Learners’ conceptualisation of the sine function during an introductory activity using *sketchpad* at grade 10 level

MICHAEL DE VILLIERS[1]
JANEESHLA JUGMOHAN[2]

Resumo
Este artigo descreve um estudo que investigou e analisou a conceitualização dos estudantes do Grau 10, sobre a função seno, durante o desenvolvimento de uma atividade introdutória, usando o Sketchpad. Na análise da compreensão dos estudantes sobre a função seno, intuições corretas, bem como concepções equivocadas em sua matemática foram identificadas. O uso do Sketchpad para introduzir a função seno mostrou ser uma atividade razoavelmente bem sucedida e significativa no desenvolvimento da compreensão de alguns aspectos significativos da função seno.
Palavras chave: *Sketchpad; Análise de compreensão; Função seno.*

Abstract

This paper outlines a study that investigated and analysed learners’ conceptualisation of the sine function during an introductory activity, using Sketchpad at Grade 10 level. In the analysis of the learners’ understanding of the sine function, correct intuitions as well as misconceptions in their mathematics were identified. The use of Sketchpad to introduce the sine function proved to be a reasonably successful and meaningful activity in developing understanding of some significant aspects of the sine function.
Keywords: *Sketchpad; Conceptualisation analysis; sine function.*

Introduction

The experience of one of the researchers teaching trigonometry at Grade 10-12 level and her observation of other teachers and their learners, support the finding that the mathematical knowledge of secondary school learners are dominated by content- and teacher-centred pedagogies (Boaler, 1997). Many learners have been observed remarking on the difficulty of learning trigonometric functions, and many colleagues at high school have expressed concern about helping learners make sense of this topic. However, very little research has been done to explore the learning and understanding of trigonometry. Many learners appear to have little understanding of the underlying trigonometric principles and thus resort to memorising and applying procedures and rules, while their procedural success masks underlying conceptual gaps or difficulties.

[1] University of KwaZulu-Natal - profmd1@mweb.co.za
[2] University of KwaZulu-Natal - janeeshla@yahoo.com

The main motivation for this study was to address the gap in the research literature on learners' learning and understanding of trigonometric concepts, specifically at the introductory level to trigonometry using dynamic geometry software.

In a study by Blackett and Tall (1991), the initial stages of learning the ideas of trigonometry, are described as fraught with difficulty, requiring the learner to relate pictures of triangles to numerical relationships, to cope with function ratios such as sinA = opposite/hypotenuse.

The findings of a study by Pournara (2001), suggest that learners need to be able to shift between ratio and function orientations, that operational conceptions of trigonometric functions may override learners' structural conceptions of trigonometric ratios, and that there are problematic aspects to the current dominant focus of teaching of algorithmic methods and procedures in trigonometry.

In a 'ratio orientation', the mathematical concept most central to a ratio orientation is the right-angled triangle. Other mathematical elements, he states, include: definitions of trigonometric ratios in terms of the lengths of the sides a right triangle; the relationships between the ratios – particularly the quotient ratios such as $\tan\theta = \sin\theta/\cos\theta$ and the inverse ratios such as $\mathrm{cosec}\theta = 1/\sin\theta$; and typical Grade 10 tasks where learners are given a point in the Cartesian plane and are asked to determine values of particular trigonometric ratios and expressions involving these ratios.

Such problems, Pournara (2001) states, usually require learners to set up a right-angled triangle and to make use of the Theorem of Pythagoras. The angle is back-grounded in a ratio orientation and it merely serves as a reference point to locate the opposite and adjacent sides of the triangle, and must be positioned in the triangle before the opposite and adjacent sides are assigned. Thereafter the angle plays no further part in the problem.

In contrast, a 'function orientation' to trigonometry is based on the notion of the input-processing-output relationship, similar to algebraic functions, or functions in general (Pournara, 2001). A strong function orientation makes explicit that the process links the input to the output, and vice versa, whereas a weak function orientation does not make the connection explicit.

A function orientation focuses on three aspects: the angle, the trigonometric operator (e.g. sin, cos, tan) and the function value. This orientation is dependant on an understanding that the trigonometric operator maps an angle to a real number in a many-to-one relationship. The trigonometric operator, according to Pournara (2001), is seen as exactly that – an operator. In the function definition, function values are not defined in terms of the sides of a triangle. He further goes on to say that a function orientation is more likely to promote a dynamic view of trigonometry than would a ratio orientation because a function orientation assumes that the independent variable – the angle in this case – can take on many values and the resulting function value reflects clearly the effect of changing the angle.

Additional mathematical elements of a function orientation include the notions of periodicity, amplitude, asymptotes and discontinuity; as well as the representation of trigonometric functions by means of table, equation, or graph. It is possible that South African learners may develop a distorted view of trigonometric functions because the trigonometry curriculum places a great deal of emphasis on algebraic solutions of trigonometric equations and only studies the graphs of *sine*, *cosine* and *tangent* functions. As a result, learners may develop a function orientation that is limited to the graphical representation of these functions, and only finding input values for given output values (solving equations). Pournara (2003), argues that this is limited if learners are to develop a broader understanding of functions and hence be able to draw links between trigonometric functions and other functions in the curriculum like linear, quadratic, cubic and exponential functions.

He further argues that the implicit function notion of trigonometry may form a more suitable foundation on which to build trigonometric concepts and he has therefore suggested that the current ratio approach to introducing trigonometry in South African schools be replaced with a function approach. For example, instead of starting with introducing trigonometry as ratios of sides of right triangles as has traditionally been the case, one might consider starting straight away with a unit circle definition of trigonometric ratios as functions of the various ratios of the coordinates x and y as a point moves on the circle.

However, a historical perspective suggests that the function approach might perhaps not be suitable as an elementary starting point for young learners. In general, a historical glimpse provides a genetic view of how humans 'learned' mathematics, and can suggest potentially viable learning trajectories as well as providing valuable insight into possible conceptual difficulties that children may encounter (compare Polya, 1981; Freudenthal, 1973, Donovan & Bransford, 2005). For example, it is worth noting that the ancient Greeks did not even have any definition of functions and obviously not the standard unit circle definition of trigonometric functions used today. In fact, from antiquity till about the Renaissance, trigonometry was essentially the study of numerical relations between the arc lengths and chord lengths of a circle (of which the right triangle is a special case). Nevertheless they developed and applied trigonometry to the solution of triangles on the plane and the sphere using various identities at a fairly advanced level. The calculations in Ptolemy's famous book the *Almagest* ("the greatest") in approximately 150 AD were so accurate that it was in use by the civilized world for over 1000 years. In this book he used the theorem named after him, Ptolemy's theorem, to calculate trigonometric tables, accurate to about 5 decimal places (Boyer, 1968, p. 183-187; Katz, 2004, pp. 89-93). This theorem is entirely geometric and a generalization of Pythagoras's Theorem, and states that the sum of the products of the opposite sides of a cyclic quadrilateral is equal to the product of the diagonals.

Regiomantus's *De Triangulis* ("on triangles") published in 1533 was possibly the first "pure" systematic treatment of trigonometry completely separated from astronomy and other real world applications like navigation (Boyer, 1968, p. 303; Katz, 2004, pp. 232-233). Much like Ptolemy, he based his trigonometry on the sine of an arc, defined as the half chord of double the arc. Interestingly, he did not use the tangent function, even though tangents were known in Europe in translations of Islamic astronomical work. Only in 1635, did Gilles de Roberval produce the first sketch of half an arch of a sine curve (Boyer, 1968, p. 390). This was the first indication that trigonometry was now slowly evolving from its computational roots towards a modern function approach.

The need for more formally clarifying what a function itself is, arose from the dramatically increasing application from the Renaissance onwards, of many different mathematical functions as well as the differential and integral calculus, to scientific problems of motion, forces, etc. This dramatic increase had been made possible by the development by Cardano, Viète, Descartes, and others, of modern algebraic symbolism and notation, for example, $y = ax^2 + bx + c$, as well as the Cartesian co-ordinate system, which simplified the antiquated methods and symbolism of the ancient Hindus, Greeks, and Arabs. According to Boyer (1968, p. 444) it is to Leibniz (1676) that the very word "function' in much the same sense as we use it today, is due (though his definition was not the set-theoretic one mostly used today).

More specifically, the need to redefine trigonometric functions also arose during the same time period with the increased study and analysis of many different kinds of periodic functions in science, from the movement of pendulums to elastic springs, biological rhythms, and other oscillating functions. This necessitated the redefining of trigonometric functions in terms of the modern unit circle definition, as the view of trigonometry simply as ratios of sides of right triangles was inadequate to model periodic functions. In fact, the *Harmonia mensurarum* of Roger Cotes (1682-1716) published posthumously in 1722 was among the first works to recognize the periodicity of the trigonometric functions (Boyer, 1968, p. 467).

Apart from the increased application of trigonometry to real world problems, mathematicians were also increasingly applying trigonometry to the solution of polynomial equations of higher order, contributing to the gradual "algebraization" of trigonometry. A famous example is the early use by Viète in 1593 to quickly finding 23 solutions to a polynomial of degree 45 in a challenge by van Roomen in the court of King Henry IV (Derbyshire, 2006, p. 87).

Euler in his seminal book on Analysis, *Introductio in analysin infinitorum* of 1748, defined a function of a variable quantity as "any analytic expression whatsoever made up from that variable quantity and from number or constant quantities". This book gave the first strictly analytical treatment of the trigonometric functions devoid of earlier geometric connections, deriving for example the power series for the sine and cosine from the binomial theorem and complex numbers, as well as expressing the sine and cosine in the well-known Euler identities of today in terms of complex numbers and *e*, for example, $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$ and $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ (Boyer, 1968, pp. 485-486; Katz, 2004, pp. 351).

This relatively late historical development of the formalization of the function concept, and associated redefining of trigonometric functions, not only suggests that this conceptual transition from the ancient definition of trigonometry as ratios of the sides of a right triangle to a function definition is not a trivial matter to be taken lightly, but that the function approach might conceptually be more sophisticated for young learners to grasp at the very start of trigonometry.

More-over, one could even question the relevance of the formal function definition at Grade 11 and 12 of trigonometric functions in terms of the unit circle, given that currently, applications to periodic functions in various real-life settings are virtually non-existing. Apart from the two examples mentioned earlier, however, there are many examples of contexts that could suitably be modelled at school using trigonometric functions such as rotating wheels, tides, the cycles of the moon, mechanical vibrations, orbiting planets, etc.

On the other hand, it seems learners should still have the concept image of trigonometry as ratios of the sides of a right triangle, as this ‘ratio orientation’ has many useful applications by itself, especially in land surveying, building construction, navigation and astronomy. Some of these have traditionally been used in teaching and are common in curriculum materials. In fact, historically these are precisely the types of contexts from which the ancient Egyptians and Babylonians developed this particular meaning of trigonometry, and which the ancient Greeks later developed further and systematized. Hence, the name “trigonometry” which comes from the Greek *trigōnon* "triangle" and *metron* "measure".

Lastly, a welcome new emphasis in Curriculum 2005 (at least “officially”) is on including historical aspects of mathematics where appropriate. Perhaps an aspect currently missing from the trigonometry curriculum at school, and mathematics teacher education courses, is some historical background about the various ways of calculating the trigonometric function values from the earliest times of Euclid and Ptolemy to the modern approach, which uses infinite series.

Research Questions

Data was collected from a high school situated in a middle-class area of Reservoir Hills (KZN), by task-based interviews and questionnaires. Given a self-exploration opportunity within *Sketchpad*, designed to begin developing both a ‘right triangle ratio’ and ‘functional’ understanding, the study by Jugmohan (2005) investigated Grade 10 learners’ conceptualization and understanding of the sine function during an introductory activity within the first quadrant. Specifically, the following research questions were investigated:

1) What understanding do learners develop of the sine function as:

a) a ratio of the sides of a right triangle?

b) a function output of an angle as independent variable or input?

c) an increasing function in the first quadrant?

d) a function that non-linearly increases from zero to one as the angle increases from $0°$ to $90°$?

2) And lastly, what understanding do they develop of the similarity of all right triangles with the same reference angle, as the basis for the constancy of trigonometric ratios?

Theoretical Framework

This research was informed by a constructivist perspective on learning (e.g. Piaget, 1970; Skemp, 1979) which assumes that concepts are not taken directly from experience, but from a person's ability to learn from and what s/he learns from an experience depends on the quality of the ideas that s/he is able to bring to that experience. According to Olivier (1989), *"knowledge does not simply arise from experience. Rather it arises from the interaction between experience and our current knowledge structures."*

The learner is therefore not seen as passively receiving knowledge from the environment. A basic assumption is that knowledge cannot be transferred ready-made and intact from one person to another. The learner is always an active participant in the construction of his or her own knowledge. This construction activity, according to Olivier (1989), "*involves the interaction of a child's existing ideas and new ideas, i.e. new ideas are interpreted and understood in the light of that child's own current knowledge, built up out of his or her previous experience."* Children do not only interpret knowledge, but they organize and structure this knowledge into large units of interrelated concepts called *schemas* by Skemp (1979) and others. Such schemas of interrelated ideas in the child's mind are valuable intellectual tools, stored in memory, and which can be retrieved and utilized. Learning then basically involves the interaction between a child's existing schemas and new ideas.

Constructivist theory is thus based on the view that knowledge is made and not passively received – it is assumed to be constructed by an active cognizing subject rather than just directly transmitted by a teacher or text. From a constructivist point of view, according to Von Glasersfeld (1987), it's not sensible to assume that any powerful cognitive satisfaction springs from simply being told that one has done something right, as long as someone else assesses 'rightness'. To become a source of real satisfaction, 'rightness' must be seen as the fit with an order one has established oneself. This cognitive satisfaction appears to be best gained through investigative work in learner-centred teaching, which is often made more effective when mediated by a computer.

The Interview and Microteaching Experiment

This study used an action-research based approach, specifically a micro teaching experiment in an interview setting where each individual learners' progress was carefully audio-taped and transcribed. The learner interviews were structured and task-based (Goldin, 2000), and one of the salient features of such interviews, is that the interviewer and interviewee(s) interact in relation to a task(s) that is/are presented by the interviewer in a pre-planned way.

During the interview, learners were also probed to determine how each child experienced and conceptualised each activity. The objective was to see what learning took place, and to analyse the nature and quality of that learning, and specifically what concepts of the sine function they had formed. A further objective was to examine to what extent *Sketchpad* had assisted in their conceptualization.

A pilot interview was first carried out. Several adaptations were made thereafter to both the activity and the questions. The introductory task to the sine function that the learners had to work through was based on a circle with arbitrary diameter drawn with *Sketchpad* within a Cartesian-coordinate system. This sketch was presented ready-made to learners, mainly to interact with and provide the basis of the introductory activity. All measurements of lengths, coordinates and calculations were clearly visible on the screen of the computer, so that learners could dynamically view the changes to these as they dragged a point on the circle.

In total six learners were interviewed in the final version and each interview was approximately 60 to 90 minutes long and each was audio taped. Although these questions were structured around the research questions, it also allowed for variation in expected responses from the learners, and further probing was done in particular cases. Learners wrote out the answers to questions at each step of the experiment.

In the final stage, the data was analysed. This required the systematic grouping, analysis and summarising of the responses, which provided a coherent organising framework that explained the way each learner produced meaning whilst working through the tasks provided.

The researcher first took each learner quickly through a brief session, which described the clicking and dragging modes for using S*ketchpad* effectively:

- ❖ POINT: Move the mouse until the tip of the cursor is over the desired object
- ❖ CLICK: Press and release the mouse button quickly
- ❖ DRAG: Point at the object you wish to drag, then press and hold down the mouse button. Move the mouse to drag the object, then release the mouse button.

The learners seemed to quickly grasp the clicking and dragging operations of S*ketchpad.* This was presumably due to the fact that all the learners had already done computer literacy at school and had computers at home, which they used for projects and assignments for school (but not previously in mathematics). They also referred to playing computer games, and thus their ability to use the mouse was good.

After the introduction, they were asked to complete a set of tables for $r=1, r=2, r=3$ and $r=4$. An example of sketch they used in order to complete the table for r = 3 is given in Figure 1. By dragging point *G* along the fixed circle, the learners could move it until the desired angle *GOB* was formed, while measurements and calculations on those measurements are dynamically updated as point *G* is moved.

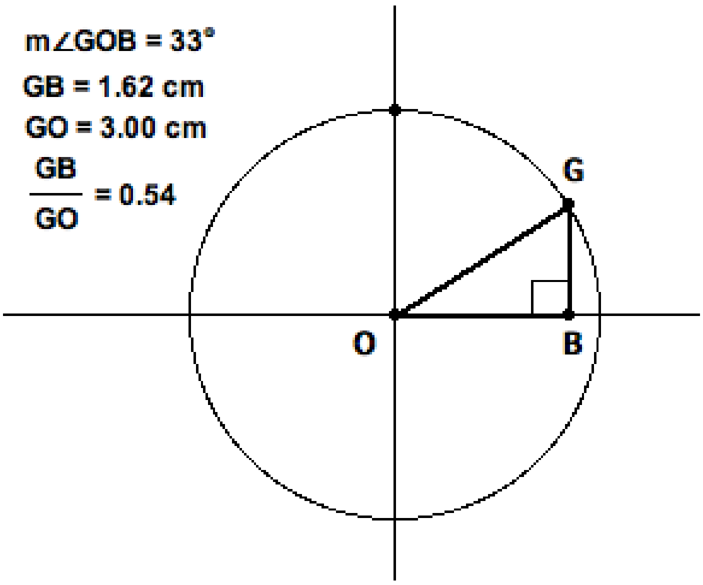

Figure 1: *The Geometer's Sketchpad Screen*

Learners initially found the information displayed on the upper left hand corner a little confusing as the labels in the sketch did not match those of the table. In retrospect, it would have been better to have used the labels θ, *y* and *r* respectively for angle *GOB*, *GB* and *GO* in the sketch, as well as for the variables displayed in the upper left corner.

To enable the learners to correctly identify the relevant variables, the interviewer needed to assist some of the learners by guided questioning as follows:

Omika: *What has to be* 10° *?*

Researcher*: What do you think* 10° *represents?*

Omika: *An angle.*

Researcher: *Now looking at the diagram on the screen, are there any angles we are dealing with?*

Omika*: Yes ….. these two* (pointing them out on the screen*).*

Researcher*: So which one do you think has to be* 10° *?*

Omika: *This one …..* (correctly pointing it out*)…… angle GOD.*

An example of a learner's completed table for *r* = 1 is shown in Figure 2.

θ	$\frac{y}{r}$	$\sin\theta$
10°	0,17	0,18
20°	0,34	0,34
30°	0,50	0,49
40°	0,64	0,65
50°	0,77	0,77
60°	0,86	0,86
70°	0,94	0,94
80°	0,99	0,99
90°	1,00	1,00

Figure 2: *Perusha's Table* 1

Results and Analysis

Question 1

After the table for $r = 1$ was completed, each learner was asked:

a) *"What do you notice about the y and r values respectively as the angle θ increases?"*

Four learners said that *y* increases, but *r* remains the same as follows:

Perusha (using the dynamic *Sketchpad* sketch): "*As it increases, y gets higher and r stays the same*".

Vishen (using the dynamic *Sketchpad* sketch): "*When it goes up (dragging point G), the y value is increasing … When the angle increases r stays the same*."

Nadeem (switching from table to using dynamic *Sketchpad* sketch): "*r value stays the same, and the y value increases*."

Mayuri (referring her completed tables): "*y and r changes as the angle increases, but the spaces in between them is not equal*."

When Mayuri was asked what she meant by the spaces in between, she replied referring to the table: "*like if this is 10, 20 (pointing at the angles), then this increases by like 20 (pointing at the difference of 0.16/0.17 between the first two ratios), but not all the time*." When asked again about what happened to the *y* and *r* values as the angle increased from 0 to 90 degrees, she then responded (smiling): "… *y is increasing yes, and r is remaining the same*."

Of note here is that Mayuri was the only student who did not use the dynamic *Sketchpad* sketch to answer this question, focussing only on the table, and apart from correctly observing that *y* increased, she clearly noticed from the table that it was a non-linear increase.

The other two students, Suren and Omika, replied that both *y* and *r* increased, apparently not realising that *r* did not increase. Neither of them checked their responses using either *Sketchpad* or the table, and may have just made a slip. However, the interviewer unfortunately did not probe this further.

Next the learners were asked the following question:

b) *"What do you notice about the values of $\frac{y}{r}$ and $\sin\theta$ in Table 1?"*

All the learners seemed more confident and clear about what was being asked, and responded by saying that $\frac{y}{r}$ and $\sin\theta$ are the same or almost the same. Four learners remarked they were 'almost the same', picking up on the small differences in the decimal displays in a few places (because of round-off errors caused by dragging the point G to the required angles rather than accurate construction). Here are some examples of their responses in this regard:

Mayuri: "*they are almost the same*."

Omika: "*… it's like the same … it's not exactly the same … some are exactly the same, but some are like below or above the value*."

Question 2

Next the learners were asked the following question before given the *Sketchpad* figure for $r = 2$ to see if they could conjecture by themselves that the ratios for the same reference angles would remain unchanged:

"What do you think will happen to the above ratios if we increase r to 2? Why?"

Initially all six learners replied that they thought the ratio $\frac{y}{r}$ would increase. The following are examples, typical of their responses:

Perusha: "*… the circle will increase … er, the … I suppose the answers for the ratios will increase. The total degrees will get higher by probably 2.* $\frac{y}{r}$ *will get higher*."

Vishen: "*… so I think if we change r to 2, the sine value and the* $\frac{y}{r}$ value *will increase proportionate to that*."

Mayuri: "*Each figure will increase by 4 … each ratio … around about the same …*"

All right triangles with the same reference angle θ are similar since they have two corresponding angles equal, i.e. θ and a right angle. Conceptually, therefore the whole of trigonometry is underpinned by this similarity of right triangles with the same reference angle θ, resulting in the ratios between different sides remaining constant, no matter the size of the right triangle.

Clearly, at this stage none of the learners anticipated this fundamental similarity, realizing that an enlargement changes all segments in a figure by the same scale factor, and therefore ratios of segments do not change under enlargements. Or in other words, that an enlargement preserves similarity. It seemed that Perusha might even have had the impression that an enlargement also changed angle size, but this was unfortunately not pursued further during the interview.

Question 3

After using given *Sketchpad* sketches to complete a similar table as before for each of $r = 2$, $r = 3$ and $r = 4$, learners were asked the following:

"For any given angle, what do you notice about the corresponding values of $\frac{y}{r}$ *in each table for r = 2, r = 3 and r = 4?"*

All six learners now observed, some with surprise as it conflicted with their initial expectation, that the ratios for the same angle were the same or almost the same, irrespective of the value of *r*. For example:

Perusha: "*They are almost the same*."

Omika (sounding surprised): "*Won't it be the same? … Because every time I hold it at 10° for example, I notice both values are the same. … The values are similar, it's either one below or one below* (referring to the 2nd decimal differences)"

Nadeem (expressing surprise): "*They are the same. The values of y/r in each table is the same. … Now I realise that the y/r will still have the same ratio, because when you increase the r to 2, y will increase as well."* Nadeem then proceeded to write the response shown in Figure 2, who was the only learner at this stage clearly expressing an awareness that $\frac{y}{r}$ remained constant, since the *y* and *r* values both increased proportionally. His written response is shown in Figure 3.

Suren (after a while): "… *Oh … they are the same … the both corresponding values are equal*."

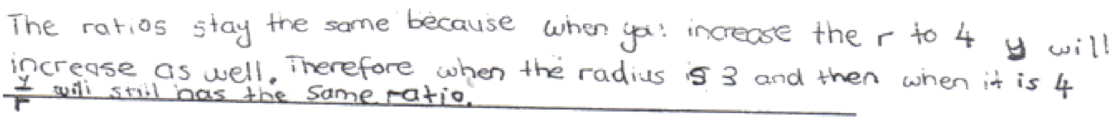
The ratios stay the same because when you increase the r to 4 y will increase as well. Therefore when the radius is 3 and then when it is 4 $\frac{y}{r}$ will still has the same ratio.

Figure 3: *Nadeem's observtion & explanation*

Question 4

To evaluate how well learners understood the underlying similarity of right triangles with the same reference angle as the basis for the constancy of trigonometric ratios, the following question was given next:

"*Find the ratio* $\frac{y}{r}$ *for the second right triangle above (the angles at A and D are equal."* (Figure 4).

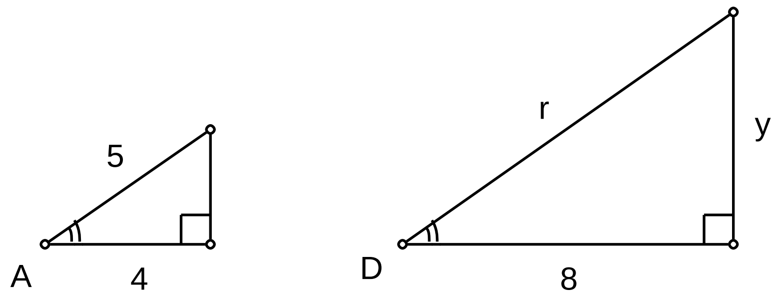

Figure 4

This question required learners to first use the Theorem of Pythagoras to find the 3rd side in the first triangle (or simply from knowing the 3, 4, 5 relationship of a right triangle). Then since scaling up a triangle maintains similarity, and therefore ratios of corresponding sides would remain constant, the ratio y/r would remain unchanged as 3/5.

This question gave all the learners a lot of difficulty, as they did not know how to proceed. Three learners asked whether they could use the computer, for example:

Suren: "*Can I use the computer?"*

Vishen (puzzled): "*… no computer?*"

Four of the learners initially confused angles with lengths which indicates that they didn't yet have a very solid conceptual understanding of the underlying concepts of trigonometry, for example:

Mayuri enquired: "*Can y be 90 degrees*?"

Vishen: "*Now I must find y … must I find out how many degrees?*"

Perusha: "*I am thinking 90 degrees divided by 8*".

Suren: "*Must I use the triangle = 180 degrees?*" When asked by the interviewer if the question required angles, he replied: "*No, we are talking about lengths* …" but nevertheless continued *"… so you can say 180° - (8 + 10) is equal to … so 180 – 18* *will give you y*", though shortly afterwards he corrected himself: "*180 minus is an angle! Oh … so no … (silence) … there could be a way to do it, but I don't know*."

Only three learners, namely, Vishen, Mayuri and Nadeem later realized, and specifically mentioned, that they needed to use the Theorem of Pythagoras to calculate the third side of the first triangle, but could not recall the theorem or how to use it. Perhaps more significantly, none of the students immediately knew that the missing side in the 1st triangle was 3, something one might have expected of Grade 10 learners as they were supposed to have encountered right triangles and the Theorem of Pythagoras as early as Grade 7 or 8, and that the 3, 4, 5 right triangle should be a very familiar example.

When told by the interviewer that the missing third side of the triangle was 3, however, all the learners, with the exception of Perusha, quickly noticed that corresponding sides were doubled, and could then determine the values y and r by doubling, $y = 2 \times 3 = 6$ and $r = 2 \times 5 = 10$, and through substitution find $\frac{y}{r} = \frac{6}{10}$. Significantly though, none of them seemed to have realized that due to the similarity of the two triangles (i.e. the second is simply an enlargement of the first by a scale factor 2) the ratio of corresponding sides $\frac{y}{r}$ would remain $\frac{2}{3}$, and that there was really NO need to determine the values of y and r for the second triangle.

This clearly shows, despite the previous activity in Question 3 of a comparison of the ratios for different r values, a still undeveloped conceptual understanding of the underlying similarity that underpins the whole of trigonometry. For example, understanding that similarity was the reason WHY the ratios of the sides of right triangles with a given reference angle, but of ANY size, always remained constant. From an instructional point of view, this implies a need for further follow-up activities to develop and solidify this understanding.

Question 5

Since only whole numbers for r was used in Question 3, the following group of questions were intended to evaluate whether learners would spontaneously generalize to other values of r:

a) "*Do you think this ratio $\frac{y}{r}$, for a given angle in a right triangle will always remain constant irrespective of how r changes?"*

b) *"What if r = 2.1? or r = π? Will it still be the same for a given angle?"*

c) *"Why? (Explain or justify your reasoning).*

All six learners answered that they thought that the ratios would remain constant for given angles and four learners were able to explain their observation to some extent in terms of proportionality as follows:

Vishen: "*When r is doubled for example, I notice that y is also doubled. y over r is then constant*."

Omika: "*Er, if you want the triangle to be bigger, you can times it by 2 and the number will get doubled or if you want it 3 times bigger, it will always be times by 3 … The number will be bigger, but if you divide it like you want it 3 times bigger, you can divide it by 3 and you will still get the same number*."

Mayuri: "*Yes, because as r is increasing the other values are increasing in proportion*."

Nadeem asked to look at a previous *Sketchpad* sketch again, and by dragging a point to change the value of *r* to observe what happened, responded as follows: "*It is the same … because as r increases, the y increases the same. So when you find y over r, it will always be the same …*"

Question 6

To explore learners' understanding of the relationship between the input and output values of the sine function so far, the learners were next given the following set of questions:

Answer the following questions:

a) $If\ \sin(angle) = \frac{1}{2}\ then\ angle = \_\_\_\_\_\_\_\_?$

b) $\sin 35° = \_\_\_\_\_\_\_\_?$

c) $Estimate\ the\ value\ of\ the\ angle\ \ if\ \frac{y}{r} = 0{,}55$

For question 6(a), three of the learners correctly obtained $30°$, Suren reading it off from his tables of values, while Nadeem and Mayuri preferred to use the *Sketchpad* sketch, dragging point *G* until the displayed ratio was 0.5, and then looking at the corresponding angle displayed by the computer.

For question 6(a), Vishen assumed the sine function was a linear function and reasoned as follows: "*The angle is 45 degrees … Because when the angle increases to 90 degrees, it is 1. Half of 90 degrees is 45 degrees*."

Omika and Perusha had difficulty interpreting Question 6(a), and may not yet have made a clear distinction between the input and output values, for example:

Perusha: "*If sin (angle) =* $\frac{1}{2}$*, the angle equals* $\frac{1}{2}$*.*"

Omika: "*The answer is* $\frac{1}{2}$."

Question 6(b), namely, $\sin 35° = ?$, required that learners now find a ratio for an input value not in their tables. Three learners Mayuri, Suren and Nadeem used *Sketchpad* by dragging point *G* until the angle was $35°$ and reading off the answer 0,57 or 0,58. Suren initially looked at the table, but after seeing there was no information for $35°$ in the table, chose to use *Sketchpad* instead.

Two students, Vishen and Omika, tried using the table and estimation, finding a value midway between the sine values for 30° and 40° as follows:

Vishen: "*OK, 0,57, because 30 and 40 have a 14 number difference, in terms of the ratio. So the answer is 0,57 using the ratio*."

Omika: "*0.58 or something … 15 divided by 2 is* $7\frac{1}{2}$*. Then 0,57 since we know for 30°* *you get 0,5*."

Like Vishen earlier, Vishen and Omika clearly also assumed that the sine function was linear, but in this case it worked reasonably well as the sine function is approximately linear in this particular short interval. This serves as a cautionary example of incorrect reasoning leading to a correct answer, and should be a reminder to teachers to probe learners' reasoning more deeply.

For question 6(b), Perusha simply replied that sin $35° = 35°$, apparently equating the angle and the ratio, and did not yet seem to clearly distinguish between input and output values.

For the final question 6(c), the learners were required to estimate the value of the angle if $\frac{y}{r}$=0,55. Three learners, Vishen, Suren and Perusha got approximately $33°$ using *Sketchpad* to check the corresponding angle when *G* was dragged until the ratio was 0,55. Of special interest here is that Perusha now suddenly, despite her earlier difficulty in the previous two questions, correctly used the computer.

Mayuri initially tried using the tables, but had some problems, at first incorrectly estimating the corresponding angle as 28°, apparently assuming that the sine function was decreasing, but after checking herself with *Sketchpad*, and dragging *G* until the ratio was 0,55, corrected her estimate to $33°$.

The other two learners preferred working only from the tables to estimate a value of about 0,33 or 0,34, for example:

Nadeem: “If 0,5 is 30 and 0,57 is 35, so 0,55 will be … I am just like … I am like blocked … (silence) … If I have to estimate, I will say 34°.” When asked to explain how he got 34°, he responded: “Mmm, like for every 0.02, there is 1 degree more … “, clearly also incorrectly assuming a linear relationship as before. On the positive side, despite this misconception, both learners in this category not only realized that the answer had to be less than 35 degrees, and more than 30 degrees, but also closer to 35 degrees as the output value of 0,55 was close to 0,57 (the output value of 35 degrees).

Concluding Remarks

Most of the students seemed to have made a solid start to the development of a healthy conceptual schema of concepts related to the sine function, given the relatively short space of time for a first introduction to trigonometry. In Question 1, all the learners observed that $\sin\theta$ was equivalent to the ratio of sides $\frac{y}{r}$, and four out of six correctly observed at this point that the sine function was an increasing function in the first quadrant. After completing further tables for different *r*, Questions

3 and 5 indicated that all the learners were able to generalize that the sine of a given angle is independent of the radius *r* of a circle, and four of learners were able to give correct explanations in terms of scaling up or down (i.e. in terms of similarity).

Despite the four learners above showing some understanding of the underlying similarity of right triangles with the same reference angle as the basis for the constancy of the sine of a given angle, learner responses in Question 4 revealed that this understanding was not yet fully developed by any of the learners. More-over, this question revealed a difficulty with recalling the Theorem of Pythagoras that, unless dealt with, may impact negatively on learners' further study of trigonometry given the fundamental role this theorem will play later on in deriving fundamental trigonometric identities like $\sin^2 x + \cos^2 x = 1$, etc.

Question 6 revealed that more than half the students had developed a reasonable understanding of the relationship between the input and output values of the sine function, finding values not given in the table, either by using the *Sketchpad* sketches or extrapolating from their tables. However, three of the learners using the tables clearly had the misconception that the sine function was linear, and though this led to a correct answer in 6(b) and 6(c) because of the small interval, it was far from the correct answer in 6(a) due to the larger interval.

To address misconception like these, it is generally recommended that teachers use the diagnostic teaching approach advocated by Bell et al (1985) and Bell (1993). In such an approach learners are put in situations (learning activities), which create 'cognitive conflict' between their expectations and the eventual outcome. (For example, giving learners more activities like those in Questions 6(b) and (c), but over larger intervals.) Rather than viewing misconceptions as something intrinsically negative, misconceptions are viewed and dealt with in this approach as an important and necessary stage of the learning process. Unless dealt with adequately, all subsequent learning will be annulled by the learner's preconceived misconception, especially if the misconception is masked by learners getting right answers with incorrect reasoning.

The usual strategy by teachers of simply 'reteaching' a correct method is mostly quite ineffective to address deep, underlying misconceptions. Instead in diagnostic teaching, learners themselves are confronted with a situation or set of situations where they are brought to realise that something is 'wrong', so that they can make the appropriate changes to their conceptual schemas. A classroom setting therefore needs to be engineered so that challenging discussions and thinking can take place, with learners encouraged to argue about concepts, present their own ideas and have their individual approaches publicly scrutinised in a non-threatening way.

This strategy of diagnostic teaching was implicitly employed in Question 2 when learners were asked, immediately after completion of the first table for $r = 1$, what would happen if r was changed. Since none of the students correctly anticipated what would happen, it conflicted with their expectation and came as quite a surprise to several of them that the sine of a given angle was independent of the radius. This strategy seems to have meaningfully contributed to their correct generalizations in Questions 3 and 5 later on.

Learners appeared to benefit from the visual illustration by Sketchpad of the functional relationship between the changing angle and the corresponding sine ratio formed by the lengths y and r. More-over, *Sketchpad* allowed learners to dynamically manipulate the figure themselves, dragging point G to form different angles and then reading off the corresponding ratios, emphasising both visually and manually the distinction, and functional relationship, between input and output values. This 'hands-on' kinaesthetic experience seemed to assist conceptualization and generalization in a positive way as shown in Question 5, and also served as a useful problem solving tool for some of the learners, as evidenced by their responses in Question 6.

The minor discrepancies between the values of the ratio y/r and the sine of the corresponding angle observed by the learners was caused by the way the sketch was set up to allow learners to drag a point to change the angle. More-over, for simplicity, the accuracy for angle measurement was set to the nearest whole number, but that meant that when dragging to an angle of 20 degrees, the angle could be lying anywhere in the interval from 19.5 to 20.4. In hindsight, it might have been better if one decimal accuracy was used for both the lengths and the ratios, as this might have minimized the observed discrepancies. However, given the discrete nature of computer measurement and calculation, discrepancies such as these are unavoidable for a figure of this kind. On the other hand, as shown by the responses of learners in Questions 3 and 5, these discrepancies did not seem to seriously impact on learners forming the desired conceptualization of the connection between the ratios and the sine of the corresponding angle. More over, the advantage of the 'hands-on' manipulation by the learners seems to outweigh the minor disadvantage of these minor discrepancies.

Acknowledgement

This research was partially funded by the National Research Council (NRF), specifically the Spatial Orientation and Spatial Insight (SOSI) Project coordinated by Proff. Dirk Wessels, Hercules Nieuwoudt and Michael de Villiers, respectively from the Universities of South Africa, North-West and KwaZulu-Natal. The views reflected in this research are those of the researchers, and do not necessarily reflect those of the NRF.

References

BELL, A. W.; SWAN, M.; ONSLOW, B.; PRATT, K.; PURDY, D. (1985). *Diagnostic teaching: Teaching for long term learning*. Report of ESRC Project HR 8491 / 1, Nottingham, UK: University of Nottingham, Shell Centre for Mathematical Education.

BELL, A. (1993). Some experiments in diagnostic teaching. *Educational Studies in Mathematics*, 24, 115-137.

BLACKETT N; TALL D 1991. Gender and the versatile learning of trigonometry using computer software. *The Proceedings of the International Group for the Psychology of Mathematics Education* XV. Assisi, Italy, (1991), Vol 1, pp. 144-151.

BOALER, J. (1997) *Experiencing school mathematics: teaching styles, sex and setting,* Buckingham: Open University Press.

BOYER, C.B. (1968). A History of Mathematics. New York: John Wiley & sons.

DERBYSHIRE, J. (2006). *Unknown Quantity: A real and imaginary history of algebra*. Washington: Plume Books.

DONOVAN, M.S; John D. Bransford, J.D. (Eds). (2005). *How Students Learn: History, Mathematics, & Science in the Classroom*. Washington, D.C.: The National Academies Press.

FREUDENTHAL, H. (1973). *Mathematics as an educational task*. Dordrecht: Reidel.

GOLDIN, G. (2000) A scientific perspective on structured task-based interviews in mathematics education research, In Kelly, A. and Lesh, R. (Eds) *Handbook of research design and mathematics and science education*, New Jersey: Lawrence Erlbaum Associates, 517-545.

JUGMOHAN, J. (2005). Learners' Conceptualization of the sine function with *Sketchpad* at Grade 10 Level. Unpublished M.Ed. thesis, University of KwaZulu-Natal.

OLIVIER, A. (1989). Handling Pupils' Misconceptions. *Pythagoras*, no. 48, pp. 17-25.

PIAGET, J. (1970). *Genetic Epistemology*. Chicago: University of Chicago.

POLYA, G. (1981). *Mathematical Discovery*. Vol 2, pp. 132-133, New York: John Wiley.

POURNARA, C. (2001). An investigation into learners' thinking in trigonometry at Grade 10 level. Unpublished M.Ed. thesis, University of Witwatersrand.

_______. (2003). Trigonometry: rethinking a weapon of maths destruction. *Pythagoras*, Vol. 3, Issue no. 58, Aug, pp. 10-20.

SKEMP, R.R. (1979). *Intelligence, Learning and Action*. New York: John Wiley & sons.

VON GLASERSFELD, E. (1987). Learning as a Constructive Activity. In C. Janvier (Ed.*) Problems of Representation in the Teaching and Learning of Mathematics.* Hillsdale, NJ: Erlbaum.