

Mathematical investigation in mathematics education in its expressions of *doing* and *teaching* mathematics

Investigación matemática en educación matemática en sus expresiones de hacer y enseñar matemáticas

Recherche mathématique en enseignement des mathématiques dans ses expressions de faire et d'enseigner les mathématiques

A investigação matemática na educação matemática em suas expressões do fazer e do ensinar matemática

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Abstract

In search of a phenomenology of Mathematical Investigation in Mathematics Education, the doctoral research of the first author of this article, guided by the second, interrogated 12 academic works that proved significant to the community affected by the topic. As a result, Mathematical Investigation in Mathematics Education could be understood as a way of doing mathematics and as a way of teaching mathematics. As a way of doing mathematics, research has shown that Mathematical Investigation values conjectures, tests, generalizations, and demonstrations as its *modus operandi*. As a way of teaching mathematics, it idealizes the scientific experience of producing knowledge in mathematics for the classroom, conceiving the students' actions as imitating, acting like a mathematician. The meanings thus opened were interpreted hermeneutically in light of the question: *What is this, Mathematical Investigation in*

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Mathematics Education? This text is an excerpt from the aforementioned research, through which we present understandings of Mathematical Investigation in Mathematics Education, expressed in the doing and teaching of mathematics.

Keywords: Philosophy of mathematics education, Hermeneutics, Doing mathematics, Teaching mathematics.

Resumen

En busca de una fenomenología de la investigación matemática en la Educación Matemática, la investigación doctoral del primer autor, guiada por el segundo, analizó 12 trabajos académicos que resultaron significativos para la comunidad afectada por el tema. Como resultado, la investigación matemática en Educación Matemática podría entenderse como una forma de hacer y de enseñar matemáticas. Como forma de hacer matemáticas, las investigaciones han demostrado que la investigación matemática valora, como su *modus operandi*, las conjeturas, pruebas, generalizaciones y demostraciones; y, como forma de enseñar matemáticas, idealiza la experiencia científica de producir conocimiento en matemáticas para el aula, concibiendo las acciones del estudiante como imitadoras, actuando como un matemático. Los significados así abiertos fueron interpretados hermenéuticamente a la luz de la pregunta: ¿Qué es esto: *Investigación Matemática en Educación Matemática*? Este texto es un extracto de la investigación mencionada, con el que exponemos comprensiones sobre la investigación matemática en la Educación Matemática, en sus expresiones del hacer y de la enseñanza de las matemáticas.

Palabras clave: Filosofía de la educación matemática, Hermenéutica, Hacer matemáticas, Enseñar matemáticas.

Résumé

À la recherche d'une phénoménologie de la recherche mathématique en éducation mathématique, la recherche doctorale du premier auteur de cet article, guidée par le second, a été menée à travers douze travaux académiques qui se sont avérés significatifs pour la communauté touchée par le thème. En conséquence, la recherche mathématique en enseignement des mathématiques pourrait être comprise comme une manière de faire des mathématiques et comme une manière d'enseigner les mathématiques. En tant que méthode de pratique des mathématiques, la recherche a montré que l'enquête mathématique valorise les conjectures, les tests, les généralisations et les démonstrations comme *modus operandi*; et, comme moyen

d'enseigner les mathématiques, elle idéalise, pour la classe, l'expérience scientifique de production de connaissances en mathématiques, en concevant les actions des élèves comme une imitation, un acte de mathématicien. Les sens et les significations qui ont été ouverts ont été interprétés herméneutiquement à la lumière de la question : « *Qu'est-ce que c'est ? La recherche mathématique dans l'enseignement des mathématiques ?* Ce texte est un extrait de la recherche susmentionnée, avec laquelle nous présentons des compréhensions sur la recherche mathématique dans l'enseignement des mathématiques, dans ses expressions de faire et d'enseigner les mathématiques.

Mots-clés : Philosophie de l'enseignement des mathématiques, Herméneutique, Faire des mathématiques, Enseigner les mathématiques.

Resumo

Em busca de uma fenomenologia da Investigação Matemática na Educação Matemática, a pesquisa de doutorado do primeiro autor deste artigo, orientada pelo segundo, interrogou-a com doze obras acadêmicas que se mostraram significativas à comunidade afeta ao tema. Como desfecho, a Investigação Matemática na Educação Matemática pôde ser compreendida como um modo de fazer e de ensinar matemática. Enquanto modo de fazer matemática, a pesquisa mostrou que a Investigação Matemática valoriza as conjecturas, os testes, as generalizações e as demonstrações como seu *modus operandi*; e, enquanto modo de ensinar matemática, idealiza, para a sala de aula, a experiência científica da produção do conhecimento em matemática, concebendo o fazer dos alunos como um imitar, um agir como o matemático. Os sentidos e significados que com isso se abriram foram interpretados hermeneuticamente à luz da interrogação: o que é isto, a investigação matemática na educação matemática? Esse texto é um recorte da pesquisa supracitada, com o qual expomos compreensões sobre a Investigação Matemática na Educação Matemática em suas expressões do fazer e do ensinar matemática.

Palavras-chave: Filosofia da educação matemática, Hermenêutica, Fazer matemática, Ensinar matemática.

Mathematical investigation in mathematics education in its expressions of *doing* and *teaching* mathematics

Introdução

The mathematics education community has highlighted Mathematical Inquiry as a potential approach to teaching and learning mathematics. Grounded in the inquiry-based paradigm, Mathematical Inquiry aims to empower students to construct mathematical knowledge in the classroom through a speculative approach. In this sense, Mathematical Inquiry proposes a way of organizing mathematics lessons and, alongside other approaches, is part of a broader set of teaching methodologies.

Beyond the didactic-pedagogical sphere, Mathematical Inquiry has become the focus of scientific research from various perspectives. Regarding academic production *stricto sensu* in Brazil, research on this topic usually confirms facts or provides *post hoc* explanations for new facts and discoveries (Wichnoski, Klüber, 2015). There appears to be a tacit acceptance of the status quo of Mathematical Inquiry in mathematics education, which is grounded in a hegemonic conception that leaves theoretical gaps and lacks understanding, in our view.

In this context, we saw the development of a thesis³ as an opportunity to bring mathematical research to the forefront and to shed light on these gaps from ontological and philosophical perspectives. This approach is not in opposition to or in alignment with what is already established but rather alongside it. It introduces new perspectives, ideas, and questions that are inherent in the constitution of knowledge. Thus, it serves as a way to view and understand mathematical inquiry in mathematics education.

With this aspiration and with our embarkation on phenomenological-hermeneutic research, we ask: What is Mathematical Inquiry in Mathematics Education? Mathematical inquiry in mathematics education can be understood in two areas of convergence: as a way of doing mathematics and as a way of teaching mathematics.

The invariants expressed in each area are called nuclei and are interpreted hermeneutically to express the meaning and significance of the subject under inquiry. In this text, we present a hermeneutics that elucidates our understanding of mathematical inquiry in mathematics education, as it relates to doing and teaching mathematics.

³ Cf. Wichnoski (2021).

Methodology

In seeking to understand Mathematical Research in Mathematics Education through significant academic works, we turn to texts authored by individuals situated within specific historical and cultural contexts. For these individuals, the world has its own meaning. Thus, what we come to understand, as well as the act of understanding itself, cannot be divorced from these aspects. This has led us to hermeneutic phenomenology.

Phenomenology stems from an approach that ultimately seeks the complete absence of a priori presuppositions. Rather than denying their existence, it interrogates them alongside the phenomenon⁴. This means not projecting theories or explanations onto what is observed but rather allowing oneself to be guided by it so that it may clarify itself. It is a return to things themselves, stripping them bare to perceive their essence. Therefore, it is an epistemology of essence (Husserl, 1989). In this sense, phenomenology looks at the phenomenon in its entirety, free from the lens of any theories that explain what is seen. Phenomenology is solely concerned with the study of essences, which become evident through phenomenological reduction in a movement contextualized within the lifeworld (Husserl, 1989).

Therefore, our task with significant academic texts was not to interpret them through the lens of a prior theory but to understand them as texts that express meaning. Thus, our interpretation was guided by listening to the texts' speech as human products. By referring to them as significant academic works, we treat them as human constructions, not mere objects of analysis. As objects of analysis, they would be static and reducible to scientific methods. However, as human constructions, they "appeal to more subtle and comprehensible modes of understanding" (Palmer, 2018, p. 21).

Thus, understanding transcended linguistic comprehension, which is usually analytical and tied to language and objective words. Instead, it advanced toward an understanding that engaged the text within and through the historical context of the subject who experienced the interpretation. For this reason, it was hermeneutic. The task of interpretation was not limited to understanding the content of significant academic works as if it were possible to control them under the subject-object framework. Rather, it involved what we experienced through lived interaction, a mutual

⁴ It refers to all things intended by consciousness and, therefore, carries within its meaning the revelation of essences—not of things in themselves, but of the “in-itself” of the thing as revealed. In Wichnoski (2021) research, the phenomenon is Mathematical Inquiry. *Educ. Matem. Pesq.*, São Paulo, v. 28, p. 01-27, 2026, e73307

interrogation between us (interpreters) and the works. Hermeneutics contributes to this dynamic and makes it possible to see and hear the text. From this perspective,

[...] Interpreting a work means moving toward the horizon of questions within which the text operates. But this also means that the interpreter moves toward a horizon where other answers are possible. It is in light of these other answers—within the temporal context of the work as it unfolds—that we must understand what the text says. (Palmer, 2018, pp. 312).

In phenomenological research, hermeneutics acknowledges and embraces the possibility of producing knowledge, breaking away from the patterns established by the positivist scientific method. By allowing the process of knowing to unfold without adhering to predetermined methodological paths, phenomenological hermeneutics does not deny the importance of such methods. However, it does not ground the rigor of research in them as something predefined. Rather, it is grounded in the journey and the lived temporality of the knowing process. In other words, scientific rigor consists of the "interweaving of happening and understanding" (Gadamer, 1999, pp. 594–595).

In this sense, interpretation is not addressed by the text; rather, it is a process of questioning and shifting direction that allows one to understand the text in terms of what has not been said—that which is given to be seen (Heidegger, 2015)—or, in other words, interpretation is allowing oneself to be guided by the text since it always has something to reveal to the reader. The ways phenomenology and hermeneutics were present in the construction of the doctoral research can be found in Wichnoski (2022) and Wichnoski e Klüber (2022), respectively.

Methodological procedures

Our doctoral research was based on 12 significant academic works on mathematics. These works have been preserved by tradition as useful sources of knowledge for the mathematics education community. Guided by our experience in academic research, we examined Brazilian dissertations and theses on mathematical inquiry to identify the works they referenced.

We accessed these works through the Brazilian Digital Library of Theses and Dissertations (Biblioteca Digital Brasileira de Teses e Dissertações – BDTD), using the search terms "Mathematical Research" and "Mathematical Investigations," and focusing on works published before 2018. This search yielded fourteen studies⁵, which differ in

⁵ The reader will find them listed in Wichnoski (2021).

academic nature: two theses and six dissertations (three academic and three professional).

We then identified and selected the works referenced in these studies that addressed mathematical research in mathematics education, without differentiating by scientific nature. We quantified them based on how often they were cited. We assumed that works cited four or more times were relevant to the field because the others had only been cited once by any of the studies. Sixteen⁶ works proved relevant and served as references for Brazilian *stricto sensu* studies on mathematical inquiry in mathematics education.

Pausing the research process allowed us to realize that the works consulted in Brazilian *stricto sensu* studies also had a set of works considered references. Taking the works that served as references in the first search level implied treating the works cited as references by the works selected in that first level as significant. Recursively, this process could be repeated as many times as desired to access further levels of selection.

Using this approach, we moved on to the second level of quantification. At this level, we only considered works that had already been quantified at the first level. In other words, we sought the number of citations among the works selected at the first level. Next, we cross-referenced these works and counted instances in which one work cited another.

For the second quantification, we selected works that had been cited at least once. This selection criterion led to the exclusion of works three, eleven, and thirteen. Thus, we used two levels of quantification, along with inclusion and exclusion criteria, to select the works. The inclusion criteria were as follows: at the first level, a frequency greater than or equal to four; at the second level, a frequency greater than or equal to one. The exclusion criteria were: at the first level, a frequency less than four; and at the second level, a frequency of zero.

We excluded one study idiosyncratically because, although it had a frequency of 1, it was cited by a study with a frequency of 0 at the second level of quantification—that is, it was cited by a study that was not cited by any others. Two other works with a frequency of one at the second level were not excluded because they intersect with works that had higher frequencies.

⁶ Ibid., note 5.

These choices were significant to us because they were made in light of our experience with the phenomenon under investigation. Furthermore, we believe these choices are significant to the Brazilian community researching mathematical inquiry in mathematics education because these sources are frequently cited in Brazilian research in this field. These studies are referred to in the research as *significant academic works* and are presented in Table 1.

Table 1.

Significant academic works in mathematical research are included in the study (Wichnoski, 2021, p. 41-42)

Encoding	Scientific Nature	Title	Authors
1	Book	Investigações Matemáticas na sala de aula	João Pedro da Ponte; Joana Brocardo; Hélia Margarida Oliveira
2	Article	As actividades de investigação, o professor e aula de Matemática	Helena Fonseca; Lina Brunheira; João Pedro da Ponte
3	Book chapter	Investigações, Resolução de Problemas e Pedagogia	Paul Ernest
4	Article	O trabalho do professor numa aula de investigação matemática	João Pedro da Ponte; Hélia Margarida Oliveira; Lina Brunheira; José Manuel Varandas
5	Book	Histórias de investigações matemáticas	João Pedro da Ponte; Hélia Margarida Oliveira; Maria Helena Cunha; Maria Irene Segurado
6	Book chapter	Divagações sobre investigação matemática e o seu papel na aprendizagem de matemática	Carlos Alberto dos Santos Braumann
7	Book chapter	Quatro funções da investigação na aula de Matemática	E. Paul Goldenberg
8	Thesis	As investigações na aula de matemática: um projecto curricular no 8º ano	Joana Brocardo
9	Dissertation	Avaliações de investigações matemáticas: uma experiência	José Manuel Varandas
10	Book	A relação professor-aluno na realização de investigações matemáticas	João Pedro da Ponte; Catarina Ferreira; José Manuel Varandas; Lina Brunheira; Hélia Margarida Oliveira
11	Book chapter	Investigando as Aulas de Investigações Matemáticas	João Pedro da Ponte; Catarina Ferreira; Lina Brunheira; Hélia Margarida Oliveira; José Manuel Varandas
12	Dissertation	Actividades de investigação na aula de Matemática: Aspectos da prática do professor	Hélia Margarida Oliveira

After reviewing the significant academic works listed in Table 1, we highlighted excerpts that demonstrated connections between the respective texts and the research question. Using these excerpts, we constructed clear, consistent, articulated assertions that explained our understanding of the texts and remained within the scope of the research. By searching for the meanings present in these assertions, we proceeded to establish units of meaning — that is, phrases expressing distinguishable meanings within the text's context. An example of this process is provided in Table 2.

Table 2.

Example of the construction of units of meaning (Wichnoski, 2021, p. 43).

Excerpts from the text	Compound statements	Units of meaning
<u>Can mathematicians' research serve as inspiration for the work that professors and students do in math classes? This general question raises the question of what constitutes mathematical research activities. It is important to know whether students are capable of investigating mathematical problems and how this can contribute to their learning.</u>	The authors begin by asking whether it is possible to conduct mathematical investigations in math classes, inspired by the work of mathematicians. This question sparks discussions about what constitutes mathematical investigation activities in the classroom.	The work of mathematicians as inspiration for educational work with Mathematical Inquiry.
<u>Research, since it constitutes an essential part of mathematical activity, is directly related to the production of mathematical knowledge and is linked to the nature of this science.</u>	In presenting the recommendations of the National Council of Teachers of Mathematics (NCTM) for mathematics education, the author explains why it is important to integrate mathematical inquiry into mathematics classes, as it is directly related to the production of mathematical knowledge and is linked to the nature of mathematics.	Mathematical research is directly related to the production of mathematical knowledge. Mathematical research is directly related to the nature of mathematics.

In moving towards more comprehensive syntheses of what is said and interpreted, we sought convergences among the units of meaning so that broader constituent elements might emerge from the individual characteristics — elements that would articulate the overall characteristics expressed in each unit. The initial convergence of these units of meaning produced the first ideas, which were still vague and were termed 'first nuclear ideas'. Table 3 illustrates the formation of a group of these initial ideas.

Table 3.

Example of phenomenological reduction performed on units of meaning (Wichnoski, 2021, p. 44)

Nuclei of ideas	Units of meaning
On Mathematical Research and Mathematics.	Mathematical research is directly related to the nature of mathematics.
	In official discourse, mathematical research is the essence of mathematics.
	In professional discourse, mathematical research constitutes true mathematics.
	Some approaches regard mathematical research as an example of true mathematics.

Looking at the entirety of the initial core ideas, converging meanings were identified, prompting further convergence and giving rise to groups of ideas broader and clearer than the previous ones, referred to as secondary core ideas. Table 4 illustrates the formation of a group of secondary core ideas.

Table 4.

Example of phenomenological reduction carried out on the initial core concepts (Wichnoski, 2021, p. 45)

Second core idea	Early nuclear ideas
On Teaching Practices Involving Mathematical Inquiry.	It speaks to the openness and diversity of mathematical research.
	Discusses the professor and Mathematical Inquiry.
	Discusses teaching, pedagogical practice, and Mathematical Inquiry.
	Discusses the presence of intra- and extra-mathematical contexts in Mathematical Inquiry.
	Discusses the student and Mathematical Inquiry.
	Discusses the phases/moments/stages of pedagogical practice with Mathematical Inquiry.

We proceeded with further convergence, articulating the meanings that emerged from each group of the second core ideas into clusters. This concluded the process of phenomenological reduction, as these clusters expressed the essentials — the invariants of the phenomenon under investigation — that remained unchanged in meaning even as they were manifested in different ways. These nuclei lend themselves to description and interpretation, which are always intertwined with the question, '*What is Mathematical Inquiry in Mathematics Education?*' The articulation of the second core ideas, which constitute the idea nuclei, is shown in Table 5.

Table 5.

Example of phenomenological reduction carried out with the second nuclear ideas

Nuclei of ideas	Second nuclear ideas
Mathematical Inquiry in Mathematics Education is a way of doing mathematics.	Discusses mathematical research and the production of mathematical knowledge.
	Discusses mathematical research and the application of mathematics.
	Discusses mathematical research and mathematics.
	Discusses the types of reasoning present in mathematical research and in doing mathematics.
Mathematical Inquiry in Mathematics Education is a method of teaching mathematics.	Discuss teaching practices related to mathematical inquiry.
	Discuss mathematical inquiry and problem-solving.
	Discuss specific mathematical inquiries.
	Discuss mathematical inquiry and the curriculum.

Throughout the entire process, we remained vigilant, setting the phenomenon aside and observing it with caution. This approach ensured that we avoided naive ways of proceeding and questioning, thereby breaking with the certainties of common sense. It is worth mentioning that the linearity of our written account of how we conducted our doctoral research does not accurately reflect the actual process. The procedures adopted did not follow a fixed, predetermined order, as the description suggests; rather, they were revisited and reoriented as the phenomenon required.

Hermeneutics of core ideas

In this section, we present our observations and experiences from the research to highlight the general structures of the phenomenon under investigation: Mathematical Inquiry in Mathematics Education. Our goal is to express the meaning of this phenomenon within the context of doing and teaching mathematics using a hermeneutic approach. We acknowledge the existence of an epistemology behind teaching, which could blur the distinction between teaching and doing mathematics. However, the phenomenon proved to be multifaceted, comprising two inseparable yet distinct aspects: the epistemological, concerned with doing mathematics, and the didactic, concerned with teaching mathematics.

Mathematical research in mathematics education as the practice of doing mathematics

Let us start with the following assertion: *Mathematical inquiry in mathematics education is a way of doing mathematics*. The term "doing" highlights an epistemological dimension that speaks to the genesis, conditions, legitimacy, and structuring through which mathematics reveals itself and becomes present in Mathematical Inquiry. As a mode of knowledge production, an expression of doing is found in mathematics itself. It manifests the nature and essence of mathematics and is a constituent of "true mathematics."

In this sense, we see a discourse on doing mathematics in mathematical research that grounds itself in nature, essence, and true mathematics. This way of articulating it rests on a conception of mathematics specific to the speaker. This leads us to ask: What is true mathematics? What is its nature? And its essence?

According to Inwood (2002, p. 54), essence "is the 'internal nature or principle' of a thing." [...] 'its basic nature,' 'its essential nature,' 'its inner being.'" Essence can be understood as the substantivization of the verb "sein," which originally signifies modes of being. Therefore, essence refers to the "being-what" and "being-as" of something. In other words, "if we inquire about 'essence' in the usual sense of the question, the question is about what 'makes' an entity be what it is, and thus about what constitutes its 'being-what'" (Inwood, 2002, p. 55).

For Aristotle, the true essence of a being or thing—which he calls "necessary essence" or "substance," is that which states what a being or thing cannot fail to be (Abbagnano, 2012). Thomas Aquinas defines essence as the common mode of being inherent to entities, a state that establishes the modes of being and how they are.

[...] something common to all natures, by virtue of which different beings are classified into different genera and species. For example, humanity is the essence of man. However, since the essence of a thing is what constitutes its genus and species, philosophers have replaced the term "essence" with "quiddity." This is why, in Book VII of *Metaphysics*, the Philosopher frequently speaks of the *quod quid erat esse*—that is, the quiddity of a thing (Abbagnano, 2012, p. 420).

The word "nature" comes from the Latin "natura," meaning "birth, characteristic, natural order," and, in turn, from "nasci," meaning "to be born, to grow, to be produced" (Inwood, 2002, p. 125). This definition refers to the innate qualities of a being or something at birth. In other words, nature designates a condition inherent to existence, which aligns with the previously explained understanding of essence.

Furthermore, nature is associated with birth as the condition in which a being is constituted and finds meaning.

Therefore, the nature of a being or thing can be understood as the essence that speaks to its existence and existence-as at birth. In other words, it is a mode of existence at birth. Thus, when saying that mathematics is linked to its nature, the idea expressed connects it to its essence, to its mode of being in its nascent state, at its origin. Heidegger (2005, p. 11) tells us that "origin means that from which and through which a thing is what it is and how it is." What a thing is, as it is, is its essence. The origin of something is the source of its essence." In light of this, we should ask: What is the true mathematics that Mathematical Inquiry exemplifies, and how is it exemplified?

First, let us consider that investigative work is directly related to experimentation. Emerging mathematical ideas result from students' attempts to understand. Similarly, observation is present in mathematics and is fundamental to generating conjectures in the initial phase, as well as to helping students articulate their understanding.

Observation and experimentation are closely associated with science and scientific research. Although these actions are central to mathematics, they establish intuitions that lead to the formulation of conjectures. However, mathematical scientists generally agree that these must be validated by proof, which devalues the experimental aspect of mathematics in favor of the deductive. In mathematical research, it seems that observation and experimentation trigger the practice of mathematics but are then abandoned in favor of processes that demonstrate and generalize.

Another characteristic of this mathematical practice, as understood in mathematical research, is the presence of intuitive thinking. In analytical psychology, intuition is a psychic function associated with the irrational realm and "consists of a perceptual apprehension of phenomena through the unconscious" (Ramos, 2005, p. 141). Thus, intuition reveals itself as a sudden, unconscious insight into the self. It is a form of clear-sightedness that does not rely on any intelligible means. Its apprehension of phenomena is an immediate occurrence in the realm of the sensible that is peculiar only to the person who intuits.

Throughout the history of philosophy, intuition has been understood as "a superior and privileged form of knowledge" (Abbagnano, 2012, p. 670), characterized by an immediate relationship between subject and object. For Descartes, intuition is a mental faculty that "extends to things [...] and to everything that the intellect

experiences with precision in itself or in the imagination" (Abbagnano, 2012, p. 670). Thus, intuition is a particular and privileged form of thought in human consciousness through which truths are apprehended by the intellect in and of themselves. In other words, the object of knowledge is immediately present.

According to Kant (2001), intuition is a pure form of thought through which the mind perceives, organizes, and comprehends. He did not view intuition as a passive faculty, but rather as a mode of creative action. Kant believed that intuition operates as a priori knowledge, grounded in reason and independent of experience.

[...] offers us a brilliant example of how far one can go in *a priori* knowledge, independent of experience. It is true that it deals with objects and knowledge only to the extent that they can be represented in intuition. But one easily fails to notice this circumstance, because that intuition itself can be given *a priori* and, therefore, is scarcely distinguishable from a mere pure concept (Kant, 2001, pp. 67).

According to Kant (2001), intuition is independent of experience. As far as mathematics is concerned, objects of knowledge are always pure, a priori, intuitive judgments and, therefore, non-empirical. For this reason, mathematics is, for Kant, the most brilliant example of pure reason. But what understanding of intuition do significant academic texts speak of? Is intuition a psychological faculty? Is it Descartes's irrational mental faculty? Is it Kant's pure reason?

Consider the analogy between intuition and one of the human body's senses, the sense of smell, as suggested by Braumann (2002, p. 24): "The 'sense of smell' (or intuition) is one of the researcher's closest allies; without it, one does not get far." How is it acquired? I don't know, but I know that experience with our own work, and that of others, helps develop it." By asking how intuition is acquired, one eliminates the possibility that it is a sudden unconscious thought, as understood by analytical psychology.

Similarly, Cartesian and Kantian ideas, which conceive of intuition as a pure form of thought independent of experience, do not hold up either. Furthermore, the idea that experience with one's own work as a mathematician and the work of other mathematicians helps develop intuition makes it clear that it "is a psychic quality [...] but one that, at any given moment, is embodied in the body of attitudes derived from mathematical experience" (Wilder, 1967, p. 606).

Therefore, we understand that significant academic texts speak of intuition as an anticipation of what is to come. However, it is not possible to know how intuition occurs.

It simply comes suddenly to those who cast their gaze upon the future. It is a unique experience that sees the diverse and diffuse and finds a common thread that, at first, is enigmatic. Poincaré's biographical accounts support this idea of intuition. For example:

[...] For a fortnight, I had been trying to prove that no function analogous to what I later came to call "Fuchsian functions" could exist. At the time, I was completely ignorant. I would sit at my desk for an hour or two every day, trying many combinations, but I could not arrive at a result. One afternoon, contrary to my habit, I had a cup of coffee and could not fall asleep. Ideas came flooding in, and I felt them slipping away until, as it were, two of them clicked into place, forming a stable combination. By dawn, I had proven the existence of a class of Fuchsian functions derived from hypergeometric series (Poincaré, 1996, p. 9) (Poincaré, 1996, p. 9).

Another aspect evident in mathematical research is abstraction, which is expressed through mathematical practice. Lexically, the term "abstraction" refers to that which is not concrete, belonging to the realm of the unreal. In philosophy, it is defined as "the process of selecting something as an object of perception, attention, observation, consideration, research, or study and isolating it from other things with which it has a relationship" (Abbagnano, 2012, p. 4). In this case, there are two meanings of abstraction: the first is that of something non-concrete, and the second is that of separation resulting from a choice.

Among the definitions of isolate are "separate" and "break away" (Ferreira, 2010). In this sense, Japiassú and Marcondes (2001, p. 7) clarify that abstraction is "the operation of the mind that isolates in order to consider separately." Abstraction can also be defined as "the process by which the mind detaches from familiar meanings of lived experience and perceptions to construct concepts" (Japiassú & Marcondes, 2001, p. 7). This gives abstraction a third meaning: an operation that breaks relationships between things, focusing on them in isolation, but not an operation of choice.

Consider what Braumann (2002, p. 10) reports: "Let $N = N(t)$ represent the population size at time $t \geq 0$, and let $N(0) = N_0 > 0$ represent the initial population. In this model, it is assumed that the instantaneous growth rate of the population size is equal to the derivative $dN(t)/dt = rN(t)$, where r is the proportionality constant." As we understand it, this excerpt indicates that the initial situation of the mathematical research in question is grounded in reality. Yet, the data are stripped of their meaning, extracted from reality, and explored as purely mathematical elements. In other words,

they are abstracted, or separated, from the physical world in the process of study. This points to a way of doing mathematics situated at the boundary between targeted reality and mathematical ideality. The resulting mathematical knowledge merely reflects the things with which mathematics is related in its own language. In other words, mathematical language is the mechanism for representing things from the real world while evading the concrete.

Inductive thinking is also present in mathematical inquiry as a way of doing mathematics. This thinking rests on observing a pattern in particular cases and generalizing it to unobservable cases, extending the characteristics made explicit by the pattern. According to Bicudo (2013), one aspect of this approach to mathematics involves moving "from the multiplicity of aspects perceived in empiricism to the unity visible within that multiplicity [...] which transcends empirical data and brings together aspects of the observed into an articulating synthesis" (pp. 21–22). In other words, the practice of mathematics in mathematical research involves and emphasizes generalization.

Hermeneutically, this epistemological schematization that guides induction "implies generalization based on causal observations and claims validity as long as no contrary instance appears" (Gadamer, 1999, p. 515). This supports the inference that "it always holds." Japiassú and Marcondes (2001, p. 103) clarify that it is a practice that "proceeds to generalization based on the repetition and observation of a regularity in a certain number of cases." According to Abbagnano (2012, p. 557),

[...] sometimes the term G. [generalization] is also applied to *induction* (see entry) or to the formulation of a hypothesis (see entry), which would more accurately be called operations of universalization. G. [generalization] is discussed primarily in mathematics. 'To extend a domain by introducing new symbols, such that the laws valid in the original domain remain valid in the broader domain.'

Significant academic works indicate that generalization, along with abstraction, is one of the primary goals of mathematical research. The term "objective" is a noun derived from the transitive verb "to objectify," meaning "to aim at" or "to intend" (Ferreira, 2010, p. 539). When an objective is considered major, it implies an intensity that transcends parameters, meaning it is broader than other objectives. Therefore, generalization and abstraction transcend mere intuition and reveal themselves as the purposes of this practice, even when intuitive. They also reveal a practice that values systematization expressed in mathematical language.

Observing and studying regularities and patterns are actions that shape this way of doing mathematics and highlight its hallmark. A hallmark is defined as a "distinctive sign of an object [...] quality" (Ferreira, 2010, p. 488). This leads us to ask: Does this practice distinguish Mathematical Inquiry from other perspectives within Mathematics Education?

Generally, patterns and regularities are associated with generalization as a means of uniting the observed aspects of multiplicity into a single, typically algebraic, unit. In other words, they involve an effort to identify the implicit rule present in particular cases and unify them into a single representation. Thus, being regular is an indispensable condition that allows one to be perceived "in a single representation and determine the multiplicity in its form" (Abbagnano, 2012, p. 989). We understand, then, that this demonstrates a way of doing mathematics that seeks a pattern capable of expressing observed regularities in mathematical language. In the context of mathematics education, this involves constructing the language of mathematics in the algebraic field and its symbolic representations, which are essentially abstract.

As significant academic works in Mathematical Inquiry have shown, mathematical knowledge is ultimately constituted by proof. This implies that practice is only validated if there is proof. According to the literature on the subject, there is evidence that formalized mathematical practice relies on deduction and logical argument. Oliveira (1998, p. 15) states: "Passing the test requires demonstrating truthfulness to cease being 'merely' a conjecture and become a property established by the mathematical method." Ponte, Ferreira, Brunheira, Oliveira, and Varandas (1999, p. 17) also state: "However, it is essential to share meanings, promote the use of more precise mathematical language, and understand the importance of proofs—in short, to move toward formalization."

The truth stated in the proof is a property derived from the mode of production itself. As dictated by the rationality of Western science, this truth is communicated in propositional statements that render it eternal. According to Bicudo and Garnica (2011, p. 85), the proof embodies "the mythical character of mathematics, fueled by an excessive proliferation of the ideology of certainty, the univocal meanings of its concepts, and its temporal-spatial eternity."

Another way to view proofs in mathematical research is as justifications that employ plausible arguments. According to Brocardo (2001, p. 118), "the focus, at the level of mathematics education, must shift from the concept of rigorous

proofs/demonstrations to a concept of proof/demonstration as a convincing argument." In this sense, proofs can be made more flexible, reducing the emphasis on mathematical rigor and formalization without diminishing their importance as a means of communication in mathematical research.

Intuition plays a role in the creation of mathematics, as revealed by Connes's account in Gravina (2001) and by Braumann and Andrew Wiles's account in Singh (2014). However, "Mathematicians hate to make a false statement. Of course, they employ intuition and inspiration, but formal statements must be absolute" (Singh, 2014, p. 8). In significant academic works, Singh's assertion becomes evident when the investigative work is directly related to intuition. However, in such works, an intuitive explanation is insufficient to validate the veracity of the conjectures. Rather, argumentation and formal proof are required, indicating a rejection of the knowledge construction process that relies on intuition.

Considering Singh's (2014) citation and the fact that the conjunction "but" links clauses with the same syntactic properties and introduces phrases denoting opposition or restriction to what was previously stated, it is reasonable to suppose that intuition and demonstration contrast with one another. Thus, absolute formal statements, understood as demonstrations, are not restricted to intuitive explanations. This brings Poincaré's statement regarding Bernard to light: "It is demonstrated with logic, but invented only with intuition" (Abbagnano, 2007, p. 583). It also brings the question of the truth of mathematical knowledge to the fore, establishing it through proofs and, therefore, the correspondence between the statement in the proof and the entity being stated.

Heidegger (2015) refers to this as the classical conception of truth, which boils down to a correspondence between the intellect and the entity (*adaequatio intellectus et rei*). This term is literally translated as "conformity of the thing(s) and the intellect" (Inwood, 2002, p. 196). According to this view, the utterance of truth lies in the statement in which the entity is merely represented. However, Heidegger (2015, p. 300) cautions that this "truth belongs to the utterance" and that truth is often not expressed in valid utterances.

Some authors agree that the mathematical activity involved in mathematical investigation follows the sequence of conjecturing, testing, generalizing, and proving (Brocardo, 2001). However, before conjectures, there are insights, imaginings, vague suppositions, intuitions, and experiments. Between conjectures and tests, there are

uncertainties. Between tests and proofs, there are further uncertainties, setbacks, errors, and falsifications. Only after bridging the gap between the insight of the result and the proof does the conjecture become valid. Thus, the practice of mathematics reveals itself as standard, with a conjecture-test-proof style as a defining characteristic. Here, we highlight the term "style," which Abbagnano (2012, p. 473) defines as follows: "a set of characteristics that distinguish a particular form of expression [...] a unity of forms, tones, and dominant attitudes within a complex variety of forms and contents."

By stating that conjectures, tests, and proofs constitute the style of this way of doing mathematics, we reveal the uniqueness of its configuration and the characteristics that give it specificity and distinguish it from others. These characteristics are a recurring assertion in significant academic works, even if not described with this adjective (style). They are necessarily present in mathematical investigation. This characterization of mathematical inquiry in mathematics education reveals that conjectures, tests, proofs, and generalizations are at its core. Being at the core means they are the heart of the process that produces mathematical knowledge.

Mathematical research in mathematics education as it relates to the practice of teaching mathematics

The term "*teaching*" comes from the Greek "*didaktiké*," which is also the origin of the term "didactics." Didactics refers to the aspects involved in the teaching process. Therefore, when we say that Mathematical Inquiry in Mathematics Education is a way of teaching mathematics, we emphasize the core elements that define it as a classroom activity. As explained in the previous section, the expression "teaching mathematics" projects the expression "doing mathematics." In terms of pedagogical practice, Mathematical Inquiry aims to create a close connection between students' experience learning mathematics and mathematical scientists' experience doing mathematics.

According to Ponte, Brocardo, and Oliveira (2013), this closeness brings authentic mathematical activity into the classroom, fostering an identity between learning and doing mathematics. In the lexicon, the adjective "authentic" refers to that which is true to the author to whom it is attributed. It is synonymous with "genuine," "legitimate," and "pure" (Ferreira, 2010). According to Abbagnano (2012, p. 108), authenticity indicates that which is proper to Being, "in contrast to the loss of oneself or one's own nature, which is inauthenticity." Inwood (2002) defines the term as a derivation of the

Greek *autos*, meaning "oneself." Therefore, authenticity indicates the realization and discovery of original ways of being by the being itself because "to be authentic is to do one's own thing, not what the impersonal prescribes" (Inwood, 2002, p. 12).

Thus, understanding mathematical inquiry as the teaching of mathematics as an authentic experience is assumed to enable students to discover their own unique relationship with mathematics and to engage in a process "made by their own hands" (Inwood, 2002, p. 11). However, when we aspire for students' lived experiences to resemble those of mathematical scientists closely, we present authenticity as something inherent, disregarding the possibility that students will open themselves to it. In other words, this approach to teaching mathematics disregards the possibility of students learning through their own ways of thinking and acting, instead encouraging them to follow the tradition of doing mathematics. Although the discourse addresses student autonomy, we do not perceive it as authentic. In this sense, the practice's impersonal nature is revealed, prompting us to reflect on the inauthenticity of this experience.

Mathematical Inquiry reveals a way of teaching mathematics that closely aligns with the work of mathematical scientists and follows configurations dictated by the tradition of a distinct scientific community. This approach gives little consideration to the phenomenon of understanding, focusing solely on one facet of the cause-and-effect relationship between doing and teaching mathematics. It accepts what this tradition dictates, raising the possibility that the experience of teaching mathematics is inauthentic.

Evidently, the authenticity of the experience of doing is related to the practice of the mathematical scientist, who follows specific, objective procedures maintained by tradition. This reveals a permeation of objectivity into the experience, reducing it to an experiment governed by a method and subject to reproducibility. Thus, "the experience suspends its own history within itself and extinguishes it" (Gadamer, 1999, p. 513). This hermeneutic opening leads us to ask: To what extent can a student have an authentic mathematical experience, or perform mathematics with their own hands, according to others' methodological principles?

Since students are expected to undergo an experience similar to that of mathematical scientists, we understand that there is an idealized projection of the latter onto the original experience in the lifeworld. Thus, students "are not free to fly as they please" (Gadamer, 1999, p. 515). They are compelled to ascend gradatim (step by step) to progressive degrees of generalization and abstraction" (Ponte, Oliveira, Cunha & Segurado, 1998, p. 17). In other words, experience manifests as a trajectory that moves from formulating questions to be investigated to generalization and abstraction. This trajectory passes through creating, testing, and refining conjectures, as well as demonstrating them.

Another issue that sparks debate in this field is raised by the following excerpt: "Can the research of mathematicians inspire the work of professors and students in mathematics classes?" (Ponte, Brocardo, & Oliveira, 2013, p. 9). Inspire means to influence (Ferreira, 2010), so the inspired person is influenced by the one who inspires. If the scientific work of mathematicians inspires Mathematical Research, does it subserviently follow the practice of doing mathematics?

It is said: "The student is called upon to act as a mathematician" (Ponte et al., 2013, p. 23). This reinforces the similarity between the activities students engage in in a school context and those mathematicians engage in in a scientific context. These activities are equivalent; thus, in this approach to teaching mathematics, students have the opportunity to experience the work of professional mathematicians. When we refer to a mathematician or mathematicians, we are referring to the action in relation to others, not to the being who exists in the world and is capable of doing mathematics. With this in mind, we see that teaching mathematics invites students to engage in mathematical practice. When students engage in mathematical inquiry, they form a small community that uses processes characteristic of mathematical investigation.

According to Gravina (2001, p. 17), "The process of mathematical creation is complex. In it, 'mathematical thinking' takes place, which is characterized by experimenting, interpreting, visualizing, abstracting, conjecturing, making mistakes, and proving." Therefore, mathematicians' approaches to mathematics are homogeneous; they all follow the same path, albeit with their own steps. In a school setting, students are also expected to follow a similar path, beginning with

experimentation and ending with demonstration. If this journey is not experienced through students' original ways of engaging with and understanding mathematics, then the content of this science will mean nothing to them because they did not "do mathematics with their own hands" (Inwood, 2002, p. 11).

We acknowledge that pedagogical guidance in the classroom is inevitable to some extent. Alongside significant academic work, this guidance manifests through attitudes that are guiding, indicative, supportive, stimulating, and informative. Therefore, guidance means moderating work by discreetly and lightly intervening; it is "giving direction to" (Ferreira, 2010, p. 257). On the other hand, guidance reveals itself as control, meaning it is up to the professor to choose which conjectures to test and which situations to deem challenging for students.

As a moderator, the professor has the opportunity to see things from students' perspectives and to value their ways of thinking. However, this "giving direction" may serve to satisfy the professor's desires, which depend on how the professor views education, the curriculum, and learning (Ponte et al., 1999, p. 25). Thus, students' views are conditioned by their professors'. Although it is recognized that students' attitudes change when confronted with information or experiences, professors determine this transformation, as they implicitly provide information and direct experiences. In other words, students' actions are governed by professors' actions. Professors provide information, structure experiences, and consequently, structure students' conceptions of mathematics.

It should be noted that students' actions in Mathematical Inquiry reaffirm the transposition of the epistemology of doing mathematics to teaching mathematics. This introduces another figure besides the mathematician: the professor. Student behavior is circumscribed by limits imposed by the professor and the scientific tradition of mathematics. It is also framed by the productive modes of mathematical scientists and professors, which may reflect the former's productive mode. Drawing inspiration from the work of mathematical scientists and using the professor's didactic and mathematical reasoning as a model may stifle students' ability to engage in investigative work and to understand mathematics as not simply given.

In this discussion, Ponte et al. (1999) state that mathematical ideas in Mathematical Inquiry arise from students' attempts to understand and experience them. However, as previously mentioned, the work of teachers and mathematical scientists serves as a model for students. It seems, then, that students merely replicate previous experiments in line with the principle of reproducibility. This points to the experiment's objectivity, which, in turn, ceases to be authentic and becomes a "controlled or directed experiment" (Abbagnano, 2012, p. 479). We see a certain objectivity in this experience, which is "guaranteed by the fact that these experiences could be repeated by anyone" (Gadamer, 1999, p. 513). This frees the experience from the person who had it and the historicity of the experience itself, in the name of the epistemological schematization of the mathematician's or professor's mathematical practice.

Some considerations

The assertion that inspired this hermeneutic text is: "Mathematical Inquiry in Mathematics Education" as a way of doing mathematics. While this statement is not new, when viewed through the lens of the question, "What is Mathematical Inquiry in Mathematics Education?" the reflections that emerge focus on how this practice manifests and the mathematics it produces. In other words, the focus is on the nature and characteristics of the mathematics that interests Mathematical Inquiry. Saying that it is a way presupposes other possibilities and the existence of other ways. The different interpretations of the expression "doing mathematics" imply different understandings.

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However, our understanding of how to do mathematics, as expressed in mathematical research within the context of mathematics education, aligns with how mathematical scientists do mathematics. Therefore, this practice values the set of processes characteristic of knowledge production in mathematics—conjectures, tests, generalizations, and proofs—as its *modus operandi*. While this way of understanding the practice of mathematics inspires teaching practice, serving as its focus, objective, and guide, the teaching of mathematics advances along the same paths. This suggests that mathematical inquiry in mathematics education, as expressed through the teaching of mathematics, is ontologically based on conjecture, testing, generalization, and demonstration.

Thus, we understand that mathematical inquiry, as expressed through the teaching of mathematics, invites students to engage in the mathematical practices of mathematicians. It is important to note, however, that in this practice, the goal (demonstration) and the method (conjecturing, testing, generalizing, and demonstrating) are predetermined. From an ontological perspective, this orientation is problematic because "it does not derive the mode of being of entities from themselves" (Machado & Silva, 2017, p. 40), but rather from a rationality that is exclusively in the method.

In the context of mathematics education, the processes of constructing mathematical knowledge through scientific practice are not the only important factors. Doing mathematics in the classroom requires multifaceted activities that embrace not only mathematical language but also other languages, as well as ways of thinking, validating, and communicating knowledge that are authentically the students' own. This does not mean excluding the modes of production of mathematics as a science when engaging with mathematical research in mathematics education. Rather, it means considering them from the school context as a practice of possibilities, in which students produce mathematics with their own hands and in their own authentic ways.

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