broadly, through notions of the ATD, as these theoretical-methodological resources highlight the questioning of knowledge, more precisely, the "version" of knowledge of the DNS that lives in the school institution as a *wise knowing* (Chevallard, 2005) naturalized and, as such, legitimized by human culture, whose treatment in school required considering didactic transpositions to make it teachable.

[14]Fragments of the text: Parce que son rapport s'est formé par l'intégration, au fil du temps, des influences exercées par les divers rapports institutionnels auxquels elle a été assujettie.

The empirical results found with 6th-grade elementary school students at a public school highlighted important aspects of the problem situation in a potentially real context by introducing the creation of a set of conditions guided by the dynamics of the didactic of the SRA, especially in the search for answers to the question they faced: How do those people and their flying saucer probably look like?

Among these conditions, we observed the installation of several auxiliary didactic systems, whose study path allowed the encounter of these didactic systems with different objects of knowings, such as the investigation of Roman numerals, the decoding of non-decimal numerals adopted in the context of the problem situation supported by Hindu-Arabic numerals, the consultation on Internet sites about data on the supposed existence of "area 51," as well as the exercise of *methodical doubt* (Chevallard, 2009) by encountering different abstract situations, some of which are not immediately accepted by the class of students as a response to the type of problem.

The context of the study of the problem situation introduced a dynamic for the mobilization of knowings from the use of the correspondence technique between numerals, decimals, and non-decimals; in addition, of course, to the revelation of traces, even if partially, of the denaturalization of decimal numerals, specifically in the defense of the answers found by the didactic systems when they realized that the numerals of beings similar to humans cannot be interpreted or read in full, as is done with decimal numerals that are endowed with names. Otherwise, the process of studies undertaken ratifies Ripoll et al.'s (2016) emphasis on "escape" from the "comfort zone" provided by the decimal base.

Otherwise, the conditions introduced by the problem situation caused changes in the quality of the students' relationships with the DNS and in their cognitive dynamics, since the "personal relationship of x with an object changes it –or is created, if it did not already exist– by the encounter of x with the object o in the institutions I where it (object) lives and where x comes to occupy a certain position"[15] (Chevallard, 2009, p. 2, our translation), since it is necessary to take into account that mathematical activity "is described in terms of *situations*

[15] Fragments of the text: Rapport personnel de *x* à un objet *o* change – ou se crée, s'il n'existait pas encore – par la rencontre de *x* avec l'objet *o* dans les institutions *I* où il vit et où *x* vient occuper une certaine position *p* qui le met en contact avec *o*.

and consists mainly in 'dealing with problems' in a wide sense"[16] (Bosch et al., 2006, p. 2, authors' emphasis, our translation).

Thus, the study of the problem situation in an unusual context included, in our understanding, "the *raison d'être* or rationality that gives meaning to mathematical activity carried out under institutional constraints that provide and limit the application of the corresponding mathematical knowledge"[17] (Bosch et al., 2006, p. 3, authors' emphasis, our translation), in our case, with the study of the DNS from non-decimal numerals, since the mobilization of knowings proved to be dependent on the situation at stake, as recommended by the ATD.

Ultimately, we feel encouraged to conduct future research into the study of other conditions to be considered in the study of decimal numerals, in order to provide more empirical answers, even if partially, to address the naturalization of these numerals in school mathematics teaching, by considering manipulative objects within the reach of basic school students, as recommended by Ferreira (2020), to not only give relevance to the practice of quantifying physical quantities but also to forward tasks that take into account the structuring of numerals in different types of groupings, above all, due to the strategic importance that this object with its genesis in social practices has for the structuring of various knowledge, including the notion of numbers, as desired by official documents and by the institution of academic mathematics.

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[16]Fragments of the text: It is described in terms of *situations* and consists mainly in "dealing with problems" in a wide sense.

[17] Fragments of the text: The "raison d'être" or *rationale* that gives sense to the performed mathematical activity.

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