

<http://dx.doi.org/10.23925/1983-3156.2026v28e74120>

The arithmetic thinking of 4th grade children about two addition and subtraction word problems: what do their responses reveal?

El pensamiento aritmético de los niños de cuarto grado al enfrentarse a dos problemas de suma y resta: ¿qué revelan sus registros?

Le raisonnement arithmétique des élèves de CM1 confrontés à deux problèmes d'addition et de soustraction : que révèlent leurs calculs ?

O pensamento aritmético de crianças de 4º ano ante dois problemas de adição e subtração: o que revelam seus registros?

João Paulo Machado Godoy¹
Universidade Federal de Goiás - UFG
Mestre em Educação

<https://orcid.org/0009-0008-9205-6074>

Riani de Souza Alvaro²
Universidade de São Paulo - USP
Graduanda em Matemática

<https://orcid.org/0009-0006-8867-5791>

Raquel Milani³
Universidade de São Paulo - USP
Doutora em Educação Matemática

<https://orcid.org/0000-0002-2015-7641>

Abstract

This research investigates children's responses to two addition and subtraction problems. Based on Critical Mathematics Education (CME), which understands mathematical learning as action, that is, a behavior marked by the subject's intentionality, we investigated the responses of 60 fourth-grade students from a public school in São Paulo in a diagnostic assessment activity. The objective was to understand the plurality of responses presented by the students, revealing their mathematical action

¹ joao_godoy@ufg.br

² riani.alvaro@usp.br

³ rmilani@usp.br

(much more than the mere correct and obedient application of algorithmic rules). The data show that there were conventional algorithms and alternative procedures, with a preference for the latter. It was possible to analyze them due to notations left on paper, the nature of the arithmetic thinking revealed in the responses, and the presence or absence of non-numerical support. Among the diversity found, we highlight the approaches related to counting by 1s or 10s, decimal or non-decimal numerical composition or decomposition, and altering the values involved to facilitate the operations to be performed. These non-algorithmic ways reveal a flexible and creative way of working with addends, minuends, and subtrahends, mobilizing knowledge about numbers, the number system, and the properties and regularities of operations. Unlike conventional algorithms, they make action-based learning evident, open up landscapes of investigation grounded in pure mathematics, and reaffirm, for the educator, the defense of the dialogical task of going where the other is.

Keywords: Critical mathematics education, Arithmetic, Elementary School, Algorithm, Addition/subtraction.

Resumen

Esta investigación analiza las respuestas de niños y niñas ante dos problemas de suma y resta. Desde la perspectiva de la Educación Matemática Crítica (EMC), que entiende las matemáticas como acción, es decir, un comportamiento marcado por la intencionalidad del sujeto, se investigaron las respuestas de 60 estudiantes de cuarto grado de una escuela pública de São Paulo, en una actividad de evaluación diagnóstica. Se buscó comprender la pluralidad de enfoques presentados, revelando su acción matemática (que va mucho más allá de la mera aplicación correcta y obediente de reglas algorítmicas). Los datos muestran la presencia de algoritmos convencionales y de enfoques alternativos, con una preferencia por estos últimos. Fue posible analizarlos a partir de los registros escritos, de la naturaleza del pensamiento aritmético manifestado en la respuesta y de la presencia o ausencia de apoyo no numérico. Entre los diversos

enfoques encontrados, destacamos aquellos que se traducen en el uso del conteo uno en uno o de diez en diez, la composición o descomposición numérica, decimal o no decimal, y la alteración de los valores involucrados para facilitar las operaciones. Estos enfoques no solo proporcionaron a los niños una mayor tasa de éxito, sino que también revelaron una manera flexible y creativa de trabajar con sumandos, minuendos y sustraendos, movilizándolo el conocimiento sobre los números, el sistema numérico y las propiedades y regularidades de las operaciones. A diferencia de los algoritmos convencionales, hacen evidente el aprendizaje basado en la acción, abren escenarios de investigación basados en la matemática pura y reafirman, para el educador, la defensa de la tarea dialógica de ir a donde está el otro.

Palabras clave: Educación matemática crítica, Aritmética, Escuela primaria, Algoritmo, Suma/resta

Résumé

Cette recherche examine les réponses d'enfants à deux problèmes d'addition et de soustraction. S'appuyant sur l'approche critique de l'éducation mathématique (ACE), qui conçoit les mathématiques comme une action, c'est-à-dire un comportement marqué par l'intentionnalité du sujet, nous avons étudié les réponses de 60 élèves de CM1 d'une école publique de São Paulo, lors d'une évaluation diagnostique. L'objectif était de comprendre la pluralité des approches présentées, révélant leur démarche mathématique (bien plus qu'une simple application correcte et obéissante de règles algorithmiques). Les données montrent l'existence d'algorithmes conventionnels et d'approches alternatives, avec une préférence pour ces dernières. L'analyse a été possible grâce aux notes écrites, à la nature du raisonnement arithmétique révélé par la réponse et à la présence ou l'absence de supports non numériques. Parmi la diversité observée, nous soulignons les approches qui se traduisent par l'utilisation du comptage de 1 en 1 ou de 10 en 10, par la composition ou la décomposition numérique décimale ou non décimale, et par la modification

des valeurs impliquées pour faciliter les opérations. Ces approches ont non seulement permis aux enfants d'obtenir un meilleur taux de réussite, mais révèlent aussi une manière flexible et créative de manipuler les termes, les soustractions et les additions, en mobilisant les connaissances sur les nombres, le système numérique, les propriétés et les régularités des opérations. Contrairement aux algorithmes conventionnels, ils rendent évident l'apprentissage par l'action, ouvrent des scénarios d'investigation basés sur les mathématiques pures et réaffirment, pour l'éducateur, la pertinence de la tâche dialogique consistant à aller là où se trouve l'autre.

Mots-clés : Enseignement des mathématiques critiques, Arithmétique, Enseignement primaire, Algorithme, Addition/soustraction.

Resumo

A presente pesquisa investiga as produções das crianças ante dois problemas de adição e subtração. Partindo da Educação Matemática Crítica (EMC), que compreende a aprendizagem matemática enquanto ação, ou seja, um comportamento marcado pela intencionalidade do sujeito, investigamos as respostas de 60 crianças do 4º ano escolar de uma escola pública de São Paulo, em uma atividade de avaliação diagnóstica. O objetivo foi compreender a pluralidade de caminhos por elas apresentados, reveladores da sua ação matemática (muito mais que o mero aplicar correto e obediente de regras algorítmicas). Os dados mostram que houve algoritmos convencionais e caminhos alternativos, com preferência por estes, tendo sido possível analisá-los a partir dos registros deixados no papel, da natureza do pensamento aritmético revelado na resposta e da presença ou não de apoio não-numérico. Dentre a diversidade encontrada, destacamos os caminhos que se traduzem na utilização da contagem de 1 em 1 ou 10 em 10, na composição ou decomposição numérica decimal ou não decimal e na alteração de valores em jogo para facilitar as operações a serem realizadas. Tais caminhos são reveladores de uma forma flexível e criativa de trabalhar com as parcelas, minuendos e subtraendos em jogo, mobilizando conhecimentos sobre os números, o sistema de numeração e

as propriedades e regularidades das operações. Diferentemente dos algoritmos convencionais, eles tornam patente a aprendizagem baseada na ação dos sujeitos, abrem cenários para investigação fulcrados na matemática pura e reafirmam, ao educador, a defesa da tarefa dialógica de ir até onde o outro está.

Palavras-chave: Educação Matemática Crítica, Aritmética, Anos iniciais do ensino fundamental, Algoritmo, Adição/subtração.

The arithmetic thinking of 4th grade children about two addition and subtraction word problems: what do their responses reveal?

Introduction

Several studies in the field of Mathematics Education use children's and teenagers' work on mathematical word problems to extract, for example, types of situations and difficulty levels of the questions, children's strategies, conceptual invariants mobilized, underlying action schemes, types of symbolic representations chosen, errors made and their meanings, among others (e.g., Gitirana et al, 2014; Nunes et al, 2009; Moro, Soares, 2005; Zunino, 1995; Kamii 1995, 2000). The marks and notations left by students on paper reveal a lot about how they deal with, interpret, and mathematically solve the problems presented to them.

When evaluating children's responses, the teacher usually seeks to determine whether the responses are right or wrong, aiming to assign a grade to the student. Confusing marks, poorly erased attempts on paper, numbers that, at first glance, don't make sense, are often discarded by the educator (or considered partially correct). This follows bureaucratically (Alrø; Skovsmose, 2021), without analyzing them in greater depth, since, often, due to the high demand for activities to correct and the fulfillment of other pedagogical tasks, or even based on non-critical pedagogical understandings, they do not reflect on the children's responses, in their particularities, seeking to make sense of their movements, even if they are erroneous. The result is, unfortunately, the discarding of the subjects themselves, their invalidation as mathematical beings in the world (Freire, 1996).

This research aims to investigate children's responses to two mathematical problems, seeking to understand the plurality of procedures they present, revealing their mathematical action, which goes far beyond the mere correct and obedient application of algorithmic rules in the symptomatic search for the right answer. We believe that its results can

contribute to the difficult (but imperatively ethical) task of each of us, mathematics educators: to go where the other is (Milani, 2017; Lins, 1999).

Theoretical basis

This research is based on Critical Mathematics Education (CME), a perspective to which we relate to, that is concerned with how mathematics in general influences society and, therefore, seeks not only to reflect on more efficient ways of learning mathematical concepts, but also on how mathematics education can develop citizenship and empower individuals (Alrø; Skovsmose, 2021, p. 17). Philosophically, we understand that there is a “mathematical way of being in the world” (Freire, 1996, our translation), configuring each of us as legitimate subjects who produce mathematics.

CME reflects on the exercise paradigm, which places mathematics teaching within a learning perspective based on the strenuous repetition of models, ultimately undermining the potential of individuals to use mathematics to interpret and create a more politically just and egalitarian society. A counterpoint to this way of thinking about mathematics teaching would be the insertion of landscapes of investigation in the classroom, valuing dialogical acts in the classroom and a multiplicity of procedures to problem solving, allowing students to construct genuine discoveries (Skovsmose, 2015).

Therefore, CME understands learning as action, which presupposes that the learner takes an active part in the learning process, with a mixture of desire/intention and openness/possibilities. Unlike other possible forms of learning, such as the exercises that soldiers do when learning to march (which certainly result in certain learning outcomes), learning as action is marked by the free and conscious attitude of the subjects (Alrø; Skovsmose, 2021), and, so, the defense of the use of landscapes of investigation in the classroom:

Investigating and exploring are conscious acts; they do not happen as forced activities. They do not take place until students actually carry out the investigations and explorations, and for this to occur, it

is assumed that the students' intentionality is part of the investigative process. (Skovsmose, 2015, p. 69, our translation)

Freire's epistemological understanding dialogues with CME insofar as, for Freire, "under the conditions of true learning, learners are transformed into real subjects of the construction and reconstruction of the knowledge taught [...]" (Freire, 2015, p. 28, our translation). By differentiating the banking concept of education from problem-posing education, Freire states that the cognoscent act is characterized by being creative, transformative, unsettling, inventive, impatient, permanent, practiced by the subject and not deposited in them.

A landscape of investigation can occur in problems based on reality (real contexts), semi-reality (artificial contexts), and pure mathematics (decontextualized problems). In this article, we are primarily interested in landscapes based on pure mathematics, which are those in which the processes of exploration and investigation occur solely on the regularities, properties, connections, relationships, and generalizations of numbers, operations, functions, and other mathematical "entities," allowing the student to make mathematical discoveries that will not be new within the body of mathematical knowledge, but will be genuine for the learner who produces them (Skovsmose, 2015).

The children who collaborated in the research were students of the first two authors (the first was the teacher and the second was an intern in the class) and, since the beginning of the school year, had been participating in a pedagogical proposal that involved teaching the additive structures with an emphasis on a variety of procedures to problem-solving, without the requirement of knowing and applying the conventional algorithm. We have brought here the analysis and discussion of their responses to two mathematical problems, which, in our view, are marked by a plurality that is only possible due to the aforementioned practice.

To assist us in interpreting the data, we also include contributions from some authors dedicated to the study of basic arithmetic teaching,

especially regarding the appreciation of alternative procedures to the conventional algorithm for solving problems.

Maria do Carmo Domite, for example, advocates the exploration of mental calculation in school, arguing that it allows the appreciation of spontaneous knowledge acquired outside of school, as well as contributing to a better understanding of the properties of numbers and the rules of the numbering system, giving children greater autonomy and developing their intuitive thinking (Domite Mendonça; Lellis, 1989; Conceição, 2021).

Constance Kamii and her partners discovered that the conventional algorithm requires children to understand the number system, which they have not yet mastered, leading to false arithmetic learning insofar as some algorithmic rules, such as "carry one" in addition or "borrow" in subtraction, are memorized by students but do not make sense to them. Therefore, they identify numerous advantages in the procedures constructed by the children themselves, valuing them in the school context (among others: Kamii; Housman, 2002).

Delia Lerner and colleagues, in the same vein, argue that conventional algorithms cause children to lose sight of the number being operated on, as they require addition or subtraction in columns, which does not occur when they use numerical decomposition procedures to solve problems. Furthermore, decomposition algorithms allow children to perform operations from left to right and also have better control of the result (Lerner; Sadovsky; Wolman, 1996; Zunino, 1995).

Terezinha Nunes and colleagues, while investigating the arithmetic practiced by children and adults in the streets and in the school, realized that there were numerous math problems that were solved satisfactorily by them in contexts such as buying and selling at open markets, but when in a school context, they did not obtain the correct results for the same problems. They thus identified a gap between school mathematics and street mathematics, suggesting the creation of bridges between formal knowledge and the practical knowledge that children already possess (Nunes; Carraher; Schliemann, 2011).

Sandra Magina, Tânia Campos, Verônica Gitirana, Alina Spinillo, among others, have contributed significantly to studies and research on mathematics teaching in the early years of schooling, notably based on Gérard Vergnaud's Theory of Conceptual Fields. Their work helps to reflect on mathematics teaching, bringing us closer to what children think and the mental structures they mobilize (Magina; Campos; Nunes; Gitirana, 2008; Nunes; Campos; Magina; Bryant, 2009; among others).

Regina Damm (2003), when examining how children thought about additive problems, found that it was necessary for students not only to select the relevant data, but also to organize the data and the operation, which requires them to represent it appropriately. They discovered that the difficulties in solving these problems were mainly due to understanding the statement, temporal relationships, and verbs carrying numerical information, noting that an appropriate representation must mark the different stages of the situation described in the statement (before, after, first, second, etc.) and the operational data according to the situation (the numbers and their increase/decrease).

All these studies, although based on different paradigms, such as Freirean popular pedagogy (in the case of Domite), classical psychogenetic perspectives such as Piaget's (in the case of Kamii), or mixed perspectives, also encompassing Rousseauian didactic contributions (in the case of Lerner) or cultural contributions (the paradigmatic "Street Mathematics and School Mathematics"), and even authors who draw on the contributions of The Theory of Conceptual Fields (Magina et al.) or The Registers of Semiotic Representations (Damm), converge in placing the subject and their genuine arithmetic thinking at the center, to the detriment of possible mechanical rules that they may or may not have memorized to solve a given problem.

Our study builds upon the contributions of these and other authors, confirming their findings at various points and expanding the discussions concerning the treatment of additive problems by establishing, as we will present in the discussions, three aspects to be considered in the analysis of mathematical problems: 1) the amount of notations left on the paper; 2)

the arithmetic strategy mobilized; 3) non-numerical support. Regarding the second aspect, we identified types of strategies for solving addition and subtraction problems when conventional algorithms are not involved, namely: a) those based on counting one by one and ten by ten; b) those that use decimal and non-decimal composition and decomposition to operate the terms; c) and those that modify the values involved to facilitate operations. This understanding suggests that, in order to understand and evaluate children who think mathematically about an additive problem, it may be important for educators to consider these processes. Theoretically, we also aim to innovate by bringing CME contributions to the analysis of responses to mathematical problems based on alternative procedures to conventional algorithms for addition and subtraction, because 1) they highlight learning based on the subjects' actions, 2) they enable the creation of landscapes of investigation based on pure mathematics and the consequent possibility of genuine discoveries, and 3) they reaffirm the defense of the dialogical task of “going where the other is”.

Methodology

This is a predominantly qualitative study in which we investigated the responses of 60 children⁴ from two 4th-grade classes at a public school located in the Butantã neighborhood of São Paulo to two math questions posed by the class teacher, answered in August 2024 without adult assistance. The first question analyzed (problem 1) asked the following: Breno has a collection of 46 miniature cars. On his birthday, he received 7 more cars. How many cars does Breno need to have 101 cars in his collection? The second question analyzed (problem 2) asked the following: To measure the 160 cm length of the teacher's desk, Amanda used a 30 cm ruler and a 55 cm piece of string. When lining up the two instruments, how many centimeters of the desk's length will be left to measure?

⁴ Upon enrollment, families sign a general authorization for this type of action, given the *unique* nature of this school, in which teaching, research, and extension converge in Basic Education. Even so, specific “Free and Informed Consent Forms” (Termo de Consentimento Livre e Esclarecido) were collected for this research.

Our research presents the following guiding questions: what mathematical procedures do children reveal in their answers? What do they say about their mathematical actions? We analyzed the answers one by one, extracting quantitative elements (number of children who thought of the same strategies, number of correct and incorrect answers, etc.) and, above all, qualitative elements, understanding the nature of the strategies used to solve the problems and how they could be grouped/categorized⁵.

According to Lüdke and André (1986), in documentary analysis, the researcher analyzes the collected documents, seeking the symbolic content of the messages. This is done using units of analysis, which can be words, sentences, paragraphs, or texts as a whole, which are highlighted by the researcher, in some cases even quantitatively (number of times they appear). Once the units have been counted and the data organized, the researcher, in a process of reading and rereading, begins to detect the most frequent themes and topics, culminating in the construction of categories of analysis that will reflect the purposes of the research. In our case, the documents analyzed (answers to the questions) were thoroughly analyzed, one by one; this process was also carried out by two different researchers (the first two authors) at different times, so that when they met, they were able to compare the categories they had designed, reworking them together, and, in the end, it was also discussed with a third researcher (third author). For Lüdke and André,

[...] the construction of categories is not an easy task. They spring, at first, from the theoretical framework on which the research is based. This initial set of categories, however, will be modified throughout the study, in a dynamic process of constant confrontation between theory and empiricism, which gives rise to new concepts and,

⁵ Qualitative analysis predominates in the study; however, quantitatively, we computed the number of children who chose this or that mathematical procedure and the number of errors and correct answers in each of these procedures, which strengthens the qualitative analyses, as it corroborates their statements (for example, in the additive structures, the complementation procedure is qualitatively less complex than that of subtraction; this qualitative analysis is corroborated quantitatively by the greater number of errors caused by the former compared to the latter). However, the quantitative data were not subjected to statistical tests and, therefore, the findings of this study are not based exclusively on them.

consequently, new areas of interest (Lüdke; André, 1986, p. 42, translated from portuguese).

The process of reworking culminated in categories that took into account the amount of notations left on paper, the nature of arithmetic thinking, and the presence of non-numerical support. The process had its ups and downs; in some cases, a given category was divided into two (as is the case, for example, with the procedure of decomposition, divided into decimal and non-decimal decomposition, with support in base-ten blocks); in other cases, we reformulated some categories, due to their similarity to others already defined (this was the case, for example, with the numerical composition procedure in cases of subtraction via complementation, which correlate with the decomposition strategy in the addition situations analyzed). In analyzing the nature of arithmetic thinking, our theoretical references were important in helping us with categorization, but we believe that our results are innovative in that we did not find the detailed information presented here in the literature.

For didactic purposes, we divided the presentation of the analysis into two groups: first, we will analyze and discuss the answers to the first part of each of the two questions (addition situations) and then the answers to the second part (subtraction situations). For the purposes of categorizing procedures, we excluded inconclusive, blank, or interpretive error responses⁶ from the universe investigated (14 to 18% of the total).

Adding 46 and 7

The first situation, **“Breno has a collection of 46 miniature cars. On his birthday, he received 7 more cars. [...]”**, is a prototypical situation of positive transformation (Magina et al., 2008), in this case, the value of 46 cars (initial state) plus 7 cars (value of the transformation), how

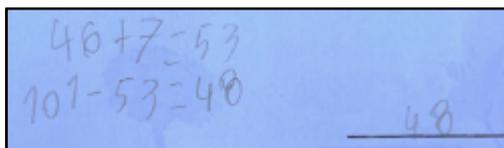
⁶ Blank or inconclusive answers made our analysis impossible, due to the very small number or absence of marks on the paper that would allow us to interpret the mathematical action of the child under investigation. We also excluded some with interpretative errors when these errors did not follow a logical line that we could understand (the child, for example, used numbers or operations out of context).

much does it add up to (unknown final state)? The final answer is given by applying the joining scheme (Nunes et al, 2009): $46 + 7 = 53$.

Analyzing the procedures chosen, it is possible to note that 72.9% of the children opted for alternative procedures to the conventional algorithm. Among these procedures, most children simply wrote the numerical expression horizontally with the final answer ($46 + 7 = 53$), or even wrote 53 directly on the paper, as part of their reasoning or already inserting the number into the operation of the second situation. We call the first procedure **mental calculation with single expression** and the second **mental calculation without notation**.

Figure 1

Example of mental calculation with single expression for solving the first situation (Research Data, 2024)



Handwritten mathematical work on a blue background. The first line shows the addition $46 + 7 = 53$. The second line shows the subtraction $101 - 53 = 48$. The number 48 is underlined.

Figure 2

Example of mental calculation without notations for the first situation (the value 53 was simply placed on paper, composing the calculation for the second situation) (Research Data, 2024)

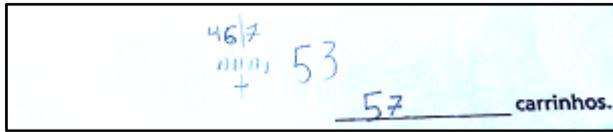


Handwritten mathematical work on a light blue background. The first line shows the subtraction $101 - 53$. The number 48 is written below it.

There was one child, G.L., who made an interesting notation of counting one by one using tally marks, which suggests that they solved the sum by counting one by one from 46 to 53, that is, $46 + 7 = 46 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 53$. We will call this method of **counting one by one with the aid of tally marks**:

Figure 3

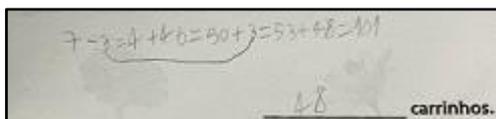
G.L.'s response (Research Data, 2024)



There was also the case of a child, T.A., who, to facilitate the sum, chose to remove 3 from 7 before adding it to 46, and then adding 3 (indicating with an explanatory arc where the added 3 was “coming from”). This is very interesting, as it reveals the child's mastery of numerical relationships (Kamii; Housman, 2002), understanding that the 7 they was working with could be turned into $4 + 3$ to facilitate operations, since doing $46 + 4$ and then $+ 3$ is easier than $46 + 7$ directly. In decomposition heuristics, according to Nunes, this is a non-decimal decomposition (decomposition in which the number is broken down into smaller numbers, not necessarily into tens and units) that seeks to be used to work with round numbers and avoid mental overload (Nunes; Carraher; Schliemann, 2011). We will call this procedure **non-decimal decomposition**:

Figure 4

T.A.'s response (Research Data, 2024)

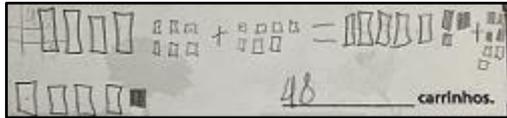


Two children, whom we will call N.S. and E.A., wrote $46 + 7$ using drawings of the base-ten blocks (4 rods and 6 units arranged horizontally, a $+$ sign in sequence, 7 units next, an $=$ sign, and the value 53 in base-ten blocks). The base-ten blocks facilitates the child's reasoning of decomposition into tens and units, which allowed them, concretely, to transform 46 into $10 + 10 + 10 + 10 + 1 + 1 + 1 + 1 + 1 + 1$, and, when adding 7 ($1 + 1 + 1 + 1 + 1 + 1 + 1$), enabled the child (who already knew how to use it) to join units together to form a ten and continue counting,

i.e., $10 + 10 + 10 + 10 + (6 + 4) + 3$. We will call this method of **decomposition using base-ten blocks**:

Figure 5

N.S.'s response (Research Data, 2024)



Thus, there was a multiplicity of non-algorithmic procedures to solving the prototype situation of positive transformation. The use of mental calculation by most children is probably due to one of these factors: 1) the child realized that the numbers were small and chose the alternative procedure because of the ease of operating them mentally and/or 2) there were two situations, the second more difficult than the first, therefore, “there would be no problem” in doing the first one mentally, since there would be some explanatory notation when solving the second situation.

On the other hand, in 27.1% of the responses, **the conventional addition algorithm** appeared for the solution of $46 + 7$. Among them, most performed the algorithm in the traditional format. One child chose to also use the place value chart (PVC).

Figure 6

Child using conventional algorithm (Research Data, 2024)

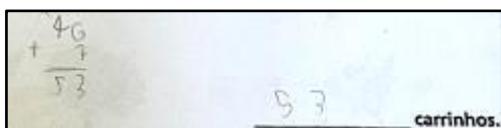
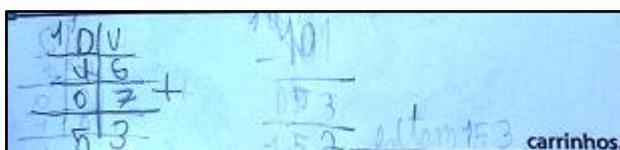


Figure 7

Child opts for the algorithm within the PLT (Research Data, 2024)



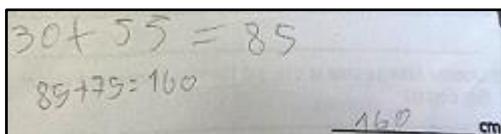
Why is it that, even for an account with small numbers, we still have 27.1% of fourth-grade children understanding that it is necessary to write the conventional algorithm? We have two explanations: 1) it is possible that they still do not understand that alternative procedures is legitimate for solving a problem on paper (legitimacy for the teacher), in which case, even if they know how to operate more quickly using mental arithmetic; 2) another hypothesis is that the use of the algorithm may give them more confidence in having the correct answer (legitimacy for themselves), even in relatively simple operations, that is, they do not trust their mental calculation to solve the problem. The algorithm in the QVL seems to us to be a symptom of the exercise paradigm (Skovsmose, 2015), the result of the flood of addition exercises using the conventional algorithm that children have been subjected to throughout their schooling.

Adding 30 and 55

The additive situation in problem 2 was: **“To measure the 160 cm length of the teacher’s desk, Amanda used a 30 cm ruler and a 55 cm piece of string. By lining up the two instruments [...]”**. This is a prototypical composition situation (Magina et al., 2008), in which the table is already being measured by the 30 cm ruler and the 55 cm string, and the goal is to find out how much of the table has already been measured: this is solved by joining (Nunes et al., 2009) $30 + 55$. When analyzing the chosen strategies, it is noted that 73.7% of the responses opted for alternative procedures to the algorithm. As in Problem 1, most children solved the first situation by **mental calculation without notation** and by **mental calculation with a single expression**. Here are some examples:

Figure 8

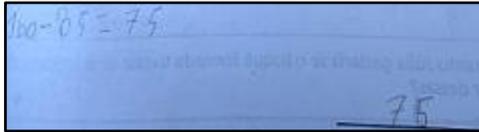
Example of mental calculation with single expression (Research Data, 2024)


$$30 + 55 = 85$$
$$85 + 75 = 160$$

160 cm

Figure 9

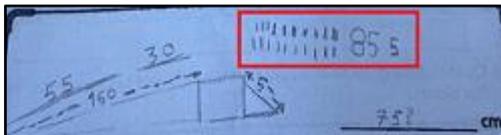
Example of mental calculation without notation for the first situation (note that the value 85 was written in the solution to the second situation, without writing how the value was calculated) (Research Data, 2024)



Furthermore, child G.L. again solved this situation by **counting one by one with the help of tally marks**. In their solution, there are two rows with 10 marks each, drawn in the upper right corner. Apparently, the child started from 55 and, to add 30, added 3 times 10, counting one by one. Thus, in the first row of 10, they reached 65 ($55 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$), in the second row, they reached 75 ($65 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$) and probably realized that if they added another 10, it would be 85, so they saw no need to draw another 10 marks:

Figure 10

G.L.'s response (Research Data, 2024)



There was also child D.A., who chose to solve the first situation using **decimal decomposition**. The child decomposed 55 into $50 + 5$, choosing to first add the tens, i.e., 30 (ruler) $+ 50 = 80$, and then adding the remaining units: $80 + 5 = 85$. Unlike non-decimal decomposition, present in Problem 1, in decimal decomposition the child necessarily decomposes the number based on orders, that is, separating units, tens, and hundreds (Nunes; Carraher; Schliemann, 2011). However, as with non-decimal decomposition, this is a method that allows numbers to be operated on by taking advantage of known numerical regularities, such as the sum of round numbers:

Figure 11

D.A.'s response (Research Data, 2024)

$30 + 55 = 85$ $160 - 80 = 80 + 5 = 85$
85

Child S.M. solved the **problem using the base-ten blocks with written form in Portuguese** (it was not the same child who opted for the base-ten blocks in the previous problem). As explained by the child, they separated the values 30 and 55 in the base-ten blocks and then added them together. Note the child's care in explaining to the educator, in Portuguese, how they thought:

Figure 12

S.M.'s response (Research Data, 2024)

Em uso o material concreto
Separei 100, 30 e 55 com 55 + 30
e depois o resultado com 160
75

It can be seen, therefore, that, as in Problem 1, there are a multitude of non-algorithmic ways to solve the first situation. Once again, there is a preference for mental calculation, which, in our view, is due to the same factors mentioned above. On the other hand, 26.3% of the children chose to use the **conventional addition algorithm** to solve $30 + 55$, as in the example below:

Figure 13.

L.N.'s response (Research Data, 2024)

$$\begin{array}{r} 30 + 55 \\ + 55 \\ \hline 85 \end{array}$$

$$\begin{array}{r} 100 \\ - 25 \\ \hline 75 \end{array}$$
 75

The choice of the algorithm to solve $30 + 55$ is, in our view, due to the hypotheses previously presented for the additive situation in the first question. Because there are two digits for each of the parts ($30 + 55$), one might think that many children would make the "safer" choice of using the

algorithm, which did not happen (in fact, the percentage of children who chose the algorithm here is slightly lower than in problem 1).

Errors in additive situations

Let us now analyze the errors in the additive situations. In the alternative procedures chosen, there were two errors in the additive situation in problem 1 and two errors in the additive situation in problem 2. These errors represent 5.8% and 7.6% of the total number of children who chose the alternative procedures in each group. Regarding the errors made by children who used the conventional algorithm, there was one error in the additive situation in problem 1 and two errors in the additive situation in problem 2. These errors represent 7.6% and 20% of the total number of children who opted for the algorithm in each group. In this sense, there was no significant difference in the percentage of errors and correct answers for either problem 1; however, in problem 2, children were more successful using alternative procedures.

In the alternative procedures, the errors that occurred were the addition of one unit or ten more or less, that is, instead of resulting in 53 for $46 + 7$, the children reached 52 and 54 when doing the mental calculation (additive situation in problem 1), and when adding $30 + 55$ (additive situation in problem 2), they reached 75 and 95 instead of 85:

Figure 14

R.Z.'s response (Research Data, 2024)



Figure 15

J.V.'s response (Research Data, 2024)

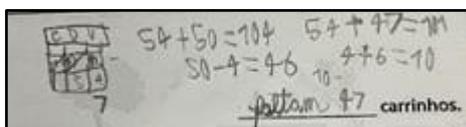
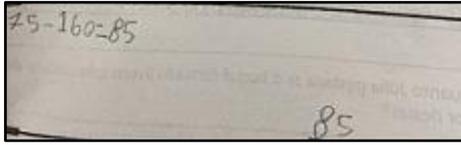


Figure 16

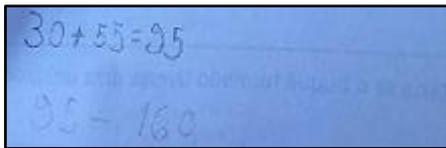
K.M.'s response (Research Data, 2024)



45-160=85
85

Figure 17

V.C.'s response (Research Data, 2024)



30+55=85
85-160

Why did these children get it wrong? We have a few hypotheses. First, we note that they did **the calculations mentally without writing anything down** (except for the last one, where the expression is written down), which requires more mental reasoning. In problem 1, the sum is of two numbers whose result is between the "knots" of the tens, which sometimes does not allow for an intuitive idea of the final answer, since the turn at the knot requires more complex reasoning of the Decimal Numbering System than the numbers within the knot interval (Lerner; Sadovsky; Wolman, 1996): when counting one by one, 47, 48, 49, thinking about what the next number is (in this case, 50), and then going back to counting one by one, 51, 52, 53, can be more demanding for the child, who may eventually make a mistake due to mental overload in the process. For Nunes, Carraher, and Schliemann (2011), the overload occurs in the mental processing of data due to the need to operate simultaneously on different decimal places (in this case, tens and units), which the decomposition strategy avoids, for example. The overload increases when some children resist using their fingers or other symbolic support for counting, doing everything in their heads (children often avoid using their fingers, either because of the need to be faster or because of the embarrassment of being observed by a classmate when raising their fingers). Another possibility for error is not counting from the successor during the counting process. We

know that Breno has 46, so when adding 7 more, it is necessary to count one by one from 47, that is, we would have 47 (1), 48 (2), 49 (3), 50 (4), 51 (5), 52 (6), and 53 (7). If the child does not have this idea consolidated, the sum may be done considering 46 as the first term, which would generate 46 (1), 47 (2), 48 (3), 49 (4), 50 (5), 51 (6), and 52 (7). In the situation of problem 2, a similar analysis can be made: counting in tens is particularly demanding from 50 onwards, due to the confusion between the oral and written expressions of the following numbers (especially 60 and 70), which sometimes causes confusion among children; furthermore, in the case of child K.M., when adding 3 “tens” (30), they may have started counting from 55 itself, that is, instead of starting from 55 and doing 65 (10), 75 (20), 85 (30), they thought of 55 (10), 65 (20), 75 (30).

Using conventional algorithms, in problem 1, child P.R. made a mistake, reaching what appears to be the number 13 or 123; in problem 2, the children reached 65 and 95:

Figure 18

P.R.’s response (Research Data, 2024)

Handwritten work showing a vertical addition of 46 and 7, resulting in 13. The text "ele já tem + 12 carrinhos." is written next to it.

Figure 19

H.F.’s response (Research Data, 2024)

Handwritten work showing a vertical addition of 30 and 55, resulting in 85. The text "30 + 55 = 85" is written above the vertical addition.

Figure 20

K.S.’s response (Research Data, 2024)

Handwritten work showing a vertical addition of 65 and 30, resulting in 95. The text "65 + 30 = 95" is written above the vertical addition.

How could $46 + 7$ equal 13 or 123? The child seems confused by the algorithmic rules, but they may have realized the absurdity, since they erased some numbers, although they did not progress from there, since they did not solve the second part of the problem. In the other two cases, the errors are due to the mistake of counting 1 by 1, which can be explained by the same reasons as the errors in mental calculation.

Subtracting 53 from 101

In the first problem, the question is asked, "**How many cars are missing for Breno to have 101 cars in his collection?**" To solve this, the child needs to understand that, beginning with 53 cars (a value already discovered after calculating the first situation), in order for Breno to have 101 cars, it is necessary to calculate how much is missing to reach 101. This is a positive transformation situation, now non-prototypical, in which the initial and final values are given and a transformation is requested (Magina et al, 2008), since Breno now has 53 cars (initial value) and asks, in order to have 101 cars (final value), how many cars will they need to gain (transformation value)? Non-prototypical transformations are more cognitively demanding than prototypical ones, since they require the child to construct an operational relationship between the schemes of joining and separating, that is, to understand the opposite relationship between the two (Nunes et al., 2009). This is made explicit by the use of the word "gain," that is, a situation of gain, which, however, involves a subtraction: $101 - 53$. It is, therefore, a counterintuitive subtraction problem. For this reason, the answer will often be given by applying the complementation strategy instead of subtraction (in this case, $53 + ? = 101$), that is, they add several other carts to the 53 carts until they reach 101 carts, thus maintaining the application of the joining scheme rather than the separating scheme (preserving the additive idea that is more intuitive in this case).

In fact, 43.9% of the children opted for complementation and 56.1% for subtraction. As this is a fourth-grade class, most of them already understand this inverse relationship, although it is noteworthy that many

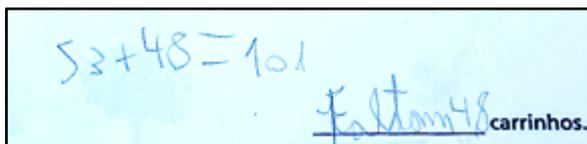
(almost half) preferred to use complementation. It is also interesting to note that, of those who opted for the addition reasoning, 83.3% were correct, while of those who used subtraction, only 65.2% were correct, which indicates greater complexity/difficulty in subtractive reasoning.

We also analyzed whether the children opted for the conventional algorithm or sought alternative ways to solve the situation: we found that 65.9% used alternative methods and 34.1% used the conventional algorithm. These percentages are similar to those in the first situation, which indicates a persistent preference for alternative methods to the algorithm. Let us now examine these strategies in detail, seeking to construct categories that help us to interpret the results qualitatively, as we did for the situations involving sums.

Regarding the children who opted for supplementation, some children simply wrote the expression $53 + 48 = 101$, without making it very clear how they calculated this value mentally. They therefore did so by **mental calculation with a single expression**. For example:

Figure 21

The child did the 48 complementation in their head, inserting it into the expression (Research Data, 2024)

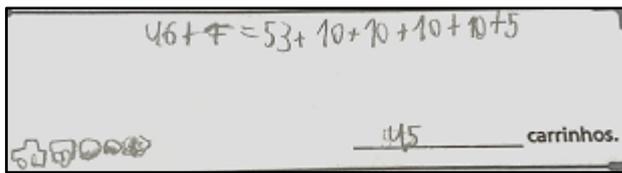


A photograph of a child's handwritten work on a light blue background. The child has written the equation $53 + 48 = 101$ in blue ink. Below the equation, there is a signature that reads "Faltanys carrinhos." The entire work is enclosed in a thin black rectangular border.

We also have two children who added 10 at a time and then added units until they reached 101. Child A.C. wrote the tens, while child M.P. wrote the partial results (63 - 73 - 83 - 93). Note that, upon reaching 93, the child mentally anticipates that adding another ten will exceed the final value, so they reflect and add only 8. We will call this type of strategy complementation by **counting/adding 10 by 10**, to draw a parallel with counting 1 by 1 (with the help of tally marks) that we see in additive situations:

Figure 22

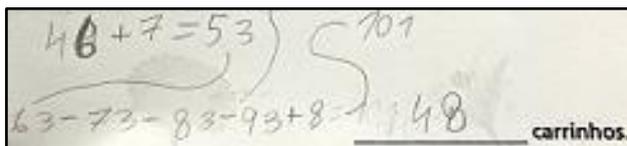
A.C.'s response (Research Data, 2024)



Handwritten mathematical work by A.C. showing the equation $46 + 7 = 53 + 10 + 10 + 10 + 5$ and the result 45 carrinhos.

Figure 23

M.P.'s response (Research Data, 2024)



Handwritten mathematical work by M.P. showing the equation $46 + 7 = 53$, a list of numbers $63-73-83-93+8$, and the result 48 carrinhos.

Three children also used the strategy of complementing by partial sums, but without going from 10 to 10, preferring faster procedures. Child A.O. adds 8 first, reaching 61, then adds 40 to reach 101 directly, and then adds everything they added to find the final answer. Child D.F. also adds in two parts, but first adds 47 to 53 to reach 100, then adds 1, and finally joins the parts with an arc, identifying the result as 48. We will call these strategies complementation by **numerical composition**, inspired here by the definition given by Nunes, Carraher, and Schliemann (2011) when analyzing children's heuristics of composition (*chunking*), which consists of grouping portions until the total is obtained. We thus draw a parallel with the numerical decomposition defined above. Domite also identifies this strategy in their writings, stating that it is especially useful in situations where one needs to find out "how much is missing" and consists of taking "small jumps" until the total is reached (Mendonça; Lellis, 1989). Both composition and decomposition seek "good" numbers, taking advantage of known numerical regularities, such as operating with round numbers, rather than operating all at once:

Figure 24

A.O.'s response (Research Data, 2024)

Handwritten mathematical work showing the equation $53 + 8 = 61 + 40 = 101$ and the result 48 . The word "carrinhos." is written at the bottom right.

Figure 25

D.F.'s response (Research Data, 2024)

Handwritten mathematical work showing the equation $53 + 7 = 60 + 40 + 1 = 101$ and the result 48 . The word "carrinhos." is written at the bottom right.

Child N.S., below, also thinks in terms of numerical composition, but based on the base-ten blocks. After solving the first situation (already explained above), they added 7 to the result 53, giving 60 (they wrote the addition $53 + 7$ in the base-ten blocks, but not the result); then added 4 more rods of 10 (giving 100, but did not write 100, only the added rods) and a unit (reaching 101, but without writing the result, only the unit). In this way, they thought by complementation, having done $53 + 7 = 60 + 40 + 1 = 101$ with the base-ten blocks and then, in their head, $7 + 40 + 1 = 48$. We are naming this strategy complementation by **composition in the base-ten blocks**:

Figure 26

N.S.'s response (Research Data, 2024)

Handwritten mathematical work using base-ten blocks to represent the equation $53 + 48 = 101$ and the result 48 . The word "carrinhos." is written at the bottom right.

We also have another possible strategy for complementation, which is to start not from the presented values, but from others whose result is already more easily known, and to "correct" them until they can represent the previous amounts. This was the case for five children. Child E.M., for example, when thinking about $53 + ? = 101$, starts from $50 + 50 = 100$ to work. They write down $50 + 49 = 99$ and starts from there to test values mentally until they reach $53 + 48 = 101$, which they write down on paper

(they may have thought: if $50 + 50$ is 100, when I subtract 1 from one of the parts, the result is 99; when I subtract another 1, the result is 98; however, when I increase one in the first part at the same time, I go back to 99... and they kept changing the values until they reached $53 + 48 = 101$). The child J.V. is more explicit in what they thought: they start by using the subtraction algorithm strategy, but does not seem to be successful in its use, as the result is 7. Dissatisfied, the child tries the complementation reasoning, starting from 54 (they do not start from 53 because, when solving the first situation, they did it in their head, but made a mistake by 1). They decided to add 50 to 54, checks that it is 104, that is, they went over the value; then they take 4 from 50 so that the result is 100, checks that $50 - 4$ is 46, then adds $54 + 46$, which is 100 (he does $4 + 6$ separately below, which is 10, so they must have used this to mentally calculate $50 + 40$, which equals $90 + 10$, which equals 100), but they do not write 100, because they know that it should be 101, so they erase the 6 and writes 7 above it, and writes the result 101. In our view, this is a type of numerical composition strategy, but one in which the child completes the calculation with values altered by themselves to facilitate the operation; therefore, we will call this type of strategy a completion strategy by **altering the values**: **Figure 27**

E.M.'s response (Research Data, 2024)

Handwritten mathematical work by E.M. showing calculations: $46+7=53$, $53+48=101$, $50+50=100$, $50+49=99$, and 48 . The name "carrinhos." is written at the bottom right.

Figure 28

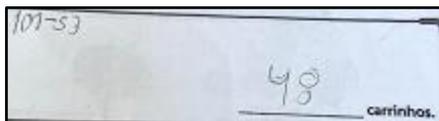
J.V.'s response (Research Data, 2024)

Handwritten mathematical work by J.V. showing a grid with numbers 1, 2, 3, 4, 5, 6, 7 and calculations: $54+50=104$, $54+47=101$, $50-4=46$, $4+6=10$, and 101 . The name "carrinhos." is written at the bottom right.

When analyzing the responses of children who chose the subtraction procedure, it is possible to make a similar categorization: first, it is possible to observe that the vast majority simply put the subtraction $101 - 53 = 48$, without making clear how they thought, as in the case with child B.A., therefore, **mental calculation based only on the expression**.

Figure 29

B.A.'s response (Research Data, 2024)



Other children use other strategies, as is the case with children D.A. and N.M., who used the strategy of **changing values**. Child D.A. thought: if it were $100 - 50$, it would be 50, but the subtrahend is 53 and not 50 (3 more), so you have to take away 3 more, leaving 47, but the minuend is greater (101 is 1 greater), so you have to add 1. It is worth highlighting the expression "if it were" used by the child, indicating to the reader the mental mobility with which they think about arithmetic. The child N.M., in turn, writes down the expression $101 - 53 = 48$, leaving on paper how they got there: they first did $100 - 50 = 50$. With the result, they do two things: first, they add 1 to the result (since the original value of the minuend was 101 and not 100), writing $50 + 1 = 51$, then they subtract 3 (since the original value of the subtrahend was 53 and not 50), writing $51 - 3 = 48$. The ability of these children to keep track of the parts that were subtracted and use the correct operation when returning them to the result is remarkable:

Figure 30

D.A.'s response (Research Data, 2024)

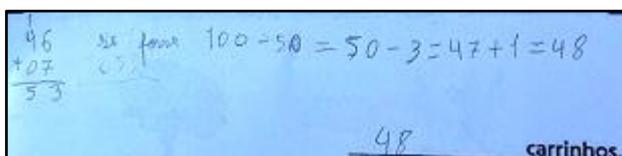
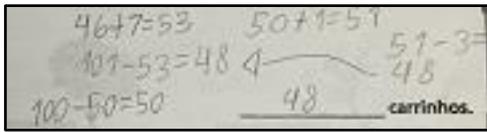


Figure 31

N.M.'s response (Research Data, 2024)



Finally, let us analyze the children who used the conventional algorithm. We found that they simply wrote down the calculation $101 - 53$ on paper and then applied the subsequent algorithmic rules. However, we highlight the case of two children, S.L. and L.R., who, although they used the algorithm, also used alternative methods. Child S.L. added 47 to 53 in the conventional algorithm, as this allowed them to reach 100. They even wrote down the answer "47 cars missing," but when they remembered that they needed to reach 101, they erased the 7 and put an 8 in its place, without, however, correcting the final answer. In the case of child L.R., the poorly erased marks reveal that there were several attempts at numbers in the algorithm until reaching 48:

Figure 32

S.L.'s response (Research Data, 2024)

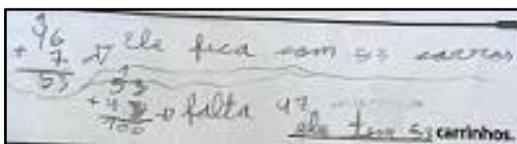
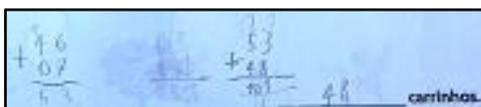


Figure 33

L.R.'s response (Research Data, 2024)



Subtracting 85 from 160

The analysis of the second situation in the second question presented similar categories. The question was: **When lining up the two**

instruments, how many centimeters of the table's length will be left to measure? This is a non-prototypical comparison problem, which requires comparing the size of the instruments, in this case, 85 cm (the result of the previous sum), and the size of the teacher's table, which is 160 cm, the unknown value being that of the relationship between the two: the referred is known, and the relationship is requested (Magina et al, 2008). Comparison problems concern those that compare two quantities, one called the referent and the other the referred, based on a relationship (+r or -r). Within the additive structures, this is considered a more difficult situation to understand than the prototype situations of transformation and composition, since, instead of using the schemes of separating or joining, a one-to-one correspondence scheme is necessary to deal with these situations. For Nunes et al. (2009), this difficulty exists because the ideas of addition and subtraction appear in the previous schemes as changes in quantities, but in comparative problems, there are no changes in quantities, and students find it more difficult to immediately think about the quantitative relationships involved in the problem. To solve it, the child needs to understand that 85 cm of the table (the value previously discovered) has already been measured and, therefore, to find out how much is left to measure, the child needs to subtract the total (160) from the value already measured (85), leaving $160 - 85$. However, as in the second situation in Problem 1, it is possible to solve this situation from the perspective of complementation, that is, starting from the 85 cm that have already been measured, the child completes the value until reaching 160 cm.

In this situation, 36.1% opted for complementation and 63.9% for subtraction. The correct answers total 69.2% of those who used complementation and 52.2% of those who used subtraction. Thus, as in

Problem 1, there is a higher error rate among children who opted for subtraction.

Now, performing a quantitative analysis of the children who chose to use the conventional algorithm or alternative procedures, we find that 70.7% of the children preferred alternative procedures and 29.3% preferred the conventional algorithm. It is noted that, in comparison with the first situation in Problem 2, there is a persistence of interest in alternative procedures (a phenomenon identical to that which occurred in Problem 1). Let us now present, qualitatively, the procedures found.

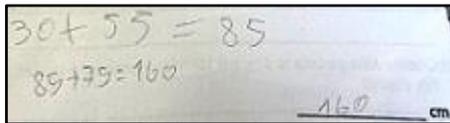
Initially, we will begin with two children whose thinking does not allow us to identify whether they opted for addition or subtraction. Child G.L., whose first part of the response we have already discussed (Figure 10), chose **mental calculation supported by drawing**. They created a graphic representation to better visualize the situation. The child's graphic representation is interesting and curious: a "ramp" whose measurements are the values at stake. We believe that this occurred due to another issue previously discussed in class, in which there was a ramp measuring 30 cm. The child probably evoked this image to, in the case at hand, schematize the part-whole relationship and perform the correspondence scheme. Although unusual (and not entirely accurate), the drawing facilitated their understanding of the problem, since they got the answer right, reminding us of the importance of two-dimensional representations, such as those presented by Regina Damm (2003), for dealing with additive problems. Child S.M., whose first part of response we have already discussed (Figure 12), used the method of **decomposition by base-ten blocks (written in Portuguese)**, a procedure already explored in the previous situation.

Now, let us analyze the children who explicitly opted for complementation.

Child J.I. was one of the children who used the **mental calculation** strategy **with a single expression**:

Figure 34

Correct use of alternative method (complementation): Child J.I. (Research Data, 2024)

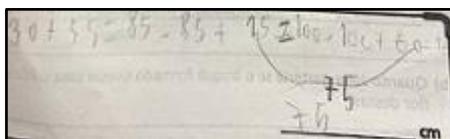


Handwritten mathematical work showing the calculation: $30 + 55 = 85$ and $85 + 75 = 160$. The number 160 is underlined.

Child D.F. was one of the children who chose the **decimal number composition** strategy. They started by adding 15 to 85 to get 100, and since 100 is 60 short of 160, the answer is $15 + 60 = 75$ (which they indicated with an arc).

Figure 35

D.F.'s response (Research Data, 2024)

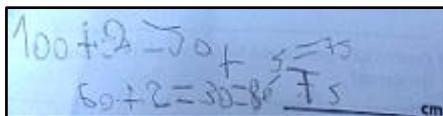


Handwritten mathematical work showing the calculation: $30 + 55 = 85$, $85 + 15 = 100$, and $100 + 60 = 160$. The number 75 is underlined.

Child C.Z. was one of the children who used the strategy of **changing values**. The child knew that they needed to do $85 + ? = 160$ and began to think about 160 as 100 and 60, remembering its halves from there: $100 : 2 = 50$ and $60 : 2 = 30$ (the child had not yet been taught the standard way of writing division in class and therefore did it their own way: to indicate it, they used the + sign with a dot at each end!). They added both together, which gave them 80, so they already knew that the teacher's desk was two 80s. Perhaps they looked for the half because they already knew that the ruler + string gave 85 and assumed that to measure the teacher's desk they would need more or less double that, i.e., another 85. When they saw that the teacher measured 2×80 , and knowing that they already had 85, i.e., 5 more, they knew that the other half should have 5 less, and therefore wrote $80 - 5 = 75$.

Figure 36

Child C.Z. using the value change strategy (Research Data, 2024)



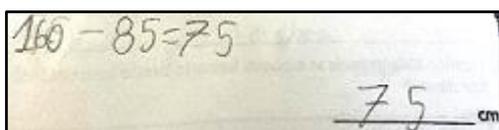
Handwritten mathematical work showing two equations: $100 + 2 = 102$ and $50 + 2 = 52$, with a result of 75 written to the right.

We will now analyze the strategies used by the children who chose to solve the problem by subtraction.

Child J.L. was one of those who used the **mental calculation** strategy **with a single expression**, represented by the expression $160 - 85 = 75$.

Figure 37

Correct use of alternative method (subtraction): Child J.L. (Research Data, 2024)

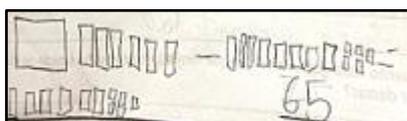


Handwritten mathematical work showing the subtraction $160 - 85 = 75$.

Child N.S. solved the problem by **decomposing** it **using base-ten blocks**. Thus, we can see the minuend as one flat and six rods: $100 + 10 + 10 + 10 + 10 + 10 + 10$ and the subtrahend as eight rods and five units: $10 + 10 + 10 + 10 + 10 + 10 + 10 + 10 + 5$. When performing the operation, however, the child reached ten less than the correct final answer, perhaps due to the need to make the exchange, and the child, not having the base-ten blocks on their desk, had to do it in their head:

Figure 38

Child N.S. performs decomposition using base-ten blocks (Research Data, 2024)



Handwritten mathematical work showing base-ten blocks and the result 65.

Child N.M. was one of those who used **decimal decomposition**. Thus, they separated 85 into 80 and 5, first subtracting 80 from 160 and then the remaining 5.

Figure 39

Child N.M. decomposing 85 into 80 and 5 (Research Data, 2024)

Handwritten mathematical work showing the decomposition of 85 into 80 and 5, and the subsequent subtraction from 160:

$$\begin{array}{l} 30 + 55 = 85 \quad 80 - 5 = 75 \\ 160 - 85 = 75 \\ 160 - 80 = 80 \quad \underline{75} \end{array}$$

Child D.P. used the **value alteration** strategy. First, they subtracted 55 (string) and then 30 (ruler) from 160. In the first case, however, they preferred to subtract 60 and correct it by adding 5 to the result. In the second calculation, $105 - 30$, they made a mistake, reaching ten less than the correct final answer, probably due to the difficulty of counting in tens in their head (in this case, backwards!) and reflecting on what number it is when passing the knot:

Figure 40

Error in using an alternative procedure: Child D.P. (Research Data, 2024)

Handwritten mathematical work showing an alternative procedure for subtracting 85 from 160:

$$\begin{array}{l} 160 - 60 = 100 + 5 = 105 - 30 = 75 \\ \underline{75} \end{array}$$

Now, let's analyze the children who used the conventional algorithm. Most of them simply set up the calculation $160 - 85$ and solved it by applying the algorithmic rules. However, as in problem 1, two children used the algorithm through complementation. Child L.R., although with an erased writing, presents a set up addition calculation. The case of child S.L. is even more curious, as the child used the strategy of **changing values**, using the algorithmic structure to investigate the value needed to complete 160 from 85. Thus, from the 85 obtained in the first situation, they verified that exactly the same amount would be missing and realized that $85 + 85 = 170$, which is 10 more than 160, but as one of the numbers is actually 85,

it is necessary to subtract 10 from the other 85, reaching the final answer
 $85 + 75 = 160$:

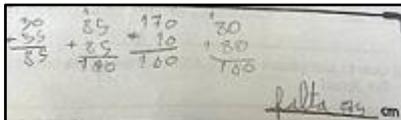
Figure 41

*Algorithm used in the completion: child L.R (notation erased by the child)
(Research Data, 2024)*



Figure 42.

Algorithm used in complementation: child S.L. (Research Data, 2024)



Errors in subtraction situations

In the first subtraction situation ($101 - 53$), of the children who chose alternative procedures, 11.5% got the question wrong and 88.5% got it right. Of those who used the algorithm, 58.8% got the question wrong and 41.2% got it right. In other words, the use of the conventional algorithm proved to be an unsuccessful strategy for most children who chose it, unlike alternative procedures to the algorithm.

The children who got the question wrong using the algorithm misapplied the algorithmic rules, for example, they used the commutative property to operate the numbers in the column, they did not know what to do with the 0 in the column, they made mistakes in their calculations (for example, without subtracting 1 from the column they were "borrowing" from), among other errors. For example:

Figure 43

Example of an error in using the algorithm, applying the commutative property: child K.S. (Research Data, 2024)

$$\begin{array}{r} 46 \\ + 53 \\ \hline 99 \end{array}$$

101
100
carrinhos.

Let us move on to the errors made by children who chose alternative procedures. Child G.L. reached 57 (we have already presented their procedure in Figure 3). In this case, there was no note of the thinking for the second situation of the problem, but the student solved the first part correctly, knowing that Breno has 53 cars, and reached 57, instead of 48. Note that $53 + 57 = 110$, which led us to the hypothesis that the student may have confused 101 with 110 when completing the problem.

One child reached 52. The student reached the expected answer in the first part of the problem: $46 + 7 = 53$, but in the second part, the student reached $53 + 52$ and mistakenly thought that this sum equals 101. We believe that the student may have thought as follows: adding 50 to the 50 in 53, we have $50 + 50 = 100$, but then thought that since it would be necessary to reach 101, and if we have the numbers 53 and 52, then $3 - 2 = 1$, so $53 + 52 = 101$. In other words, the student made a mistake when adjusting the 50 during completion, based on a mental numerical composition strategy:

Figure 44

R.M.'s response (Research Data, 2024)

$$53 - 52 = 1$$

carrinhos.

Another child, D.M., reached 47. It is impossible to know how this child reached 47, but we assume that, having been wrong by only 1, it was a counting error (however, it is impossible to know whether this error occurred in the first or second situation):

Figure 45

D.M.'s response (Research Data, 2024)

$$40 + 7 = 47$$

carrinhos.

Child D.P. final answer is 46. It is also impossible to know how they reached 46. The writing of $53 + 2 (1) = 55$ is particularly interesting because, in our view, it reveals the "beginning" of complementation: it seems that the child, starting from 53, added 2 to arrive at a number that was easier to perform the following calculations, in this case, 55. However, the absence of further writings reveals that the child did everything in their head, having made a mistake of 2 in the answer (which was 48), possibly due to a counting error.

Figure 46

D.P.'s response (Research Data, 2024)



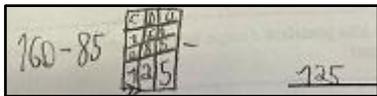
It is interesting to note that in all these cases, the children opted for **mental calculation without notations**.

In the second subtraction situation ($160 - 85$), of the children who opted for alternative methods, 61.5% got the answer right, while 38.5% got the question wrong; among those who solved it using the conventional algorithm, 75.0% got the answer right and 25.0% got the question wrong. It is interesting to note that despite the similarity between the two problems under discussion, while in the second situation of Problem 1 the alternative methods proved to be a more successful strategy, in the situation we are now analyzing (second of Problem 2), the algorithm proved to be more efficient.

In the cases of children who made mistakes when using the conventional algorithm, it is possible to see that in all three cases the error occurred due to a lack of mastery of the algorithm's processes, as there was a mistaken application of commutativity and/or an error due to the difficulty of correctly operating two exchanges (in the tens and hundreds, as the numbers required), resulting in 125, 85, and 145.

Figure 47

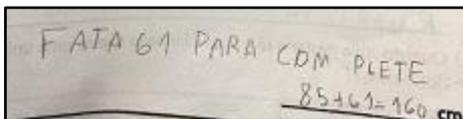
Example of error in using the algorithm: Child A.C. (Research Data, 2024)



Let us analyze the errors of the children who chose to use alternative procedures. Child J.P. made a mistake in the second situation when solving the problem using **mental calculation with a single expression**. It should be noted that they reached the correct answer in the first situation and apparently solved the second situation by complementation, that is, how much is missing to reach 160 cm from 85 cm. It may be that the child tried to complete the calculation by counting in tens and ones and got lost at some point and reached 61 instead of 75; it may also be that the child tried to put the algorithm in their head, which would somehow justify the 1 in 61, which, added to the 5 in 85, would mistakenly result in 6 in 160.

Figure 48

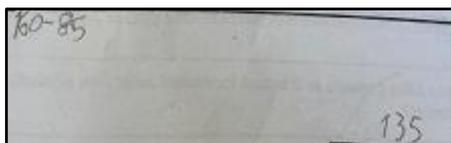
Mental calculation error: Child J.P. (Research Data, 2024)



Child B.A. wrote the expression $160 - 85$, correctly representing what was expected in the second situation, but indicated the answer 135 cm. Due to the lack of notation in this strategy, it is impossible to verify how the error occurred, but because of similar errors during the resolution of the subtraction by algorithm, we hypothesize that the child performed the algorithm in their head and mistakenly applied the commutative property, that is, $0 - 5 = 5$, $6 - 8 = 3$ (calculation error) and $1 - (\text{nothing}) = 1$, so $160 - 85 = 135$.

Figure 49

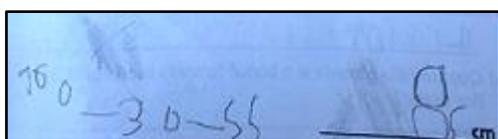
Error in using an alternative procedure: Child B.A. (Research Data, 2024)



Children D.M., A.V., and P.R., whose solutions are shown in the figures below, did not separate the problem into two situations, solving the second situation directly from the perspective of subtraction, and, perhaps because of this, got the result wrong:

Figure 50

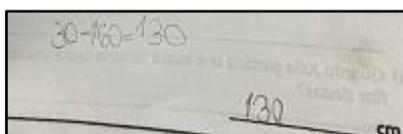
Error in using an alternative procedure: Child D.M. (Research Data, 2024)



Two children made mistakes due to an incomplete solution. Apparently, the children thought of first doing $160 - 30$ (ruler), and then, based on the result found, subtracting 55 (string), but they did not do the second step, reaching an incomplete answer.

Figure 51

Error in using an alternative procedure (Research Data, 2024)



We also highlight the errors made by children who chose to use the **numerical composition/decomposition** method. Child E.M. reached the correct result in the first situation. For the second situation, the child preferred to solve it from the perspective of complementation: from 85 to reach 100, 15 is missing ($85 + 15 = 100$), it is understood that, then, the child knows that from 100 to reach 160, 60 is missing, but instead of adding $15 + 60$ to obtain the value needed for the complementation, the child did $60 - 15 = 45$.

Figure 52

Error in using an alternative procedure: Child E.M. (Research Data, 2024)

Handwritten mathematical work on a blue background. At the top, there are two equations: $15 - 160$ and $85 + 15 = 100$. Below these, the calculation $30 + 55 = 85$ is written. At the bottom, the number 145 is written and underlined.

Child D.A., on the other hand, first added $30 + 55 = 85$ and then broke down 85 into $80 + 5$, first operating $160 - 80 = 80$, but then, instead of subtracting the remaining 5 , they added it, reaching $80 + 5 = 85$.

Figure 53

Error in using an alternative procedure: Child D.A. (Research Data, 2024)

Handwritten mathematical work on a blue background. The top line shows two equations: $30 + 55 = 80 + 5 = 85$ and $160 - 80 = 80 + 5 = 85$. At the bottom, the number 85 is written and underlined.

Finally, we highlight the error made by child A.O., who used the procedure **of changing values** based on the perception that $160 = 80 + 80$. We think the child's idea was as follows: knowing that 85 cm of the table had already been measured, as $85 = 80 + 5$, it was necessary to subtract 5 from the other 80 to maintain equality, and therefore $80 + 80 = 160 \Rightarrow (80 + 5) + (80 - 5) = 160 \Rightarrow 85 + 75 = 160$. However, instead of maintaining equality, the child simply added 5 to the result (160), which probably arose from the need to add the 5 that had been subtracted from 85 when doing $80 + 80 = 160$.

Figure 54

Error in using an alternative procedure: Child A.O. (Research Data, 2024)

Handwritten mathematical work on a white background. The top line shows the equation $80 + 80 = 160 + 5 = 165$. At the bottom, the number 85 is written and underlined.

Discussions

According to the analysis of the strategies adopted by the students in relation to problems of the additive field, we can conclude that there is a variety of ways to solve them, strategies that go far beyond the

conventional algorithm (although it was also chosen by some children). The emphasis given since the beginning of the year in this class on teaching based not on conventional algorithms, but on encouraging a multiplicity of procedures, was certainly decisive for this plurality to be revealed in the responses.

In quantitative terms, we find that most of our subjects preferred alternative procedures to conventional algorithms for dealing with the two additive situations and the two subtractive situations (72.9%, 73.7%, 65.9%, and 70.7%, respectively, preferred alternative procedures). The success rate tends to be higher for those who choose alternative procedures, since, in all four situations, the error rate did not exceed the success rate, being low in the first three situations and slightly higher in the last (5.8%, 7.6%, 11.5%, and 38.5%, respectively). In the case of those who opted for algorithms, we can state that the success rate is lower, since the percentage of errors is higher in all situations, except for the fourth (7.6%, 20%, 58.8%, and 25%), highlighting the third situation, which had a higher error than success rate. It was also found that, in both subtractive situations, the majority opted for the subtraction procedure rather than the complementation procedure (56.1% and 63.9% versus 43.9% and 36.1%, respectively), with a higher success rate among those who opted for complementation (65.2% and 52.2% versus 83.3% and 69.2%), indicating the greater complexity/difficulty of dealing with subtractive reasoning.

Qualitatively, it was possible to categorize non-algorithmic strategies based on 1) the small or large number of notations left on paper, 2) the nature of the arithmetic thinking revealed in the answer, and 3) the presence or absence of non-numerical support. Regarding the amount of notation, we classified those with few notation as **pure mental calculation** (when there is only the final answer) or **mental calculation with single expression** (when there is only the expression and the answer). Some

children leave marks in addition to these, although it still does not fully reveal their thinking. When there was enough writing to reveal their thought, it was possible to observe procedures that translate into a) **the use of counting by 1s or 10s**; b) **decimal or non-decimal numerical composition or decomposition**; and c) the **alteration of values at stake** to facilitate the operations to be performed. Finally, it was possible to observe responses in which these procedures were supported not **only by numbers in their conventional notation**, but also by **unconventional notations (arrows, arcs, isolated numbers, etc.)**, by **the base-ten blocks**, by **drawings**, or even **by the Portuguese language**.

Most children opted for mental calculation. While this is a fast strategy, it is also prone to a greater number of errors due to the mental overload it demands of the individual, as observed in the incorrect answers. Errors of 1 or 2 more or less, or 10 or 20 more or less, were the most common in this particular case. Pure mental calculation or calculation supported by expression presents a problem for the researcher: it is impossible to know what the subject actually thought arithmetically to reach the final answer. They may have moved one by one; they may have decomposed; they may even have mentally formulated the algorithm. Research that seeks to qualitatively understand this procedure must necessarily include other data production instruments, such as interviews with the subjects, which was not possible in this case.

Regarding the three strategies found: 1) counting by 1s or 10s, 2) decimal and non-decimal composition and decomposition strategies, and 3) the strategy of changing values, in our view, these are progressively more complex strategies in terms of dealing with numbers, operations, their regularities, and properties. The first strategy seems to us to be the least complex and most time-consuming, from the point of view of addition and

subtraction operations, as they are linked to the primitive idea of counting. Obviously, counting in tens is more difficult than counting in ones; however, it is an extension of this primitive counting, since the original idea remains: to repeatedly subtract from the subtrahend or add to the first term the values +1, -1, +10, or -10 until the value of the minuend or the other term is exhausted. On this subject, the literature indicates that young children like to count one by one, even after learning the school algorithm (Kamii; Joseph, 2004). Furthermore, counting in tens becomes a resource in classes where children are free to think and discuss, a resource that comes as an extension of counting one by one (as investigated by Lerner, Sadovsky, Wolman, 1996).

The strategy of decimal or non-decimal decomposition/composition is, in our view, a little more advanced, as it requires the child to make a more complex numerical interpretation of the numbers involved, since there is an intentionality to this composition/decomposition that is only possible for those who have mastered certain aspects of the numbering system and certain numerical regularities/properties. The child T.A., for example (Figure 4), when adding $46 + 7$, *knew* that 7 had a 4 (and a 3) and also *knew* that adding $46 + 4$ was easier because it would result in a round number, and therefore did $46 + 4 = 50 \Rightarrow 50 + 3 = 53$. Then, when they see that they need to reach 101, they do $53 + 47 = 100$, because *they knew* that both added together would give a round number (100) and *they knew* that it would be easier to add from there, doing $100 + 1 = 101$. They therefore "freed" themselves from the endless process of counting by 1s or 10s: they counted 4 in a row, then 3 in a row, then 47 straight, and then 1. It is, therefore, a strategy that requires the child to see the numbers and "break" them down into smaller numbers, but without necessarily "crumbling" them into ones and tens. To do this, they anticipate certain operations, using certain regularities and mastery of certain fundamental

facts. As this strategy is more demanding than the previous one, it is natural for children to make some mistakes, as was the case with children E.M. and D.A. (Figures 55 and 56). For Nunes, decomposition is a process in which subjects seek ways to "round numbers because round numbers are not only more easily memorized [...], but also because they help to avoid the overload that would occur in the mental processing of data [...]" (Nunes; Carraher; Schliemann, 2011, p. 80, our translation).

Finally, the strategy of changing values seems, in our view, to be the most advanced, since the child, here, unlike in the previous procedures, detaches themselves from the values at stake, modifying them at will. Child D.A. calculates $100 - 50$, which was not the case (it was $101 - 53$), but *if it were* $100 - 50$, they know it would equal 50. Since it was not 50 that had to be taken away, but 53, the child subtracted 3 from the result ($50 - 3 = 47$), and since the total value was 101 and not 100, he added 1 to the result ($47 + 1 = 48$). It is noticeable that the composition/decomposition reasoning is still present, but the child no longer obediently sticks to the numbers presented, as in previous strategies. The expression "if it were," written in Portuguese by the child D.A. themselves on paper, is significant because it confirms the breaking free of the constraints of the presented numbers!

We drew inspiration from Lerner and colleagues in designing these possible qualitative advances, as they state, *literis*:

it is necessary to set in motion all possible resources to ensure that children who count (or add) by ones approximate their procedure to that of those who add by tens and that this progress towards more economical "if... then" type strategies. The search for regularities linked to operations makes this progress possible... and something more. (Lerner, Sadovsky, Wolman, 1996, p. 149, our translation)

Regarding errors, the procedures of decomposition/composition and changing values, being more demanding strategies, not surprisingly

presented some errors on the part of some children in their application, especially in the confusion when "correcting" the altered values or adding instead of subtracting (or vice versa). This reveals, in our view, that the process of appropriating these strategies is not a procedure of uniform ascension: the child who alters values (for example), although mathematically knows that this is possible, may execute it with errors. We believe, however, that it is only by trying and willing to take risks that they open up possibilities for educational work that allows them to identify their mistakes and move forward.

The errors identified in the use of conventional algorithms were consistent with those reported in the literature (e.g., Batista, 1995; Zatti, Agranionih, Enricone, 2010): incorrect application of the commutative property in subtraction, lack of mastery of the "borrowing" rule, forgetting to decrease the value of the "borrowed" decimal place or getting confused when performing the second "borrowing" operation, among others.

Comparing the errors in each of the strategies (alternative procedures and algorithms), we can interpret the errors resulting from the use of alternative procedures as errors that arise mainly from the complexity of the numerical concepts involved in their processing, that is, when presenting alternative reasoning, the child becomes confused in the logic and loses track of the linearity of the reasoning, or gets lost during mental calculation, since, depending on the strategy, they need to master numbers, fundamental facts, counting, properties of operations, numerical regularities, the functioning of the numbering system, etc. This, of course, will depend on the alternative procedure chosen, but there seems to be a greater chance of errors with procedures that require less notations and/or mobilize more numerical concepts (such as mental calculation without notations and the strategy of changing values). When choosing algorithms, on the other hand, we interpret errors as errors in procedural rules, that is,

the child, because they do not master the step-by-step algorithmic operation, gets the answer wrong. The big problem is that, in this case, even children who get the answer right may not be advancing in their arithmetic thinking, since their merit lies solely in getting the algorithmic rules right (if they cannot use the algorithm, they may want to count one by one!).

We highlight that, in two cases, algorithms were used to solve the situation through typical alternative procedure strategies (S.L. and L.R.'s responses). This shows the possibility of using the algorithm as a support tool for advancing in other arithmetic strategies, allowing for flexible thinking. However, viewing the algorithm in this way is not so common, since we only obtained these examples.

Finally, we must once again bring up the understanding of learning as action. It seems to us that the plurality revealed in the responses indicates an action by the subjects who, intentionally and inventively, set out to find solutions to the problems presented, revealing in their responses the nuances of their own reasoning, their own attempts to make sense of the problems. Although in some cases the children got the answer wrong, each participant's movement revealed their own conceptualizations about the decimal number system, its properties, and regularities. This picture could be completely different if the classroom in question valued the mechanical application of conventional algorithm rules (we would have a collection of identical answers: numbers aligned vertically, column by column). It is important to understand that these were problems based on the exercise paradigm; however, the freedom experienced by the subjects in solving them, based on the variety of categorized procedures, indicates, in our view, evidence of landscapes of investigation based on pure mathematics, which can be problematized and encouraged in a classroom that aims to be critical.

Conclusion

In general, the children who collaborated in the research showed a greater preference for using alternative procedures instead of conventional algorithms in all situations analyzed. This preference, however, is even greater in additive situations when compared to subtractive situations. The large number of alternative procedures to the algorithm reveals that the class appropriated original ways of solving problems, taking risks, thinking for themselves, in a creative and flexible way.

Our text concludes by proposing a paradigm shift in the teaching of arithmetic: it is necessary *to go where the other is* in terms of how that individual thinks about addition and subtraction in problem-solving situations. This should mean focusing less on whether the child knows how to “borrow” in the subtraction algorithm and more on whether the child uses the subtraction strategy in comparison situations, or knows only the complementation strategy, for example; it means worrying less about whether the subject puts ones under ones, tens under tens, hundreds under hundreds in the addition algorithm and more about which strategy the subject *actually uses* to add: whether they need to count one by one or already know how to count by 10s, or even already know how to decompose or alter the numbers involved!

We therefore consider alternative approaches to conventional algorithms an opportunity to develop logical reasoning and construct mathematical knowledge, which seems to align with CME. They are more respectful of the learning processes of the individuals involved, open themselves to dialogue in the classroom, and enable new constructions through the exploration of numbers, the numbering system, and operations in their richness of relationships, properties, and regularities. Although it is not immune to errors, even these, in this proposal, are interesting, as they reveal the subject's fumbling with these characteristics, allowing the

educator to mediate in order to promote their progress in mastering the underlying concepts. Meanwhile, the conventional algorithm, by its own hermetic nature, hinders this development because it is given *a priori* and forces the subject to comply with its rules, hindering this learning, with the mistakes made by children signaling to adults the failure in the application of a rule, demanding its correction in a generally absolute and bureaucratic manner (Alrø; Skovsmose, 2021).

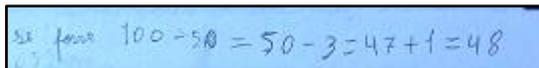
We are not condemning the use of and work with "carry" and "borrow" in school addition and subtraction, nor are we denying the possibility of problematizing it in the classroom, making it a source of learning. However, given the richness revealed by children in the various ways they found to solve addition and subtraction problems, it would be a didactic mistake, in our view, to prematurely introduce and reinforce these algorithms, since, as Nunes, Carraher, and Schliemann (2011) explain, they do not preserve relative value and require a non-intuitive decomposition.

Going to where the other is, we insist, is not about knowing at what point in the algorithmic rules the subject stopped, in order to help them move forward. It is about effectively understanding how that individual *deals with the numbers involved* in a given mathematical problem (not necessarily school-related!), how they would solve it if they did not have paper and pencil, for example, or if it were not a calculation to be shown to the teacher. It is from this point that we invite them to move forward, learning to explore numbers as if they were their allies, their friends on a journey, available, manipulable, inviting! It is thus possible to create landscapes of investigation based on various non-algorithmic procedures, procedures that were not imposed as truths, but rather created in a context of freedom, autonomy, and legitimization of the meanings produced by the children. The landscapes of investigation are fields full of diversity of reasoning, legitimization of choices, and expression of preferences. The

children's authorship shows that they can take responsibility for their learning. Notations such as D.A.'s for the calculation $101 - 53$ inspire us, for the freedom with which they manipulate the numbers and the confidence in writing their thoughts, and therefore, we conclude this work with it:

Figure 55

D.A.'s response (shown in Figure 30) (Research Data, 2024)



se. para $100 - 50 = 50 - 3 = 47 + 1 = 48$

References

- Alrø, H., & Skovsmose, O. (2021). *Diálogo e aprendizagem em educação matemática* (3ª ed.). Autêntica.
- Batista, C. G. (1995). Fracasso escolar: Análise de erros em operações matemáticas. *Zetetiké*, 3(4), 61–72.
- Conceição, A. R. C. (2021). *O cálculo mental para ensinar: Uma análise de produções de Maria do Carmo Santos Domite (1980–1995)* (Dissertação de mestrado, Universidade Federal de São Paulo).
- Damm, R. (2003). Representação, compreensão e resolução de problemas aditivos. In S. Machado (Org.), *Aprendizagem em matemática: Registros de representação semiótica* (cap. 2). Papyrus.
- Freire, P. (1996). Entrevista concedida a U. D'Ambrosio e M. do C. Domite. YouTube. https://youtu.be/1OTbuid4_G0
- Gitirana, V., Campos, T., Magina, S. M. P., & Spinillo, A. G. (2014). *Repensando multiplicação e divisão: Contribuições da teoria dos campos conceituais*. PROEM.
- Kamii, C., & Housman, L. (2002). *Crianças pequenas reinventam a aritmética: Implicações da teoria de Piaget* (2ª ed.). Artmed.
- Kamii, C., & Joseph, L. (2004). *Young children continue to reinvent arithmetic (2nd grade): Implications of Piaget's theory* (2nd ed.). Teachers College, Columbia University.
- Lerner, D., Sadovsky, P., & Wolman, S. (1996). O sistema de numeração: Um problema didático. In C. Parra & I. Saiz (Orgs.), *Didática da matemática: Reflexões psicopedagógicas* (pp. 73–155). Artmed.
- Lins, R. C. (1999). Por que discutir teoria do conhecimento é relevante para a educação matemática. In M. A. V. Bicudo (Org.), *Pesquisa em educação matemática: Concepções & perspectivas* (pp. 75–94). Ed. UNESP.

- Magina, S. M. P., Campos, T., Nunes, T., & Gitirana, V. (2008). *Repensando adição e subtração: Contribuições da teoria dos campos conceituais* (3ª ed.). PROEM.
- Mendonça, M. do C. D., & Lellis, M. (1989). Cálculo mental. *Revista de Ensino de Ciências*, (22), 50–57.
- Milani, R. (2017). “Sim, eu ouvi o que eles disseram”: O diálogo como movimento de ir até onde o outro está. *Boletim de Educação Matemática*, 31(57), 35–52.
- Moro, M. L. F., & Soares, M. T. C. (Orgs.). (2005). *Desenho, palavras e números: As marcas da matemática na escola*. Editora da UFPR.
- Nunes, T., Campos, T. M. M., Magina, S. M. P., & Bryant, P. (2009). *Educação matemática: Números e operações numéricas* (2ª ed.). Cortez.
- Nunes, T., Carraher, D. W., & Schliemann, A. D. (2011). *Na vida dez, na escola zero* (16ª ed.). Cortez.
- Skovsmose, O. (2015). *Um convite à educação matemática crítica*. Papirus.
- Skovsmose, O. (2023). *Critical mathematics education*. Springer.
- Vergnaud, G. (2009). *A criança, a matemática e a realidade: Problemas do ensino da matemática na escola elementar*. Editora UFPR.
- Zatti, F., Agranionih, N. T., & Enricone, J. R. B. (2010). Aprendizagem matemática: Desvendando dificuldades de cálculo dos alunos. *Perspectiva*, 34(128), 115–132.
- Zunino, D. L. (1995). *A matemática na escola: Aqui e agora* (2ª ed.). Artmed.