Beyond linear proportionality: an epistemological framework for teaching area-volume relationships in contemporary mathematics education

Más allá de la proporcionalidad lineal: un marco epistemológico para la enseñanza de relaciones área-volumen en la educación matemática contemporánea

Au-delà de la proportionnalité linéaire : un cadre épistémologique pour l'enseignement des relations aire-volume dans l'éducation mathématique contemporaine

Para além da proporcionalidade linear: um framework epistemológico para o ensino de relações área-volume na educação matemática contemporânea

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Abstract

This article proposes an integrative epistemological framework for teaching area-volume relationships in basic education, addressing the conceptual difficulties inherent in the transition from linear to multidimensional thinking. Grounded in complex proportional thinking, mathematical modeling, and contemporary mathematics education approaches, the proposal articulates theoretical principles with innovative didactic strategies. We critically analyze the "linear proportionality illusion" that permeates traditional geometry teaching, proposing instead an approach based on real phenomena and mathematical modeling. The presented

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framework offers significant contributions to teacher education and the development of complex mathematical competencies, positioning itself within the current debate on the epistemology of school mathematical knowledge.

__Keywords:__ Mathematics education, Proportional thinking, Epistemology of mathematics, Mathematical modeling, Geometry teaching.

Resumen

Este artículo propone un marco epistemológico integrador para la enseñanza de las relaciones entre área y volumen en la educación básica, abordando las dificultades conceptuales inherentes a la transición del pensamiento lineal al multidimensional. Fundamentada en el pensamiento proporcional complejo, la modelización matemática y los enfoques contemporáneos de la educación matemática, la propuesta articula principios teóricos con estrategias didácticas innovadoras. Analizamos críticamente la "ilusión de la proporcionalidad lineal" que impregna la enseñanza tradicional de la geometría, proponiendo en su lugar un enfoque basado en fenómenos reales y modelización matemática. El marco presentado ofrece contribuciones significativas para la formación docente y para el desarrollo de competencias matemáticas complejas, situándose en el debate actual sobre la epistemología del conocimiento matemático escolar.

Palabras clave: Educación matemática, Pensamiento proporcional, Epistemología de las matemáticas, Modelización matemática, Enseñanza de la geometría.

Résumé

Cet article propose un cadre épistémologique intégrateur pour l'enseignement des relations entre l'aire et le volume dans l'éducation de base, abordant les difficultés conceptuelles inhérentes à la transition d'une pensée linéaire à une pensée multidimensionnelle. Fondée sur la pensée proportionnelle complexe, la modélisation mathématique et les approches

contemporaines de l'éducation mathématique, la proposition articule des principes théoriques avec des stratégies didactiques innovantes. Nous analysons de manière critique « l'illusion de la proportionnalité linéaire » qui imprègne l'enseignement traditionnel de la géométrie, proposant à la place une approche basée sur des phénomènes réels et la modélisation mathématique. Le cadre présenté offre des contributions significatives à la formation des enseignants et au développement de compétences mathématiques complexes, se situant dans le débat actuel sur l'épistémologie des connaissances mathématiques scolaires.

Mots-clés : Éducation mathématique, Raisonnement proportionnel, Épistémologie des mathématiques, Modélisation Mathématique, Enseignement de la géométrie.

Resumo

Este artigo propõe um *framework* epistemológico integrador para o ensino das relações entre área e volume na educação básica, abordando as dificuldades conceituais inerentes à transição do pensamento linear para o multidimensional. Fundamentada no pensamento proporcional complexo, na modelagem matemática e em abordagens contemporâneas da Educação Matemática, a proposta articula princípios teóricos com estratégias didáticas inovadoras. Analisa-se criticamente a "ilusão da proporcionalidade linear" que permeia o ensino tradicional de geometria e propõe-se em seu lugar uma abordagem baseada em fenômenos reais e modelagem matemática. O *framework* apresentado oferece contribuições significativas para a formação docente e para o desenvolvimento de competências matemáticas complexas e situa-se no debate atual sobre a epistemologia do conhecimento matemático escolar.

Palavras-chave: Educação Matemática, Pensamento proporcional, Epistemologia da matemática, Modelagem matemática, Ensino de geometria*.*

Beyond linear proportionality: an epistemological framework for teaching area-volume relationships in contemporary mathematics education

Introduction

The understanding of area-volume relationships represents one of the most persistent epistemological challenges in Mathematics Education. The cognitive tendency to reduce complex relationships to simple proportions (an illusion of linear proportionality) constitutes a deeply rooted obstacle, as characterized by Brousseau (2002).

A common and counterintuitive example that encapsulates this challenge is the well-known Painter's Paradox, according to De Bock et al. (2002): when the linear dimensions of a solid (such as a cube) are doubled, its surface area increases by a quadratic factor (four times), while its volume increases by a cubic factor (eight times). This means that to paint the external surface of a cube twice as large, the painter will need four times more paint; however, this same cube, if hollow, will have an internal capacity eight times greater.

This paradox crystallizes the fallacy of applying linear reasoning to multidimensional phenomena, with documented consequences in low performance on national and international assessments (Organisation for Economic Co-operation and Development [OECD], 2018).

Although recent research has advanced in understanding these challenges from neuroscientific (Dehaene, 2020), cultural (González et al., 2005), and technological (Borba & Villarreal, 2005) perspectives, a gap persists in the systematic articulation of these different knowledge domains.

Just as understanding volume requires transcending the two-dimensionality of area, incorporating a new dimension and integrating previous concepts into a more complex structure, theoretical advancement requires crossing disciplinary boundaries, articulating isolated foundations into coherent, three-dimensional systems of understanding.

Overcoming this historical obstacle demands more than isolated interventions; it requires an integrative theoretical-methodological

framework to guide teaching practice and the design of learning environments.

It is in this context that this article proposes and develops an Integrative Epistemological Framework (IEF) for teaching area-volume relationships in basic education. Grounded in a critical synthesis of complex proportional thinking (Lobato & Ellis, 2010), mathematical modeling (Bassanezi, 2002), and contemporary approaches, the framework articulates three interrelated pillars: (1) the cognitive foundations of multidimensional thinking; (2) pedagogical strategies based on modeling and investigation; and (3) the mediating role of digital technologies in reconfiguring learning.

The specific objectives of this work are: to analyze the epistemological and historical bases of the linear proportionality illusion; to present the principles and components of the IEF; to discuss how the integration of cognition, pedagogy, and technology can overcome historical learning obstacles; and to establish implications of the proposal for teacher education and research in Mathematics Education.

This study is configured as theoretical research of a qualitative nature, whose design includes: procedures of critical and integrative literature review, conceptual analysis and theoretical synthesis (Bakker, 2020), aiming to propose the IEF. The construction of the IEF started from an interdisciplinary articulation between cognitive sciences, critical pedagogical theories, and studies on educational technology.

The article is organized as follows: after this introduction, proportional thinking is examined to deconstruct the linear illusion. Next, the role of digital technologies in reconfiguring the teaching of these relationships is analyzed. The central section details the IEF proposal, culminating in a discussion of its implications for contemporary Mathematics Education.

Through this approach, we seek to offer a coherent path to foster multidimensional, critical mathematical thinking, essential for the challenges of the 21st century.

In the 21st-century educational context, marked by rapid transformations, the development of Higher-Order Thinking Skills (HOTS) has consolidated itself as one of the central pillars of contemporary education (Li et al., 2024). These competencies, involving critical thinking, logical reasoning, creativity, and problem-solving ability, are indispensable for success in an increasingly knowledge-oriented society. As global educational paradigms evolve from simple mechanical memorization to investigative and reflective learning, data from Brazil's national assessment indicate a significant challenge: only 3.7% of high school students are able to apply mathematical concepts in new situations, demonstrating real proficiency in the practical use of acquired knowledge, according to the 2025 Brazilian Basic Education Yearbook[^3].

Proportional thinking: deconstructing the linear illusion in teaching area-volume relationships

The contemporary understanding of proportional thinking in teaching area-volume relationships has evolved beyond mere algorithmic application, revealing itself as a domain of multifaceted complexity (Melo Vieira & Santos, 2019). Research at the interface of Mathematics Education, cognitive sciences, and educational technology demands a deep epistemological review of the assumptions that underlie traditional pedagogical approaches (Lobato & Ellis, 2010).

The neurocognitive bases of this difficulty are a crucial starting point. Studies in cognitive neuroscience indicate that the human brain is biologically predisposed to process linear relationships more efficiently than quadratic or cubic ones (Dehaene, 2020).

This predisposition explains the persistence of the so-called "linear proportionality illusion," which is not a simple conceptual failure but an intrinsic characteristic of our cognitive apparatus, therefore requiring specific pedagogical strategies for its overcoming.

In this context, Cognitive Load Theory (Sweller, 2020) offers a valuable framework for understanding instructional challenges. Understanding multidimensional relationships imposes significant extraneous cognitive load, which can be managed through careful instructional design and the strategic integration of visualization tools. Complementarily, the embodied cognition approach (Abrahamson et al., 2023) revolutionizes our understanding by demonstrating that complex mathematical concepts are constructed through bodily interaction with the environment. Understanding area-volume relationships thus emerges from the coordination between gestures, perceptions, and actions in rich and technologically augmented environments.

Overcoming these obstacles is enhanced by the integration of digital technologies. Dynamic geometry environments (Hoyles & Noss, 2009; Sinclair & Moss, 2012) allow students to concretely experience the effects of scale transformations, developing more refined intuitions about the non-linear nature of these relationships.

Thus, the increasing use of 3D dynamic geometry applications in Elementary and High School has demonstrated not only their potential to facilitate the learning of spatial geometry but also the existence of various factors influencing the effectiveness of this approach.

Beyond visualization, educational artificial intelligence (AI) (Holmes et al., 2019) introduces the possibility of personalization, with tutoring systems capable of detecting error patterns based on excessive linear thinking and offering personalized feedback.

Parallelly, the social and critical dimensions of learning are emphasized by Critical Mathematics Education (Gutstein, 2006), which advocates for contextualizing the study of area-volume relationships within issues of social and environmental justice, such as analysis of population density and environmental impact.

This perspective dialogues with the funds of knowledge approach (González et al., 2005; Oughton, 2016), which highlights the importance of connecting formal knowledge to the mathematical knowledge present in students' daily and community practices.

In the realm of teaching practice, research on teacher noticing (Sherin et al., 2021) underscores the importance of developing teachers' capacity to identify students' specific difficulties with non-linear relationships and respond to them. This *expertise* allows for more precise and pedagogically appropriate interventions. The theory of boundary objects (Akkerman & Bakker, 2020) assists in this process by explaining that objects such as physical models, graphical representations, and symbolic notations, which facilitate movement between different representational systems, are crucial for building robust understandings.

Finally, design-based research methodology (Bakker, 2020) consolidates these perspectives by providing a path for the development of educational interventions. Its iterative cycles of planning, implementation, and analysis allow for creating approaches that effectively respond to identified learning difficulties in real contexts (Kripka & Ribeiro, 2025).

In summary, contemporary theoretical perspectives converge on the need for a multifaceted and integrative approach. Overcoming the linear proportionality illusion demands not only punctual interventions but also a broad restructuring of the teaching of complex mathematical relationships, considering the cognitive, bodily, social, technological, and didactic dimensions in an articulated manner.

Historical challenges, neuroscientific advances, and technological perspectives

Proportional thinking consolidates itself as one of the most complex and epistemologically challenging domains of contemporary Mathematics

Education. Its roots date back to the classical problems of Greek mathematics, where the famous "Delian problem," the duplication of the cube with ruler and compass, already testified to profound human difficulties with non-linear relationships (Netz, 2003).

This historical challenge, unsolvable for centuries within Euclidean paradigms, reveals that the so-called "linear proportionality illusion" is not a modern deficiency but an intrinsic characteristic of the development of mathematical thought throughout history.

In contemporary times, international research corroborates the persistence of these obstacles. Studies indicate that proportional thinking develops through hierarchical levels of cognitive sophistication, with quadratic and cubic relationships being the most complex and counterintuitive (Lobato & Ellis, 2010).

This complexity is reflected in the results of the Programme for International Student Assessment (PISA) 2022, in which only 12% of Brazilian students demonstrated adequate mastery of problems involving area-volume relationships (OECD, 2023), signaling the urgency of more effective pedagogical approaches.

In part, this occurs because an alarming number of students believe that the relationships between length and area, or length and volume in similar geometric figures, are linear, rather than quadratic or cubic; and these students apply the linear scale factor to determine the area or volume of an enlarged or reduced figure.

The cultural dimensions of proportional thinking manifest themselves in a particularly rich way when considering the mathematical knowledge present in traditional Brazilian communities. Practices such as canoe building, agroforestry management, weaving, or pottery intuitively mobilize complex non-linear relationships, often ignored or undervalued in the formal curriculum.

In this sense, the contrast between the inadequate use of the linear model by students and the proportional sophistication present in traditional cultural practices evidences that school difficulties do not stem from a universal cognitive deficit, but from a misalignment between cultural modes of producing mathematics and the way proportional knowledge is legitimized in formal education.

Ethnomathematics research developed with *quilombola* (maroon) groups reveals that artisanal practices incorporate and ground sophisticated understandings of geometric and proportional relationships, transmitted intergenerationally, but rarely recognized in formal school environments (Correia & Santos, 2021). This perspective aligns with the funds of knowledge approach (González et al., 2005), which advocates for mobilizing knowledge originating in the daily practices of communities as valuable pedagogical resources for meaningful and socially relevant learning.

Parallelly, cognitive neuroscience has elucidated the biological bases underlying these difficulties. Evidence demonstrates that the human brain processes linear relationships significantly more efficiently than non-linear ones, explaining the persistence of the tendency towards linearization even in individuals with advanced mathematical training (Dehaene, 2020). Figure 1 shows a representative diagram of human memory.

Functional neuroimaging studies corroborate these observations, revealing distinct patterns of cortical activation for different types of proportional relationships. Such findings have profound educational implications, indicating that overcoming inadequate linear tendencies requires pedagogical interventions specifically designed to confront deeply ingrained intuitions (Stavy & Tirosh, 2000).

In this context, digital technologies emerge as transformative tools. Dynamic geometry environments facilitate the perception of dimensional invariants and covariances, promoting more robust understandings of area-

volume relationships (Andrade et al., 2023; Silva & Pazuch, 2024). The direct manipulation of dynamic representations appears to facilitate the construction of more accurate intuitions about non-linear relationships.

Figure 1

Human Memory: A Proposed System and Its Control Processes
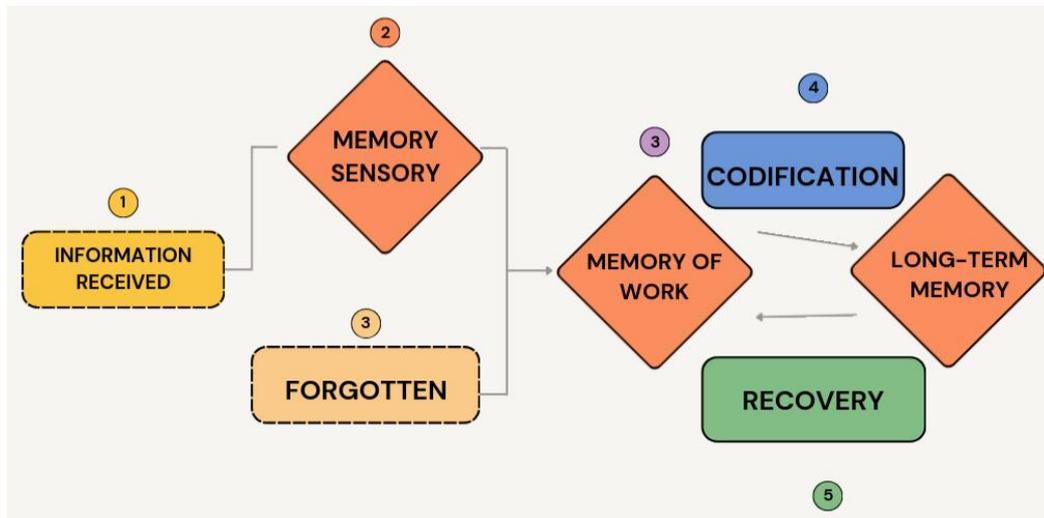
Note. Adapted from Atkinson and Shiffrin (1968).

Advances in augmented and virtual reality offer even more promising possibilities, creating frameworks for immersive learning experiences that connect sensory perceptions with mathematical abstractions (Baccaglini-Frank et al., 2024).

In teaching practice, such understandings demand profound pedagogical transformations. It is argued that teachers who understand the epistemological nature of the concepts they teach develop instructionally more effective practices to promote conceptual understanding (Artigue et al., 2024; Ponte & Chapman, 2022). Approaches grounded in design-based research (Cobb et al., 2021) offer valuable *frameworks* for the development of educational interventions, emphasizing iterative cycles of planning, implementation, and refinement based on empirical data from real learning contexts.

Future research perspectives need to articulate contributions from multiple disciplines. It is argued that understanding the development of proportional thinking would benefit significantly from investigations integrating perspectives from Mathematics Education, cognitive sciences, and educational design studies (Abrahamson et al., 2023). Longitudinal studies tracking the development of proportional reasoning throughout basic education could identify critical points and inform the design of pedagogically appropriate interventions.

The continuous integration of emerging technologies remains a dynamic frontier for research and practice. As noted, such technologies not only expand representational possibilities but also reconfigure the very ways of thinking and learning mathematics (Borba, 2012).

This reconfiguration proves particularly relevant for complex proportional thinking, where dynamic and interactive representations can facilitate understandings that transcend the limitations of traditional approaches.

Overcoming the epistemological challenges of proportional thinking therefore requires a multifaceted approach that articulates historical, cultural, neuroscientific, and technological contributions.

The application of concepts or contents to different contexts is also necessary, as when students can understand a certain concept well, they are able to apply it in many other situations, developing mathematical skills from real and meaningful experiences. This broadening of application contexts paves the way for a more integrated understanding of knowledge, where concepts cease to be treated in isolation.

Only through this interdisciplinary integration will it be possible to develop the robust conceptual understandings necessary to navigate the complex mathematical challenges of the 21st century, in which non-linear

relationships prove central to understanding complex phenomena in science, technology, and society.

Digital technologies and the reconfiguration of area-volume relationship teaching

Digital technologies are radically reconfiguring possibilities for teaching and learning area-volume relationships, signaling a paradigmatic transformation in contemporary Mathematics Education. Dynamic geometry environments (such as: GeoGebra 3D, Cabri-géomètre, and Cinderella) allow mathematical explorations that transcend the limitations of traditional static representations.

The direct manipulation of three-dimensional objects in these environments facilitates the perception of geometric invariants and dimensional covariances, promoting conceptually more robust understandings of the non-linear relationships between linear measures, area, and volume (Drijvers & Sinclair, 2024). This capacity for dynamic visualization is crucial for overcoming the "linear proportionality illusion" that persistently affects the understanding of these complex relationships.

In the Brazilian context, research corroborates these findings. Bairral and Barreira (2017) observed that the active exploration of geometric solids in a digital environment allowed students to identify patterns and regularities that often go unnoticed in traditional expository approaches. The ability to vary parameters in real time and immediately observe the effects on area and volume contributes significantly to the development of more accurate mathematical intuitions and to overcoming alternative conceptions based on inadequate linear reasoning.

Beyond dynamic geometry, the frontiers of augmented and virtual reality present even more innovative possibilities. Immersive environments allow learning experiences that connect sensory perceptions with

mathematical abstractions, substantially facilitating the understanding of elaborate spatial relationships (Mijač et al., 2024).

Parallelly, educational AI introduces an additional dimension of transformation through the personalization of the learning process. Intelligent tutoring systems demonstrate the capacity to detect error patterns and offer contextualized *feedback* that confronts alternative conceptions in an individualized manner (Leung et al., 2024).

These systems can help people learn better and achieve their learning goals, as they provide intelligent tools capable of supporting different educational styles and needs (Santos et al., 2024).

At the same time, such technologies pave the way for creating new modes of teaching and learning, as well as innovation in assessment methods and educational management. For this potential to materialize, educators need to actively transform their way of thinking, exploring new forms of articulation between AI and pedagogical practices, to promote a deep integration between technology and teaching and drive innovative educational development.

Recent research in machine learning applied to Mathematics Education, including multi-centric studies with Brazilian participation, indicates that AI-based systems can identify with high accuracy students who tend to apply inadequate linear reasoning to volume problems, enabling early and targeted interventions.

However, the effective integration of these technologies demands profound transformations in teacher education and curriculum development. Borba & Villarreal (2005) argue that digital technologies not only expand representational possibilities but also fundamentally reconfigure the very ways of thinking and learning mathematics. This reconfiguration requires teachers to develop not only technical

competencies but mainly epistemological understandings about how technologies mediate the construction of mathematical knowledge.

Continuing education programs that integrate digital technologies have shown promising results in the national context, training teachers both in the instrumental use of these tools and in developing deeper understandings of the epistemological challenges of proportional thinking.

In this process, it becomes essential to promote spaces where teachers and teacher educators can collaboratively build technological knowledge, emphasizing the need for playful experimentation with resources and critical analysis of their pedagogical potential.

Furthermore, the United Nations Educational, Scientific and Cultural Organization – UNESCO – (2023) has highlighted the importance of carefully evaluating AI tools before their implementation, as such evaluations are decisive to ensure that these systems do not compromise the integrity of assessment processes nor raise ethical concerns related to transparency, bias, or data protection.

Thus, the integration between continuing education, practical exploration, and careful evaluation of digital technologies contributes to a more responsible, reflective adoption aligned with the real needs of basic education. This solid implementation base is crucial to face the next challenges and opportunities on the educational horizon.

Looking to the future, teaching area-volume relationships in light of digital technologies presents extraordinary opportunities accompanied by significant challenges. The integration of big data, learning analytics, intelligent tutoring systems, and learning design offers unprecedented possibilities for understanding learning trajectories and optimizing educational interventions.

However, as Selwyn (2013) warns, it is crucial that technological development be guided by pedagogical and ethical considerations, not

merely by technical imperatives. Equity challenges in access to technologies demand special attention in the Brazilian context, where regional and socioeconomic disparities may limit the democratizing potential of these innovations.

Research in Critical Mathematics Education has highlighted the importance of ensuring that technological innovations do not exacerbate existing educational inequalities, arguing that the democratization of access must be accompanied by systematic efforts to empower educators and students in the critical and creative use of these tools.

The transformative potential of digital technologies for teaching area-volume relationships resides, therefore, not only in their technical capabilities but also in the way they mediate new forms of engagement with fundamental mathematical ideas.

When adequately integrated with pedagogical approaches informed by robust research and guided by commitments to educational equity, these technologies can contribute decisively to overcoming historical epistemological obstacles and to the development of deeper and more meaningful mathematical understandings.

The IEF: principles and components

The critical analysis developed in the previous sections shows that overcoming the linear proportionality illusion demands a systemic and articulated approach that transcends punctual interventions. In response to this need, we propose an IEF for teaching area-volume relationships.

The IEF is conceived not as a rigid sequence of activities but as a dynamic ecosystem of interconnected principles, grounded in the premise that deep understanding emerges from the synergy between cognition, pedagogy, and technology.

The IEF is organized into three fundamental pillars: cognitive, pedagogical, and technological, which, acting in a coordinated manner, aim to promote a robust and multidimensional understanding (see Figure 2).

Each pillar contributes an essential dimension to this understanding, as explained in Figure 2 and described in the following paragraphs.

The cognitive pillar, the base of mental processing, provides the foundations about how the mind learns these complex relationships. It recognizes and directly confronts the innate obstacles to human thinking, such as the tendency towards linearization (Dehaene, 2020) and the limitations of working memory (Sweller, 2020). It acts as the foundation. Without a strategy that consciously confronts linear intuition, manages cognitive load, and engages the body in learning (embodied cognition), any pedagogical or technological approach may fail to restructure students' deeply ingrained conceptions.

Figure 2

Integrative Epistemological Framework (IEF)
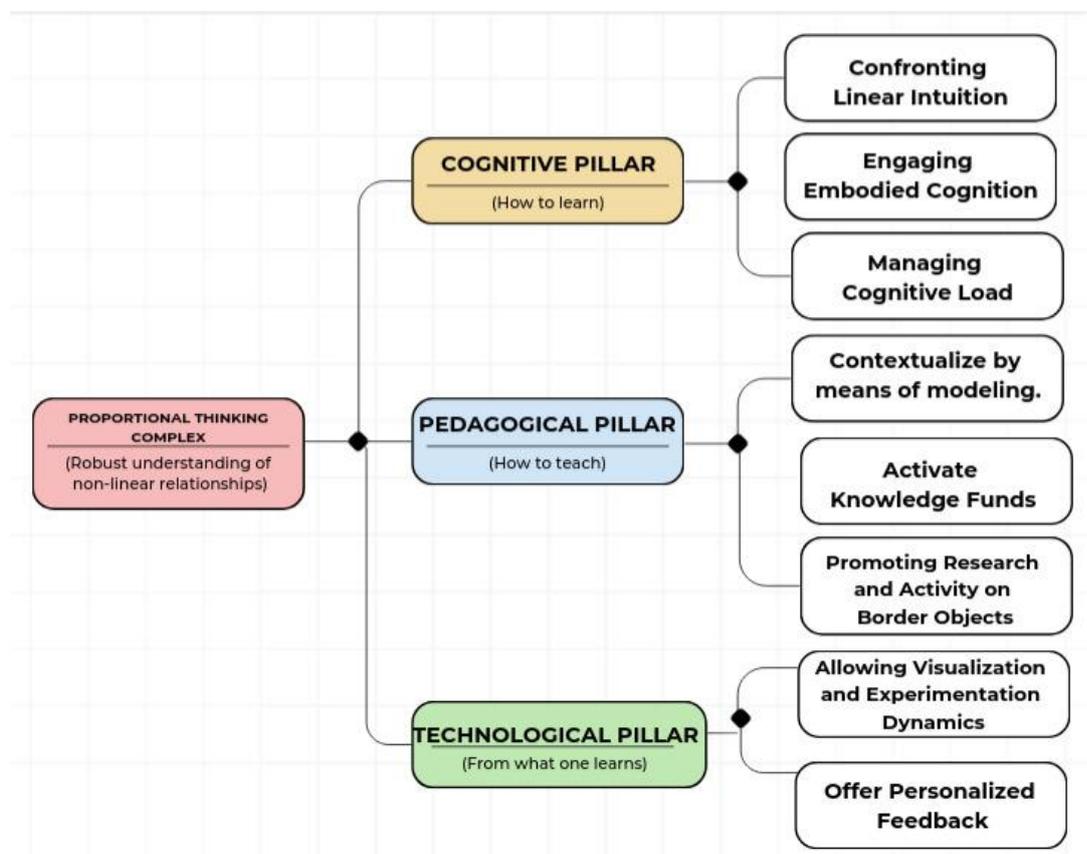
The pedagogical pillar, the bridge to meaning-making and critique, translates cognitive foundations into intentional teaching actions. It is responsible for contextualizing knowledge, making it meaningful through modeling real-world problems, valuing community knowledge (funds of knowledge), and stimulating investigation. It acts as the engine of meaning. It is this pillar that answers the question "why learn this?", connecting area-volume relationships to issues of social, environmental, and technological justice (Gutstein, 2006). It ensures that the development of complex thinking is not an end in itself but a tool for critically reading the world.

The technological pillar positions itself as an essential mediator of learning. Through digital technologies, such as dynamic geometry environments and intelligent tutoring systems, it materializes abstractions, enabling the visualization and experimentation of relationships that cannot be fully understood with static resources. It acts as a catalyst and facilitator. It operationalizes the principles of the other two pillars: it materializes cognitive conflicts (supporting the cognitive pillar) and enables investigations and modeling in complex contexts (operationalizing the pedagogical pillar).

The cognitive pillar: foundations of multidimensional learning

This pillar is grounded in findings from cognitive sciences about how the human mind processes, understands, and internalizes complex, non-linear mathematical relationships. Its objective is to provide the scientific basis for designing learning experiences compatible with the brain's architecture, facilitating the overcoming of innate obstacles.

The cognitive pillar is composed of three principles: Principle 1 - Confront Linear Intuition; Principle 2 - Engage Embodied Cognition; and Principle 3 - Manage Cognitive Load.

Principle 1, confront linear intuition, recognizes the brain's neurocognitive predisposition to simplify complex relationships, treating them as linear (Dehaene, 2020). The resulting pedagogical strategy is the intentional creation of productive cognitive conflicts, situations where the

student's linear prediction is challenged by empirical or visual evidence proving the quadratic or cubic nature of the relationship.

For example, consider the following situation: after students linearly predict that the volume of a cube will double if its edge doubles, they are asked to construct the cube with concrete material (such as linking cubes) to empirically verify that the volume octuples. The dissonance between prediction and concrete evidence creates a fertile didactic situation (Brousseau, 2002).

Or, in a problem about the increase in the area of a circle, students may predict that doubling the radius doubles the area. Using a dynamic geometry environment to visualize the area growing by a factor of four provides immediate visual feedback that confronts the inadequate linear intuition.

Principle 2, engage embodied cognition, leverages the deep connection between body and mind, positioning sensorimotor interaction as a fundamental resource for abstraction (Abrahamson et al., 2023). Activities involving gestures, manipulation of objects, and actions in physical space are designed to build a corporeal base that facilitates the understanding of abstract mathematical concepts.

To understand the relationship between the height of a cylinder and its volume, students can use transparent cylindrical containers with the same base and different heights. Filling these containers with water or sand makes the direct linear relationship between height and volume a physical and visual experience, contrasting with the cubic relationship involved in varying the radius.

Or, also, the gesture of "unrolling" the lateral surface of a cylinder to form a rectangle is a bodily action that helps understand the formula for lateral area (circumference length x height), connecting a three-dimensional representation with its two-dimensional net.

Principle 3, manage cognitive load, considers the limitations of working memory in processing multidimensional information (Sweller, 2020). This principle guides instructional segmentation, initially presenting

problems that vary a single parameter and then gradually introducing the simultaneous variation of multiple dimensions, thus avoiding overload and facilitating solid knowledge construction.

For illustration, in the study of prisms, problems are introduced that vary only the height, keeping the base area constant. Once understanding of this linear relationship is consolidated, variation of the base area is introduced, keeping height constant. Finally, problems involving the simultaneous variation of both parameters are presented.

Problems such as using graphical representations that isolate variables. A graph showing the volume of different cylinders as a function of the radius (with constant height) evidences the cubic curve, while a graph of volume as a function of height (with constant radius) evidences a straight line, helping the student discriminate the nature of each relationship.

The pedagogical pillar: strategies for knowledge mediation

This pillar translates cognitive foundations into intentional didactic actions, creating learning environments that are simultaneously investigative, meaningful, and socially relevant. It is responsible for operationalizing theory in educational practice.

The pedagogical pillar is composed of the principles: Principle 4 - Contextualize through Modeling and Critical Issues; Principle 5 - Activate Funds of Knowledge; and Principle 6 - Foster Investigation and Boundary Object Activity.

Principle 4, contextualize through modeling and critical issues, links formal mathematical content to real-world phenomena and socially relevant issues, such as environmental justice and sustainability. This approach, aligned with Critical Mathematics Education, gives meaning to knowledge and develops students' capacity to use mathematics to read and interpret their reality critically (Gutstein, 2006; Skovsmose, 1994).

As an example, one can investigate the area-volume relationship in the context of energy efficiency in residences. Why does a house with a more spherical shape (smaller surface area relative to volume) than another with the same volume have a lower heating cost? This question connects mathematics to sustainability problems.

Or, one can critically analyze the cost-benefit relationship in choosing product packaging. A cylindrical can that is "taller and thinner" *versus* "shorter and wider" for the same volume implies different amounts of material (surface area) and, therefore, different environmental impacts and costs. Students are challenged to mathematically model the problem to make an informed decision.

A common confusion between representation and modeling of knowledge and reasoning consists of believing that explanations are inherent to a specific representational formalism and that their relevance depends exclusively on the expressive capacity of that formalism.

However, a limited formalism will never allow explaining phenomena that exceed its expressiveness. For example, a system described only by linear differential equations will never be able to represent and, therefore, explain the non-linear and self-oscillating behavior of certain dynamic systems.

Similarly, in Mathematics Education, if the teacher uses only a formalism based on linear proportionality, this formalism will not allow explaining phenomena whose structure is non-proportional, such as the growth of area (which grows with the square of the dimension) or volume (which grows with the cube).

Thus, trying to explain why "doubling the side of a square does not double the area," using only proportional reasoning, will inevitably lead to an incomplete or mistaken explanation. Therefore, the choice of formalism, whether linear, geometric, algebraic, or functional, is decisive for

constructing comprehensible explanations adequate to tasks and learning styles.

Principle 5, activate funds of knowledge, builds bridges between school knowledge and the mathematical knowledge already present in students' daily, cultural, and community practices. It connects formal knowledge to the mathematical knowledge present in students' communities (González et al., 2005), promoting a culturally relevant and meaningful Mathematics Education. By valuing and incorporating this knowledge, learning becomes culturally relevant, part of a collective heritage, and meaningfully anchored in lived experience.

For example, in a community with a pottery tradition, explore volume and surface area relationships in modeling clay pots. The experienced artisan knows, intuitively, that a wider, shorter pot heats up faster in the kiln (greater contact area) than a tall, narrow pot of the same capacity. The lesson can start from this practical knowledge to formalize the underlying mathematical relationships.

Similarly, in agricultural communities, calculating the volume of silos or cylindrical water tanks is common practice. The pedagogical activity can start from this knowledge to investigate, for example, how many water tanks of a certain size are equivalent to a silo of greater capacity, exploring the cubic scale.

Principle 6, foster investigation and boundary object activity, creates inquiry-based learning spaces where students are challenged to test hypotheses, explore patterns, and generalize relationships (Akkerman & Bakker, 2020). To do this, it promotes movement between different representational registers (physical, graphical, symbolic, digital) through boundary objects or artifacts, such as physical models and software, which facilitate communication and meaning construction between different perspectives.

For example, students can be provided with an "investigation kit" containing physical models (linking cubes), digital spreadsheets (to record measurements and generate graphs), and dynamic geometry software. The student is challenged to predict, test, and generalize the relationship between the edge and volume of the cube, using these different objects to build a multimodal understanding.

Also, the same packaging optimization problem can be approached through physical mock-ups (tangible boundary object), simulations in GeoGebra (digital boundary object), and algebraic equations (symbolic boundary object). The translation between these registers is where conceptual understanding consolidates.

The technological pillar: mediation for deep understanding

This pillar positions digital technology not as an end in itself but as an essential mediating tool to operationalize cognitive and pedagogical principles (Borba & Villarreal, 2005).

The technological pillar is composed of the principles: Principle 7 - Enable Dynamic Visualization and Experimentation; and Principle 8 - Offer Personalized Feedback.

Principle 7, enable dynamic visualization and experimentation, uses digital environments, such as dynamic geometry software, to make dimensional covariances visible and manipulable (Bairral & Barreira, 2017). These environments allow students to vary parameters in real time and immediately observe effects on area and volume, transforming static relationships into interactive experiences that challenge intuition and reveal patterns.

One possibility is when a student can manipulate a slider to alter the radius of the base of a cylinder and observe in real time how the height adjusts automatically to maintain constant volume. This direct manipulation

makes the inverse dependency relationship between the variables (for constant volume) immediately tangible.

And, in an augmented reality environment, students can "stretch" a virtual cube with their hands, observing how the volume indicator grows in a non-linear (cubic) manner relative to the linear growth of the edge. This immersive experience creates a powerful link between bodily action and mathematical abstraction.

Principle 8, offer personalized feedback, leverages the potential of intelligent tutoring systems and adaptive learning environments to identify specific error patterns, such as applying linear reasoning to non-linear situations.

The principle consists of using this data to provide contextualized feedback and personalized interventions that directly confront each student's alternative conceptions, guiding them in restructuring their thinking. It leverages the potential of Educational AI (Holmes et al., 2019) to identify and intervene in specific alternative conceptions in real time.

It can be illustrated with an intelligent tutoring system that can detect when a student systematically multiplies the volume of a sphere by two when doubling its radius. The system can then automatically present an animation decomposing the sphere into infinitesimal pyramids, visually illustrating the cubic nature of the relationship.

Another context is: in an adaptive platform, if a student makes a typical linearization error when calculating the volume of a pyramid, the system can offer a scaffolding problem (pedagogical or learning support), such as filling a pyramid with water and pouring its contents three times into a prism of the same base and height, so that they discover the relationship V_pyramid = (1/3) V_prism.

Operational synthesis: the framework in action

The strength of the IEF lies in the interaction between its pillars, which do not act in isolation (Almeida & Kowalek, 2024). Figure 3 illustrates in a dynamic and integrated manner how the IEF can be operationalized in the classroom, using a packaging optimization project as an example. This case demonstrates how the three pillars of the framework (cognitive, pedagogical, and technological) act synergistically and non-linearly, promoting deep and contextualized learning.

Initially, the pedagogical pillar is activated by Principle 4 (contextualization through modeling and critical issues), by situating the problem in the real context of sustainability and production costs. Students are challenged to compare cylindrical cans that are "tall and thin" with "short and wide," considering fixed volume and material and environmental implications.

By using spreadsheets and dynamic geometry software to test different dimensions, learners engage the technological pillar, under Principle 7 (dynamic visualization and experimentation), allowing them to manipulate variables and observe area-volume relationships in real time.

Concomitantly, a cognitive conflict arises between the initial linear intuition ("a larger package should cost proportionally more") and the data generated by the modeling. This crucial moment activates the cognitive pillar through Principle 1 (confront linear intuition), creating a productive dissonance that drives conceptual restructuring.

The discussion about environmental impact reinforces the pedagogical pillar again, while the transition between physical representations, spreadsheets, and modeling in GeoGebra characterizes a boundary object activity (Principle 6), facilitating translation between registers and the construction of multimodal understanding. Figure 3 shows the IEF Packaging Optimization Project.

Figure 3

IEF Packaging Optimization Project
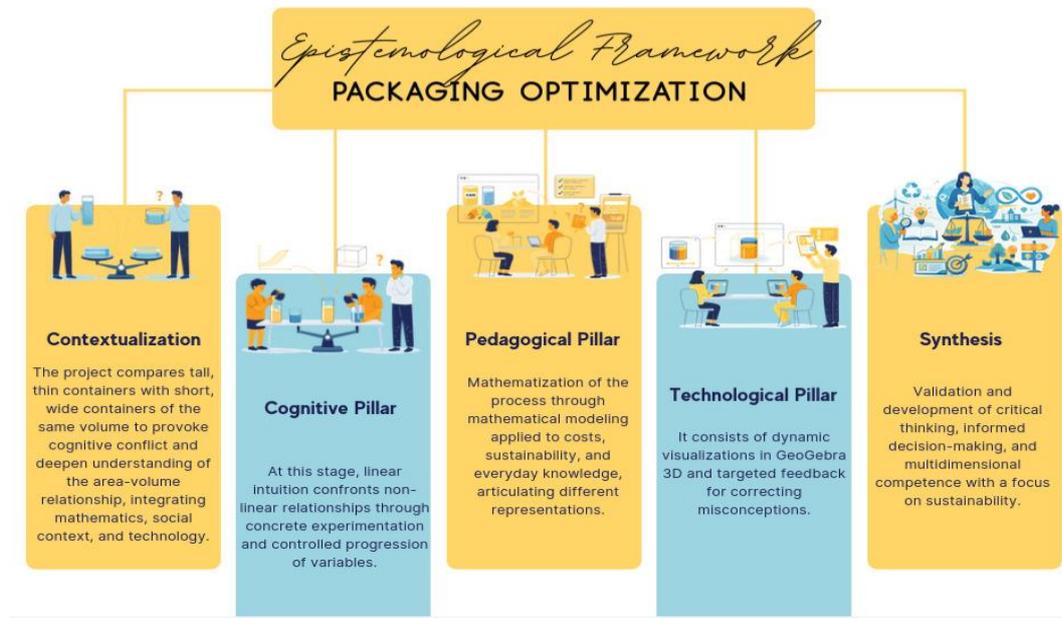
Thus, its interactive and recursive nature is evidenced: the pillars do not follow each other linearly but feed back into each other, creating a continuous cycle of investigation, reflection, and critique. This project, as a didactic example, concretizes the shift from teaching based on memorizing formulas to multidimensional learning, empowering students to apply complex proportional thinking to real and socially relevant problems.

Final considerations: contributions and horizons

This research started from a historically consolidated epistemological challenge, the "linear proportionality illusion," and culminated in the proposal of an IEF for teaching area-volume relationships.

The main contribution of this article lies not in the discovery of a new learning obstacle, but in the systematization of a theoretical-methodological path for its overcoming. The IEF offers a coherent structure that articulates, in an innovative way, domains traditionally fragmented: the neurocognitive bases of learning, critical and investigative pedagogical strategies, and the mediating role of digital technologies.

The framework demonstrates that overcoming inadequate linear thinking is more than a conceptual acquisition; it is an epistemological reorganization of the student's gaze on the quantitative world.

By integrating the confrontation of intuitions with socially relevant contextualization and intelligent technological mediation, the proposal transcends the specific scope of area-volume relationships.

It provides a theoretical prototype for teaching other complex and counterintuitive mathematical concepts, where multidimensionality and non-linearity are central.

The IEF, in its current form, is a robustly grounded proposal but lacks systematic empirical validation. This limitation opens a promising range of future investigations in the areas of Implementation Studies and Teacher Education, in different cultural contexts; and in the exploration of new technological strategies, namely:

1. Implementation Studies (Design Research): Design-based research is the natural next step. Cycles of elaboration, implementation, and analysis of teaching sequences based on the IEF in real classrooms would allow for its refinement, identifying its points of success and operational difficulties, and generating specific design principles for the basic education context.

2. Investigations on Teacher Education: It is crucial to investigate how teachers appropriate this framework. What knowledge (content, pedagogy, technology) and what epistemological beliefs are necessary for a teacher to operationalize the IEF? Studies involving initial and continuing teacher education, using the framework as a planning and analysis tool, are a field of research.

3. Research in Different Cultural Contexts: The funds of knowledge proposal, integrated into the IEF, demands local and specific investigations. How can the framework be adapted and enriched

based on the mathematical knowledge of riverside, indigenous, urban, or rural communities? Ethnographic and participatory research can answer this question, making the proposal even more inclusive and relevant.

4. Long-Term Impact Evaluation: Longitudinal studies that follow students subjected to interventions based on the IEF could evaluate the permanence and transferability of complex proportional thinking to other domains of mathematics and science, investigating whether overcoming the linear illusion in a specific context generalizes to others.

5. Exploration of New Technological Frontiers: The rapid advancement of generative artificial intelligence and educational metaverses presents new questions. How can these technologies be leveraged (or capitalized on) to create even more immersive and personalized modeling and cognitive conflict environments, aligned with the principles of the IEF?

Although the IEF offers a robust theoretical path, its large-scale implementation faces challenges such as inequality in access to digital technologies, initial and continuing teacher education, and time and resource constraints in Brazilian public schools. Future research should consider these factors to adapt the framework to real contexts.

In summary, this article does not offer a definitive solution but proposes possibilities for academic research. The IEF invites the Mathematics Education community to a collective effort of investigation, development, and critique. The challenge of forming citizens capable of thinking multidimensionality and critically in a complex world is monumental. Through bold theoretical proposals and their subsequent submission to the test of practice, a decisive step is taken to face it.

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