ISSN 1516-5388

EDUCAÇÃO MATEMÁTICA PESQUISA

v. 3 – n. 1 – 2001

educ

EDUCAÇÃO MATEMÁTICA PESQUISA v. 3 – n. 1

educ

EDUCAÇÃO MATEMÁTICA PESQUISA

revista do programa de estudos pós-graduados em educação matemática

puc-sp

EDUCAÇÃO MATEMÁTICA PESQUISA

revista do programa de estudos pós-graduados em educação matemática

puc-sp

Correspondência:

Revista do Programa de Estudos Pós-Graduados em Educação Matemática

Rua Marquês de Paranaguá, 111 – CEP 01303-050 – Consolação – São Paulo – SP

Fone: (11) 3256-1622 ramal 202

Fax: (11) 3159-0189

E-mail: pegedmat@exatas.pucsp.br

Expediente: de segunda a sexta-feira das 10h30min às 12h e das 13h30min às 17h30min

EDUCAÇÃO MATEMÁTICA PESQUISA

revista do programa de estudos pós-graduados em educação matemática

puc-sp

ISSN 1516-5388

Educ. Mat. Pesqui., São Paulo, v. 3, n. 1, pp. 1-106, 2001

educ

2001

Catalogação na fonte – Biblioteca Reitora Nadir Gouvêa Kfouri / PUC-SP

Educação Matemática Pesquisa : Revista do Programa de Estudos Pós - Graduados em Educação Matemática / Pontifícia Universidade Católica de São Paulo - n.1 (março de 1999) - São Paulo : EDUC, 1999 - semestral
ISSN 1516-5388

1. Educação Matemática Pesquisa - periódicos. I. Pontifícia Universidade Católica de São Paulo. Programa de Estudos Pós-Graduados em Educação Matemática

EDUC – Editora da PUC-SP

Direção
Maria Eliza Mazzilli Pereira
Denize Rosana Rubano

Coordenação Editorial
Sonia Montone

Revisão
Sonia Rangel

Revisão de Inglês
Carolina Muniz Ventura Siqueira

Editoração Eletrônica
Artsoft Informática

Capa
Sara Rosa

educ

Rua Ministro Godói, 1213
Cep 05015-001 - São Paulo - SP
Fone: (11) 3873-3359 / Fax: (11) 3873-6133
E-mail: educ@pucsp.br
Site: www.pucsp.br/educ

ASSOCIAÇÃO BRASILEIRA DAS EDITORAS UNIVERSITÁRIAS

Associação Brasileira de Editores Científicos
ABEC

ABDR
EDITORA AFILIADA

Projeto Editorial

A revista *Educação Matemática Pesquisa*, do Programa de Estudos Pós-Graduados em Educação Matemática da PUC-SP, de regularidade semestral, constitui um espaço de divulgação de pesquisas científicas da área.

O projeto editorial da revista prioriza artigos científicos, inéditos no Brasil, da área de Educação Matemática. Mais particularmente, aqueles relacionados às linhas de pesquisa do Programa: A Matemática na estrutura curricular e formação de professores; História, Epistemologia e Didática da Matemática; e Tecnologias da Informação e Didática da Matemática. A prioridade dada às linhas descritas não é extensiva aos referenciais teóricos, ao contrário, procura-se contemplar a diversidade.

Serão acolhidos, também, artigos que favoreçam o diálogo entre Educação Matemática e áreas afins, como a Matemática, a Epistemologia, a Psicologia Educacional, a Filosofia, a História da Matemática e a História das Disciplinas, entre outras.

A seleção dos artigos faz-se mediante a aprovação de dois pareceristas do conselho editorial ou *ad hoc*. Os pareceres serão enviados aos autores.

Os artigos são apresentados sempre na versão original, com resumos bilíngües (português e inglês ou francês).

O projeto editorial prevê, ainda, que os volumes da revista contenham uma ou mais modalidades, como análises ou relatos de pesquisa, comunicações (ciclo de palestras, conferências), entrevistas, depoimentos ou resenhas científicas.

Em cada número, haverá indicações sucintas das dissertações e teses produzidas no Programa, no semestre de edição.

Conselho Editorial

Editorial Project

The journal Educação Matemática Pesquisa *of the Post-Graduation Program in Mathematics Education of the Catholic University of São Paulo (PUC/SP) is published every semester with the aim of providing a space for disseminating scientific research in the area.*

The policy adopted by the editors is to prioritise scientific articles which have not been published in Brazil, related to Mathematics Education, particularly those addressing the lines of research of the program: Mathematics, curriculum structure and teacher training; History, Epistemology and Didactics of Mathematics; and Information Technology and the Didactics of Mathematics. The priority given to the described lines is not restricted to theoretical references; on the contrary, it is hoped that the journal will reflect the diversity that characterises research in Mathematics Education.

The editors also encourage the submission of articles which open dialogues between Mathematics Education and related areas, such as Mathematics, Epistemology, Educational Psychology, Philosophy, History of Mathematics and its teaching, amongst others.

In order to be selected, articles should receive two favourable reviews. Referees will be chosen from the editorial committee or they will be ad hoc *reviewers. Authors will receive copies of the reviews.*

Articles will be presented always in the original language of the author along with abstracts in Portuguese and English or French.

The journal can also include works of various different types, such as: research reports, papers based on lectures or conferences, interviews, commentaries on issues pertaining to research, critiques of articles and books, literature reviews and theoretical analyses.

Each issue, will also include brief descriptions of the dissertations and theses produced in the Program during the semester of the edition.

Editorial Committee

Sumário

Editorial

Filosofia da Educação Matemática e aprendizagem de conceitos matemáticos compõem os temas deste volume. Além deles, é apresentada de forma sucinta a produção discente do Programa de Estudos Pós-Graduados em Educação Matemática da PUC-SP referente ao primeiro semestre de 2001.

No artigo, "B. Russell's 'introduction to mathematical philosophy'", Michael Otte, renomado pesquisador alemão, analisa e aponta falhas na conceitualização de "número" de B. Russell, conceitualização essa fundamentada na lógica e na teoria de conjuntos. A análise de Otte tem como alvo a educação matemática.

Os pesquisadores Tsamir, do Departamento de Ciência da Educação da Universidade de Tel Aviv, e Bazzini, da Faculdade de Educação da Universidade de Turim, no artigo denominado "Can x=3 be the solution of an inequality? A study of italian and israeli students" descrevem soluções dadas por estudantes de Israel e da Itália para inequações algébricas, identificando similaridades nas respostas corretas e, também, nas incorretas, nos dois países. Norteiam-se, em suas análises, pelas concepções algorítmica, intuitiva e formal de um conhecimento matemático.

"Propostas curriculares, planejamento de ensino, práticas de classe e conhecimentos de alunos do ensino infantil sobre a adição" é o título do terceiro artigo, das pesquisadoras brasileiras Cristina Maranhão e Sílvia Sentelhas. Nele, as autoras focalizam o processo de passagem da contagem à adição no nível da educação infantil. Analisam tanto como se dá a construção, pelas crianças, do conhecimento relativo à operação de adição no campo dos números naturais, como qual deve ser a ação do professor para favorecer esse processo.

Completam esse periódico os títulos das dissertações produzidas no Programa durante o primeiro semestre de 2001.

Editores

Editorial

Philosophy of Mathematics Education and the learning of mathematical concepts make up the themes of this volume. Additionally, the volume presents, in a succinct form, the production of students of the Program of Post-Graduate Studies in Mathematics Education of PUC/SP during the first semester of 2001.

In the article "B. Russell's 'introduction to mathematical philosophy'", Michael Otte, the well-known German researcher, describes and analyses Russell´s conceptualisation of "number", which is based on logic and set theory. Otte's analysis, intended for the mathematics education community, identifies weaknesses in Russell´s conceptualisation.

The researchers Tsamir, of the Science Education Department at the University of Tel Aviv, and Bazzini, from the School of Education at the University of Torino, in the article "Can x=3 be the solution of an inequality? A study of Italian and Israeli students", describe the solutions given by Israeli and Italian students to problems involving algebraic equations. They identify similarities in both correct and incorrect answers in the two countries. In particular, they distinguish in their analyses between algorithmic, intuitive and formal conceptions of mathematical knowledge.

"Curriculum proposals, teaching plans, classroom practices and pre-school students' knowledge of addition" is the title of the third article, written by the Brazilian researchers Cristina Maranhão and Sílvia Sentelhas. In this paper, the authors focus on processes accounting for the passage from counting to addition. They analyse how children manage to construct knowledge relative to the addition operation in the field of natural numbers, and also indicate actions on the part of teachers that might be effective in supporting these constructions.

Titles of the dissertations produced by the students of the Program during the first semester of 2001 complete this journal.

Editors

B. Russell's "introduction to mathematical philosophy"

MICHAEL OTTE*

Abstract
Bertrand Russell is an important and interesting figure, undoubtedly the most read, honored, and reviled English-speaking philosopher of the twentieth century. And his "Introduction to Mathematical Philosophy"[1] is no less fascinating. Indeed, Russell's above-mentioned work, published in 1918, has sometimes rightly been called "an admirable exposition of the monumental work Principia Mathematica".
The main object of Russell's book is number, and everything belonging to number, to arithmetic, and to the logic of arithmetic. The foundations of arithmetic always remained the focus of Russell's interest in logic and mathematics, and his views had a profound influence on the reform movement of mathematics education that began around 1960. Since the beginning of the 19th century, mathematics showed a strong tendency towards arithmetization, because space and the continuum seemed beset with seemingly intractable otherness. By the end of the 19th century, even number appeared not to be so transparent and so immediately given anymore and the question "what numbers are" arose to explain and to complete the foundations of arithmetic itself by means of logical analysis and set theoretical construction.
Key-words: mathematical fundaments; new math; number concept construction.

Resumo
Bertrand Russell foi um intelectual importante do século XX, sem dúvida o autor mais divulgado e mais lido da atualidade. Ele foi o filósofo mais elogiado, assim como o mais injuriado de sua época. Seu livro Introduction to Mathematical Philosophy, *publicado em 1918, não só foi uma introdução fascinante em sua obra prima* Principia Matemática, *que revolucionou os fundamentos de toda a matemática, mas foi também o texto mais influente na reforma educativa chamada Matemática Moderna dos anos 1960-1970. O tema principal desse livro é o "número". Russell tentou construir o conceito do número em termos da lógica e da teoria de conjuntos. É proposta deste artigo tanto analisar como apontar falhas e ganhos dessa tentativa.*
Palavras-chave: *fundamentos de matemática; matemática moderna; construção do conceito de número.*

---

* Pesquisador do Departamento de Matemática da Universidade de Bielefeld, Alemanha. Professor visitante do Programa de Estudos Pós-Graduados em Educação Matemática da PUC-SP. E-mail: michaelontra@aol.com

1 "Introdução à Filosofia Matemática" – tradução para o português da 10ª impressão, de 1960, por Giasone Rebuá e publicado por Zahar Editores, no Rio de Janeiro, em 1963.

I. Presentation

Bertrand Russell is an important and fascinating figure, no doubt the most read, most honored, and most reviled English-speaking philosopher of the twentieth century. He was born in Wales in 1872 and died there in 1970. His lifetime thus extends from the times of Bismarck and Queen Victoria to the atomic bomb and the cold war period after 1950.

Russell's "Introduction to Mathematical Philosophy" is no less fascinating reading and it is a book of the kind only someone like him was able to write, while in prison with no resources at his disposal, compelled to jettison all technical ballast and to take the subject in a bold fashion trying to provide a readable, almost popular presentation of the subject. Indeed, Russell's "Introduction to Mathematical Philosophy" of 1918 has sometimes and rightly been called "an admirable exposition of the monumental work *Principia Mathematica*". The book is even more, because it builds upon the results of all of Russell's fundamental work since about 1900. And it is at the same time something else, as it provides a most original, lively and self-contained introduction to the foundations of mathematics and of epistemology. In contains, in fact, as it were in a nutshell all the fundamental questions of mathematical epistemology.

In contrast to standard present day texts on the philosophy of mathematics, Russell always allows the reader a glimpse into his thinking, as it develops, without trying to cover up his presumptions and errors. The reader can literally experience Russell's enthusiasm in dealing with his subject. Russell has always been an independent spirit to whom narrow specialization remained alien, and who strove to simultaneously keep up with all the developments in mathematics, philosophy, in the experimental natural sciences and in politics as well. And during his lifetime he wrote on an impressive variety of fields, in most cases very lively, a fact which made many of his books popular science in the very best sense of the term. Not only his comprehensive studies enabled him to do so, but also his conviction that his true vocation was to write.

As Russell is one of the eminent protagonists of modern scientific empiricism, and one of the founders of the analytical philosophy of mathematics dominant until today, his writings give the rare chance of a unique and deep insight into the philosophical, scientific, technological, and political turmoil characteristic for the first half of the 20th century.

While the program of the logical world view, as inaugurated, among others, by Frege and Russell, and later to be continued by their "disciples" – Carnap and Quine – may seem to be very dry, technical and perhaps even objectivisticly inhuman, the underlying intentions were firmly anchored in a humane belief in the merits of scientific progress and enlightenment.

The orientation towards a strictly logico-scientific mode of thought intended to counteract dumb irrationalism and traditionalism, as it became prevalent after World War I. As Rudolf Carnap expressed it in 1928, logical philosophy is supported by the belief that the future belongs to a mentality which "demands clarity everywhere, but which realizes that the fabric of life can never quite be comprehended", and which consequently puts "clarity of concepts", "precision of methods", and "responsible theses" into the service of a progress in cognition by cooperation (Carnap 1968, preface to the first edition). In view of the fact that the philosophy of mathematics and the present analytic theory of science, while adopting Russell's "technical achievements", do not utter a single word about the philosophical and historical concerns inseparably linked to it, it would seem appropriate to point out the historical origins of modern set-theoretical mathematics and epistemology, and the cultural and political context of their emergence.

II

The main object of Russell's book is number, and everything belonging to number, to arithmetic, and to the logic of arithmetic. The foundations of arithmetic always remained the focus of Russell's interest in logic and mathematics. Since the beginning of the 19th century, mathematics showed a strong tendency towards arithmetization, because space and the continuum seemed beset with seemingly intractable otherness. By the end of the 19th century, even number appeared not any more so transparent and so immediately given and the question arose to explain "what numbers are" and to complete the foundations of arithmetic itself by means of logical analysis and construction. Russell's concern is at first not so much with clarification of the ontological foundations of arithmetic, resp. of mathematics as a whole, but rather with differentiation of the conceptual apparatus, and specification of proof methods. As his argument unfolds it became obvious, however, that the existence claims on which to build, themselves were very insecure.

To answer the question "what is a number" a "new logic" was necessary. Russell was first confronted with this new logic at the International Congress of Mathematics in Paris in 1900, in the person of Peano. And however impressed he was by Peano's logical precision, he did not perceive the form of logic grown from the algebraic and axiomatic approach to be in agreement with his own natural disposition and with his own interests focussed on the arithmetization of mathematics. Russell himself describes the development as follows:

> In itself, mathematical logic had for a long time not been a novel discipline any more at this period (i.e. about 1900). [...] Boole had published his *Laws of Thought* in 1854; C.S. Peirce had elaborated a logic of relations, and Schröder had published a comprehensive work in Germany in three volumes in which he summarized everything achieved up to then. Whitehead had treated the Boolean calculus in the first part of his own *Universal Algebra* (which had appeared in 1898, treating besides the above authors Grassmann, De Morgan and others, my insertion, M.O.). Most of these works were known to me, but I did not have the impression from them that they made the logical grammar of arithmetic appear in a new light.

Nevertheless, Russell remembers,

> the Congress was a turning point in my intellectual life, because I there met Peano. I already knew him by name and had seen some of his work, but had not taken the trouble to master his notation. In discussion at the Congress I observed that he was always more precise than anyone else, and that he invariably got the better of any argument upon which he embarked. As the days went by, I decided that this must be owing to his mathematical logic. (Russell 1967, 144)

The logicians of the algebraic school like Peirce, Peano or Schröder, it is true, proceeded mathematically, transposing the mathematical laws to the realm of logic, resp. understanding logic as a universal algebra, without intending to revise the methods of mathematics. Modern axiomatic method represents nothing but the ultimate step towards a mathematization of all phenomena and areas of reality with mathematics itself now being finally mathematized. The concept of number, in particular, has always marked the heart of mathematical thought. Still it seems legitimate to ask why arithmetic was to be axiomatized only in the second half of the 19th century, that is more than 2000 years after Euclid's axiomatization of geometry.

Now mathematical philosophy and foundational research grew out from two rather different ways of thinking in mathematics itself. The first one, to which the name of set-theoretical reductionism seems appropriate, began with Bolzano (1781-1848) and Cauchy (1789-1857) and culminated in the achievement of Russell. The other, usually called axiomatic or postulational method, originated in the works of Poncelet (1788-1867) and Grassmann (1809-1878), who continued and radicalized the approach of his father Justus Grassmann (1779-1852) and reached full development in the works of Peano and Hilbert (1861-1943). While set-theoretic reductionism was mainly the work of analysts and philosophers, the new structural or axiomatic thinking was primarily established by geometers, algebraists and engineers.

It were mainly problems of teaching and communication which had always led to algebraic exposition of analysis and which after this seemed no longer appropriate enforced a search for new foundations of the number concept itself. Until then these two traditions do not seem to have become aware of their real difference, or of the possibility to propose themselves as two alternative foundational perspectives, not least because modern axiomatics itself came in two rather different forms, that of Pasch (1843-1930) on the one hand and that of Hilbert on the other.

And only after people began to inquire into the foundation of arithmetic towards 1870, looking for specific logical justifications of the latter, they were compelled to become concerned with the traditional epistemological systems. The self-referential character of the process of mathematization as it addresses mathematics itself, and this means arithmetic now, inevitably leads to raising fundamental logical and epistemological problems. In the foundations of geometry one had, since Descartes always, explicitly or implicitly, made use of number and arithmetic. Any mathematization of the continuum requires the number concept in some or other ways (see: Hölder 1899; and for an alternative Grassmann 1844). And whereas Xenon's Paradoxes for a long time had been considered mere sophisms or anachronisms, which may admittedly have some charm, they have since the 19th century become the focus of an intense and profound debate, because now the consistency of a discrete picture of the world itself grew into a serious problem (see W. C. Salmon (ed) 1970).

When dealing with the foundations of pure mathematics and of arithmetic in particular people often change all their epistemological attitudes as well as their methodological rules and become more or less obsessed with finding ultimate truths about the nature of reality. Russell provides a shining example of this attitude. On the one hand, Russell makes it clear to us, for example, that we cannot get farther in our own grasp of natural phenomena than to a mathematical representation of the relations and the relational structures – the "Book of Nature" has been written in a mathematical language, as Galilee already stated – and that we have to admit that we know much more about "the form of nature than about its content". On the other hand, where mathematics and in particular the concept of number is concerned, Russell makes an effort to answer "what is" questions in an absolute and definite way.

While he responds to the "non-mathematical mind", to whom the abstract character of our physical knowledge appears unsatisfactory, by saying that these abstractions may personally not please her or him, but are useful from a practical as well as theoretical viewpoint, he considers the opposite to be true with regard to mathematics. And against the complaints of commonsense empirism or artistic imagination, he justifies the mathematical representations of phenomena by their fertility and their uses:

> Abstraction, difficult as it is, is the source of practical power. A financier, whose dealings with the world are more abstract than those of any other 'practical' man, is also more powerful than any other practical man. He can deal in wheat or cotton without needing ever to have seen either: all he needs to know is whether they will go up or down. This is abstract mathematical knowledge, at least as compared to the knowledge of the agriculturist. Similarly the physicist, who knows nothing of matter except certain laws of its movements, nevertheless knows enough to enable him to manipulate it. After working through whole strings of equations, in which the symbols stand for things whose intrinsic nature can never be known to us, he arrives at last at a result which can be interpreted in terms of our own perceptions, and utilized to bring about desired effects in our own lives. (Russell 1971a, 138f)

In contrast to that, he would like to constitute these mathematical abstractions themselves by pure reasoning and by ensuring their applicability in a completely *a prioric* way in a somewhat Kantian spirit. We should reduce, he seems to recommend, everything in science and daily life as far as possible to number and arithmetic. What number itself is, and what the concept of number means, however, shall be determined independently of any application by pure thought and logical analysis. Sense perceptions and numbers have been forming the foundations of our cognitions since Descartes, Russell says. By clarifying the meaning of the number concept, logic is now replacing arithmetic, bridging the gap between empirism and mentalism.

While Russell believes that science can only determine its domain of investigation up to an isomorphic mapping, structural isomorphism hence marking the limits of our possibilities of symbolic representation of the world and of our mathematical descriptions of the same, and while he thinks that this might even be an advantage in the realm of the natural sciences, because "in the mathematical treatment of nature, we can be far more certain that our formulae are approximately correct than we can be to the correctness of this or that interpretation of them" (Russell, 1971a, 131), he would nevertheless not like to see mathematics be described simply by its own structure.

Where the understanding of nature is concerned, Russell states the insight "that, as reasoning improved, its claims to the power of proving facts grow less and less" (Russell 1971a, 135). This is precisely what he refuses to accept with regard to mathematics.

This difference in attitude and cognitive behavior seems due to the fact that in mathematics we have, differently from the natural sciences, simultaneously to secure the existence of our universe of discourse as well as to construct the conceptual means of dealing with the existent. Bateson has affirmed that from an epistemological point of view we seem to have the option of treating as real either objects resp. events or signs resp. messages. And he has given the following description of these two options:

> The difference between the Newtonian world (the world of objects, my insertion M.O.) and the world of communication is simply this: that the Newtonian world ascribes reality to objects and achieves its simplicity by excluding the context of the context – excluding indeed all meta-relationships – a fortiori excluding an

> infinite regress of such relations. In contrast, the theorist of communication insists upon examining the meta-relationships while achieving its simplicity by excluding all objects. This world, of communication, is a Berkeleyan world. (Bateson 1973, 221)

Now mathematics seems to belong to both worlds and the explanation of mathematics resp. foundational research on mathematics thus hence shares the exigencies resulting from both.

III

Russell begins his work with a chapter on "the series of natural numbers", in which the latter are introduced on the basis of Peano's axioms, the concept of ordinal number being the only one to have a role here. There is no mention of cardinality in the beginning.

> The five primitive propositions which Peano assumes are:
> (1) 0 is a number.
> (2) The successor of any number is a number.
> (3) No two numbers have the same successor.
> (4) 0 is not the successor of any number.
> (5) Any property which belongs to 0, and also to the successor of every number which has the property, belongs to all numbers".
> (pp. 5-6)

It might be suggested, Russell says at the end of this chapter,

> that, instead of setting up '0' and 'number' and 'successor' as terms of which we know the meaning although we cannot define them, we might let them stand for *any* three terms that verify Peano's five axioms. They will no longer be terms which have a meaning that is definite though undefined: they will be variables.
> (pp. 9-10)

This is, indeed, the common understanding of the axiomatic approach. It can also be expressed by saying that arithmetic is not about concretely existing things, but rather about general relations or ideal objects (cf. for this also section VII). But this point of view does not satisfy Russell.

On two accounts, Russell says, Peano's approach

> fails to give an adequate basis for arithmetic. In the first place, it does not enable us to know whether there are any sets of terms verifying Peano's axioms; [...] In the second place [...] we want our numbers to be such as can be used for counting common objects, and this requires that our numbers should have a *definite* meaning, not merely that they should have certain formal properties. (p. 10)
> [...] if we start from Peano's undefined ideas and initial propositions, arithmetic and analysis are not concerned with definite logical objects called numbers, but with the terms of any progression. We may call the terms of *any* progression 0, 1, 2, 3, [...], in which case 0, 1, 2, [...] become "variables". To make them constants, we must choose some one definite progression; the natural one to choose is the progression of finite cardinal numbers as defined by Frege. (Russell 1954, 4)

According to the Frege's definition of number which Russell, without knowing it beforehand, had rediscovered on his own after the Paris conference

> [...] primitive terms are replaced by logical structures, concerning which it is necessary to prove that they satisfy Peano's five primitive propositions. This process is essential in connecting arithmetic with pure logic. We shall find that a process similar in some respects, though very different in others, is required for connecting physics with perception. (Russell 1954, 4)

Logic on this account is interpreted in an utterly realistic way.

The minimal requirements to justify the concept of number, or the propositional function "x is a number" are showing that it has instantiations, that it is not "empty" in the Kantian sense. We have to understand "number as the number of a quantity" and to provide an application for the concept thus defined by demonstrating the existence of sets of arbitrary cardinality. This can obviously only be done axiomatically. In doing so, however, the notion of axiom must not be understood in the Peano-Hilbertian sense; the term must rather be conceived of according to the classical Euclidean tradition, that is as an intuitively evident truth and as a precondition of mathematics. This is why Russell introduces an "axiom of infinity".

We have to ascertain or render plausible that there are in fact infinite collections or sets in the world to be able to found number (p. 77). Infinite sets of arbitrary cardinality, Russell assumes, do exist and thus can be referred to in our reasoning. Set-theory must not be understood here as a merely formal *facon de parler*, it should rather be seen as an intuitive model of reality. Arithmetical intuition is hence replaced by set theoretical intuition. This might appear strange as the axiomatization of arithmetic has been caused by the feeling that we might be unable to intuit or fully grasp number and must therefore settle for the formal laws numbers satisfy. Russell now apparently replaces number by the intuitive concept of set as foundation of these formal laws. Nearly half a century later, mathematical education worldwide tried to repeat this move, with little success.

It is instructive to compare Russell's concern with that of Rudolf Carnap, on whom Russell exerted a crucial influence in the 1920s. Carnap calls it one of the objectives of his project "Der logische Aufbau der Welt" to establish a "cognitive-logical system" of concepts and objects which permits to derive or "constitute" all concepts "step by step from certain basic concepts", the result being "a genealogy of concepts in which every concept finds a secure place" (Carnap 1968). But the main characteristic of Carnap's book is its structuralism:

> Each scientific statement can in principle be so transformed that it is nothing but a structure statement. But this transformation is not only possible, it is imperative. For science wants to speak about what is objective, and whatever does not belong to the structure but to the material (i.e., anything that can be pointed out in a concrete ostensive definition) is, in the final analysis, subjective. (Carnap 1968, 29)

Theories make, according to this view, statements only about structure. Theoretical objects are determinable only up to the structure of their relationships. Structure is exactly what can be represented by mathematics and logic. There seems something wrong with this view, because we cannot grasp structure directly and without the mediation by some incorporation or application of it. And we would always have to observe the influence of these mediations. In contrast to Carnap, and to logical positivism as such, Russell, in his analysis of mathematics, is not concerned simply with structural descriptions, but his own logical analysis is always mixed with the notion of an absolute notion of truth and meaning. It is this concern which conveys a certain weight to the interpretation of the deductive systems, and in this connection an eminent role is due to the example of the interpretation of axiomatized arithmetic. The concept of set now serves to "link arithmetic to pure logic", and hence to interpret the axiomatical characterization in logical terms (Russell 1954, 4). In this connection, Russell also speaks of an overestimation of number. By giving up number, one attains "a gain in logical purity".

Russell is aware that his own understanding of theory, application, and truth, is not "provable", and that "it seems to be a matter of individual taste whether one accepts or rejects what is called the realistic hypothesis" (a.a.O.). For his own part, he is interested in realism and in the meaning of the word "truth", tying this interest to the question whether there are "elements, or constructions built up from such", which meet the conditions established in a given system of axioms. In this sense, he deals with numbers and with their logical constitution in a set theoretical context.

If one believes that the origin of axiomatization lies in mathematization, accepting Hilbert's conviction that "everything which can be an object of scientific thought (falls to) the axiomatic method and hence immediately to mathematics, as soon as it is ripe for the formation of a theory" (Hilbert 1964, 11), Russell's concern seems perfectly consistent. For arithmetic, to be axiomatizable, must be conceived of as a specific object field. The axiomatic theory alone, just like the natural laws, does not supply complete descriptions of its object field. It is only the application or interpretation of the axioms resp. of the natural laws which yields proper knowledge. Laws as well as axioms as such represent only possible knowledge. The undefined terms, which appear in the axiomatical characterizations, do not refer to specific singular objects, but rather serve to represent possible connections between indeterminate or general objects, free variables. Axioms in the modern sense therefore are considered mere axiomatic schemes.

To complete an axiomatic description one thus has to also indicate the intended applications. As the intended applications of arithmetic are not finite, however, the majority of mathematicians has tried to ascertain the existence of infinite sets by epistemological arguments, and not by ontological ones. If we intend, for instance, to constructively generate such sets, we might have to postulate the power of the mind for endless repetition of certain of its operations, e.g. of the operation of counting. Dedekind was not ready to imagine a straightforward axiomatic definition of number, because after recognizing the essential characteristics of such a system "the question arises: does such a system exist at all in the realm of our ideas" (Dedekind in his letter to Keferstein in 1890). Dedekind tried to provide an infinite totality of things by ascribing to us human subjects the ability to infinitely repeat certain ideas or mental actions, like adding another stroke. Dedekind considered his thought experiment to be a logical existence proof and he was not concerned, as Russell was, with the meaning of individual number symbols.

Doubts with regard to the plausibility of such a construction respecting with its assumptions arise if one is not willing to rigorously separate between our inner mental world and the outer empirical world. Otherwise, one could "prove" Euclid's parallel axiom precisely in this manner, by iteratively drawing infinitely further two parallel line segments of say 1-inch length. Why does the possibility to count an infinite set appear to us actually more plausible than such an illustration of geometry's parallel axiom? Why is there so much more confidence in conceptual constructions than in intuitive representations?

The answer could be the following: Either these two ideas represent only possibilities without any existence claims involved, then there is no difference. Or one of these, counting, is something mental and not subjected to any limitations, resp. to purely logical ones, while the other, i.e. geometric construction, has to respect objective constraints. Then the conditions for the two forms of control would nevertheless have to be analyzed first and one would have to develop a notion of (objective or logical) possibility. In contrast to this, many have been convinced, together with Frege that

> in arithmetic we are not concerned with objects which we come to know as something alien from without through the medium of the senses, but with objects given directly to our reason and which being their own kind are utterly transparent to it. (Frege 1884, §105)

The logical paradoxes proved this to be a mere illusion and showed that our mental world is no less complex and intransparent than outer reality and that there are constraints to our mental activities as there are to our concrete actions. In short, it is not possible to conduct a proof of existence without further assumptions. We do not know how Russell might have considered the difference between number and space, as representations of our inner resp. outer world. In any case Russell accordingly thinks that one can never attain infinite totalities by mere enumeration, and he considers it an empirical fact "that the mind is not capable of endlessly repeating the same act". Russell says: "So the reader, if he has a robust sense of reality, will feel convinced that it is impossible to manufacture an infinite collection out of a finite collection of individuals" (p. 135). We cannot prove that infinite numbers or sets exist. This is probably consensus today while the explanations given are very diverse and controversial.

Russell's critique of Dedekind's construction concerns also the relationship between object and concepts, which is relevant both for logic and for epistemology. While one must postulate, on the one hand, that concept and object are to be distinguished, even of different logical type, that is belong to different categorical strata, concepts, on the other hand, appear in the next step of iteration as extensions of second order concepts. As the assumptions necessary to justify such notions are by no means logical axioms, the situation becomes intransparent. Russell therefore emphasizes "that concepts do not have a factual existence in the common sense" and hence cannot be treated like things, which is, however, contrary to common mathematical practice.

Indeed the topologist Salomon Bochner rightly conceives of the iteration of abstraction as of the distinctive feature of the mathematics of the Scientific Revolution of the 17th century.

> In Greek mathematics, whatever its originality and reputation, symbolization [...] did not advance beyond a first stage, namely, beyond the process of idealization, which is a process of abstraction from direct actuality, [...] However [...] full-scale symbolization is much more than mere idealization. It involves, in particular, untrammeled escalation of abstraction, that is, abstraction from abstraction, abstraction from abstraction from abstraction, and so forth; and, all importantly, the general abstract objects thus arising, if viewed as instances of symbols, must be eligible for the exercise of certain productive manipulations and operations, if they are mathematically meaningful. (Bochner 1966, 18)

Russell also rejects all attempts at proving the existence of an infinite set for the reason that they violate the requirements of his own theory of types. On the other hand, as was said, mathematical development demands just this.

IV

What then is Russell's theory of types about? In order to repair certain paradoxes of logic and set theory, Russell had introduced the rule "Whatever involves *all* of a collection must not be one of the collection" (Russell 1971, 63) An all-concept referring to a totality cannot belong to the totality. Now Dedekind, in his "Was sind und was sollen die Zahlen?" (cf. Dedekind 1969, 14), had founded the proof for the existence of infinite sets on the antinomial set (of all things) "which can be object of my thought" (Russell's set of sets that are not members of themselves certainly is a possible object of thought). Cantor had called Dedekind's attention to the inconsistency in the latter's foundation of number already in 1899.

Despite the fact that it is not the principle of type classification in itself, but rather its too rigid, reifying interpretation, which is problematic, Russell himself characterizes it as only a negative move, as a principle of interdiction. In particular certain limitations are to be established with regard to the range of the quantified variables in propositional functions. An expression like "all propositions are either true or false", for instance, makes no sense anymore. And with respect to arithmetic it becomes clear that formal axioms could characterize number only in connection with a class of intended applications. Hence existence claims become the main concern of the philosophy of mathematics as they had already been since the beginning of the 19th century in mathematics proper. Existence can, however, be assured only by means, like ostension, which Carnap had outruled as being subjective. In this sense ideas, concepts, laws or other universals, strictly speaking, do not exist. Do Russell's sets exist?

Another consequence is, that mathematical axioms must be understood in analogy with natural laws, as hypothetically conditional propositions, something which Russell, in contrast to other logicians, seems to accept, even if he repeatedly vacillates in doing so (cf. Gödel, 1944, 127). Russell adopts a realistic position as expressed, for instance, when stating that

> logic is concerned with the real world just as truly as zoology, though with its more abstract and general features. To say that unicorns have an existence in heraldry, or in literature, or in imagination, is a most pitiful and paltry evasion. What exists in heraldry is not an animal, made of flesh and blood, moving and breathing of its own initiative. What exists is a picture, or a description in words [...] There is only one world, the "real" world [...] The sense of reality is vital for logic. (pp. 169-170)

Hence, Russell would like to ensure that the mathematical (resp. the logical) concepts be obtained by abstractions, like the empirical ones, rather than constructively. This is why he postulates the existence of infinite sets.

Russell accepts – and this might seem astonishing for an empiricist position – that thoughts and feelings are real, while considering the "objects" with which our thoughts and feelings are concerned, to be widely unreal. Shakespeare's thoughts for example, "in writing Hamlet are real. But it is of the very essence of fiction [...] that there is not [...] an objective Hamlet" (p. 169). "'A unicorn' is an indefinite description which describes nothing. It is not an indefinite description which describes something unreal" (p. 170). Such a kind of realism may blunder easily with respect to the existence claims of objects described. In the interest of the intended applications of mathematical theories the question arises which ontological status, then, mathematical, i.e. ideal objects possess.

On the one hand, type theoretical distinctions are quite self-evident and common. The totality of chairs is no chair, and the class of red things is not a red thing, but a hypostatic abstraction like redness, or a propositional function like "x is red", or whatever. While concept and object, menu and meal, map and territory are easily distinguished, this distinction, on the other hand, becomes a relative one from the perspective of mathematical activity and its dynamical development. The assumption of absolute boundaries between essence and existence, or between sign and object, is problematical from an epistemological point of view, as only that may appear as motive and driving force of cognizing activity which somehow appears to really exist. The map, or the mathematical model being instruments of exploration first of all, may also become objects of cognition in just the same way they were means of study before. Or to take the example of number theory: numbers are established in measurement as relations between quantities, but are in number theory considered as definite objects of investigation.

It is indeed mathematics which distinguishes itself for the fact that the process of abstraction is continued indefinitely and recursively, and the number of semantic levels appears to have increased quite considerably again in computer science as compared to traditional mathematics. This means it is the complexity of mathematics, of the "map" itself, which leads to problems. Hence the limitations of type theory. The theory of types touches the very same problem of contextualization and interpretation which had stimulated Russell's objections against modern axiomatic.

If one conceived the theory of types in an absolute or "ontological" sense, the natural numbers themselves already would all be of different type, not only on Dedekind's account, but also according to the set theoretical interpretation which Russell himself gave of them. Arithmetic activity, by contrast, requires them all to be of the same type. Russell attempts to meet this requirement later, by introducing additional logical axioms.

But numbers and sets of numbers or number concepts are of different logical type. We cannot proceed inductively in arithmetic, no more than in any empirical field, to gain general number concept. Is then a variable only a placeholder for the individual numbers, and must propositions containing it be interpreted in the sense of an infinite logical conjunction? Or do ideal mathematical objects exist and if so, in what sense? The two views coexist in mathematics. The difference between an affirmation about a general number and an affirmation about all numbers must be introduced into mathematics, as even Russell himself admits, because deduction depends on free variables, or ideal objects. The natural laws cannot be understood as infinite conjunction either, say according to the pattern "stone *a* drops", and "stone *b* drops", and "stone *c* drops", etc. We cannot give individual names to all the stones or all the atoms in the universe, no more than we could name all numbers or all triangles. Peano's axioms must indeed be understood meaning that if a number is given it obeys the axiomatic description and the axioms are also not to be interpreted as an infinite conjunction of definite statements.

> For example, the statement that if a is a numerical symbol, then $a+1=1+a$ is universally true, is from our finitary perspective

> incapable of negation. We will see this better if we consider that this statement cannot be interpreted as a conjunction of infinitely many numerical equations by means of 'and' but only as a hypothetical judgement which asserts something for the case when a numerical symbol is given. (Hilbert in: Benacerraf/Putnam (eds.) 1983, 194)

A natural laws' field of intended application must remain open and indeterminate in a certain sense, and it develops together with the evolution of theory. And even in arithmetic we have to accept non-standard models as soon as we confine ourselves to logical descriptions. On the other hand, however, the object field of a theory cannot remain completely indeterminate, for the natural laws alone do not yield any definite knowledge. Russell's efforts at awarding an ultimate meaning to numbers thus show that a theory's development and application are intertwined in a very complicated way. It is true that geometry and arithmetic are treated differently in general. Frege, for instance, who did influence Russell deeply held the view that in arithmetic there need be no appeal to intuition whereas geometry cannot be divorced from intuition. The analogy between mathematical axioms and natural laws would then apply only to geometrical axiomatic and consequently the notion of variable would also be different in arithmetic and geometry. In arithmetic variables were placeholders for definite entities and hence Russell's axiom of infinity.

Russell indeed seems to assume that the axiom of infinity is ontological in nature, and that it is more plausible to assume the existence of infinite sets than to adopt the opposite finitist hypothesis, even though neither can be proved. It is precisely a case of an axiom in the classical sense. And Russell himself appeals to intuitive plausibility. According to our "empirical evidence and the divisibility it would seem favorable to assume that there is an infinite number of objects in the universe", but we cannot know anything about it *a priori*, and "whether the axiom is true or false, there seems no known method of discovering" (p. 143). Strangely enough Russell fails to see that infinite divisibility actually does not mean anything but Dedekind's postulate of the infinite iterability of one and the same operation, would one not want to assume an unknown world of things in themselves, like Kant.

V

A comparison to Kant is in fact suggestive here. Kant's understanding of mathematics is based on the exemplary character of the Euclidean method as discussed at length in Aristotle's *Analytics.*

According to Aristotle mathematical definitions give the essence of a thing but do not say anything about its existence. Hence the existence of a thing must either be given directly in intuition or it must be proved showing *why* a thing is, that is, providing reasons for its existence. Aristotle says that this is particularly the case in geometry, where only points and lines must be assumed to exist, while the other notions must be given through some construction. Thus what cannot be intuited must be constructed in intuition.

In his Critique of Judgment Kant raises the question whether it is possible to provide a general characterization of the human intellect in difference to all other modes of knowing and he discovers it in the fact that we cannot know intuitively but rather are discursive minds with depend on concepts. God's intelligence is intuitive he cannot, comments Cassirer,

> think of a thing without, by this very act of thinking, creating and producing the thing. ... Human knowledge is by its very nature symbolic knowledge. It is this feature, which characterizes both its strength and its limitations. And for symbolic thought it is indispensable to make a sharp distinction between real and possible, between actual and ideal things. A symbol has no actual existence as a part of the physical world; it has a "meaning". (Cassirer 1962, 28th printing 1977, 56; 57)

Never shall we humans possess a direct, immediate access to the things in themselves. We can only think by means of symbols or concepts, but "concepts without intuitions are empty". Kant writes:

> I can think what I please, provided only I do not contradict myself; that is, provided my conception is a possible thought, though I may be unable to answer for the existence of a corresponding object in the sum of possibilities. But something more is required before I can attribute to such a conception objective validity, that is real possibility the other possibility being merely logical. (Kant, B XXVII)

To exhibit the real possibility of a mathematical conception on has to construct it. "The construction of a conception is the presentation a priori of the intuition which corresponds to the conception" (Kant, B 741). This corresponds to Aristotelian method.

The objectivity of mathematical knowledge is based on the legislative character of pure intuition in the process of knowledge application and concrete experience, according to Kant. It is well known that Kant inquires how it can be possible that the *a priori* conclusions of mathematics lead to real knowledge. And he answers this question by attempting, in a way quite similar to Russell, to determine the conditions of the applicability of mathematical judgments *a priori* and in principle. Kant localizes these conditions in pure intuition, which, differently from mere logical analysis and conceptualization, furnishes objective possibility. Following Kant, we have to distinguish, in a way which seems to anticipate Russell's theory of types, between the general conditions of applicability, and the application itself, between the conditions of the possibility of experience on the one hand, and individual experiences on the other, for Hume has taught us that inductive inference from the particular to the general is not possible "without regulative (not constitutive) principles of experience *a priori*".

The general conditions of the possibility of experience have to be determined *a priori*, quite in the sense of Russell's undertaking, and these *a priori* explanations refer to the conditions which have to be met by objects of possible experience. An epistemology from the perspective of the finite human subject compels us to define ontological questions epistemologically, and Kant thus determines the reality and objectiveness of experience from its productive conditions rather than on the basis of metaphysical assumptions. We obtain our cognitions, however, by intuition, even if we may not, like empirism does, identify intuition with the perception of mere sense data.

Ernst Cassirer, too, compares Kant's views to those of Russell, writing in this connection:

> that our concepts have to refer to intuitions means that they have to refer to mathematical physics, and to prove fruitful in how they are formed. The logical and mathematical concepts shall no longer form the tools we use to erect a metaphysical world of thought. They have their function and their legitimate application

> merely within empirical science itself. It is this, their very limitation, which assures reality to them. In this sense, Kant has formulated the supreme principle of all synthetic judgments: *"The conditions of the possibility of experience in general are likewise conditions of the possibility of the objects of experience, and have for that reason objective validity in an a priori synthetic judgment* (B 197). (Cassirer, 1907, 43)

Frege or Russell would perhaps respond to this by saying that there is objectivity in our reasoning and judging without pre-established objects. I hence understand, says Frege for example,

> objectivity to mean independence from our feeling, perceiving and intuition [...] but not independence from reason. For to answer the question what things are independent of reason would be as much as to judge without judging, or to wash the fur without wetting it. (Frege 1884, § 26)

Hence there is a strong similarity between Frege, Russell and Kant, apart form the question of intuition. Intuition and space were meant to justify existence claims and were indispensable, because for Kant the (possible) existence of objects was a prerequisite of knowledge and judgement. Frege, as we have seen, wanted to get rid of this prerequisite. But he could not define the identity of concepts or functions without it. Frege had no solution at all to the identity of concepts, which, although buried, remains a virulent issue to this day. And if our grasp of identity and hence "of the extensional equivalence depended on intuition, Frege's account would in its essentials collapse into Kant's" (Potter 2000, 80). This is in fact also the case with Russell's approach as based on the axiom of infinity.

It must be recalled that Russell, in his "Problems of Philosophy" of 1912, formulated a fundamental epistemological principle which strongly reminds of Kant's emphasis on the indispensability of intuition. He states there that "every proposition which we can understand must be composed wholly of constituents with which we are acquainted" (Russell 1997, 58), rather than knowing them merely by description. In the preceding chapter above, one might remember, we have seen that the existence claims involved in our descriptions of the world mark the cornerstone of Russell's realism.

In contrast to Kant, Russell proposes a logical construction of the theoretical concept, a construction, in which the axiom of infinity, replaces Kant's forms of pure intuition. Like Kant, Russell wants to constitute the applicability of mathematical concepts *a priori*. And both consider this possible by investigating into the structure of a separate world – the world of intuition vs. the world of logic.

And however strongly Russell repeatedly criticizes Kant for making too much use of intuition while entering too little upon logic and logical analysis, Russell's philosophy can nevertheless be justly characterized, as has often been done, by the formulation "away from Kant and back to Kant". Is the idea of set in the Russell-Cantor conception of mathematics not indeed just the substitute for space? And must not the set theoretical models, which supplement axiomatic mathematics, be understood in a way similar to how geometrical intuition was to be understood in its relationship to Euclid's axiomatic?

In particular Peano's fifth axiom, the axiom of mathematical induction, does not belong to logic in the proper sense (first order predicate logic) and cannot be replaced by logical axioms either (not even by infinitely many, a fact proved only in 1934 by Thoralf Skolem). In 1900 already, mathematicians like Poincaré had interpreted this axiom as an indication to the synthetic character of arithmetic in Kant's sense. Russell considers Poincaré's views to be erroneous, for

> mathematical induction is a definition, not a principle. There are some numbers to which it can be applied, and there are others [...] to which it cannot be applied. We *define* the 'natural numbers' as those to which proofs by mathematical induction can be applied, i.e. as those that possess all inductive properties. (p. 27)

This definition, however, depends on the foundation of the concept of number on the notion of set, and thus on the axiom of infinity, and on other "disputed" assumptions like the axiom of choice, which Russell himself considers to be "unprovable" (p. 117, see for this also Heinzmann, 1993).

VI

While we now know, having followed Russell's argument so far, that the concept of "number" is not "empty", to express it in Kantian terms, because there are finite and infinite sets to which it may be referred, we do not yet know its concrete meaning, its content, except for the fact that we have to assume that it is a concept of set theory in the general sense. It is one thing to postulate that infinite sets exist, and another to establish a correspondence between numbers and sets. We have to concretize this correspondence in particular in such a way that the entities thus given will respect Peano's five axioms. For this purpose, Russell constructs a set-theoretical model the details of which need not interest us here.

"Awareness of universals is called conceiving, and a universal of which we are aware, is called a concept" (Russell 1997, 52). We do not only perceive individual gradations of yellow, we also are aware of universals, general ideas like yellowness. This universal is the subject in judgments like ‚yellow is different from blue". In what, then, does awareness of a universal like "yellow" differ from one like "three"? Number words do not function as predicates one might answer. This is not a very profound answer, however.

Russell notes that "number" is the characteristic in numbers, just like "man" is the characteristic in men, or "triangle" the characteristic in triangles. "A number is anything which is the number of some class" (p. 19). Russell defines the individual numbers whose essence is summarized in the general concept of number as designations of respectively determined equivalence classes of sets, and "the number of a class is the class of all those classes that are similar to it" (18). In this determination, the concept of class or set is taken as a yet undefined basic concept, resp. we act as if sets were real things, and were, in fact, the only existing things.

Everybody who has been to school during the last 30 years will feel this to be somehow familiar, the catchword being "New Math". It is even more interesting to note that this reform movement which represents an attempt to create a place for modern mathematics in common sense, an attempt that was certainly as fascinating and heroic as it was ill advised, had been divided in itself from the very outset. One main current, which intended to reduce the meaning of all concepts to logic and set theory, was from the beginning opposed by another more oriented toward structure and axiomatic.

One of the most eminent reformers, for instance, Georges Papy from Belgium, held a lecture at the Düsseldorf Academy in 1967 in which he pointed out in particular the importance of the axiomatic method. Axiomatic, he said,

> has sometimes been presented as the greatest mathematical discovery of the 20th century. It is axiomatic indeed which offers a key to modern pedagogy. It is important that we understand each other well at this point. There are different axiomatic methods, and different kinds of axiomatic expositions. The most perfect and highest among them is the formal axiomatic representation. The objects are not defined and have a role in theory only by means of the abstract relations which have been introduced by the axioms.

One should not say, Papy continues, that this mode of thought was inappropriate for beginners. "One will quite conversely take the physicist's point of view who in his best moments often formulates axioms without realizing it, just like Molière's Monsieur Jourdain formulated prose" (our translation).

Here again, we have an echo of the connection between mathematization and axiomatization indicated by Hilbert. Russell's conception is thus confronted, in all the fields where we have to do with mathematics, with the other view that numbers and all universals of the like have to be conceived of functionally, in their role as means of thought. Rudolf Carnap says that logical empirism assumes "that the object and its concept are one and the same". This identification being a "functionalization" of the object rather than intending a reification of the concept (Carnap 1968, 10). And Moritz Schlick, founder of the "Vienna Circle", thinks that

> to make use of concepts in the business of reasoning requires no other of their properties than that certain judgments are valid (e.g. the axioms about the basic concepts of geometry). For exact science, which joins conclusion upon conclusion, the concept, consequently, is indeed nothing but that about which certain judgments hold. Thereby it is hence to be defined. (Schlick 1979, 51)

In the axiomatic approach, the objective seems reduced to the function of making the notion of truth definable.

Russell was not a logical positivist, and he called the predominant trend of logical positivism a new kind of scholasticism which, by concentrating too strongly on formal structure, "may forget the relation to fact that makes a statement true" (Russell 1966, 380). Russell, while deeply interested in logical analysis, always warned against the formalism of thought.

> A logically perfect language, if it could be constructed, would not only be intolerably prolix, but, as regards its vocabulary, would be very largely private to one speaker. That is to say, all the names that it would use would be private to that speaker and could not enter into the language of another speaker. [...] For in a logically perfect language, there would be only a single word for every simple object. [...] That is one reason why logic is so very backward as a science, because the needs of logic are so extraordinarily different from the needs of daily life. One wants a language in both, and unfortunately it is logic that has to give way, not daily life. (Russell 1966/1998, 198)

The intention to reduce arithmetic to logic, Russell says, only follows "a rule which is recognized in all of mathematics" and which has the objective of "making the results of a given process of deduction applicable". The mathematician, however, does not share the intentions of the logician in trying to dissect the concepts into their elementary components. The mathematician does not seek a language at all, but rather a model or a structure. The mathematician generalizes, as a rule, by weaken the hypotheses of his theorems, thus making them true about more structures and by making his structures less conditional in order to be able to enlarge the range of their possible applications. This is effected by the axiomatic foundation of mathematical theories.

Quite in contrast to Russell's view, one would claim it is precisely under aspects of application that the manifold interpretability of axiomatic systems makes the strength and significance of the axiomatic method. It is in particular difficult to understand why the meaning of all concepts should be totally fixed prior to all application. In the end concepts would be nor more than complete description of individual entities. Mathematical axiomatic, in contrast, is often characterized by Hilbert's pronouncement according to which, in the axioms of plane geometry, the terms 'point' and 'line' could just as well be substituted by the terms 'beer jug' and 'table'; "in other words: each and every theory can always be applied to an infinite number of systems of basic elements" (Hilbert in a letter to Frege dated Dec 29^{th}, 1899). Still it is true, we must find at least one application, to solve the problem of existence.

Even intuitionalistically and constructivistically minded logicians and philosophers, who do not assign any functions at all to the axiomatic method, emphasize its significance for a praxis of application. In particular, the method seems well-suited to clarify the structure of theories, and "to fuse theories which were originally quite different by uncovering relationships of isomorphy" (Heyting 1934, 30). Systems of axioms thus are properly speaking meta-theoretical entities; this aspect has a rather important role in present-day mathematics because a large part of applications and processes of mathematization occur within mathematics itself (like in the search for inter-theoretical relations).

Only when we have a special intended application in mind, the problem of how to construct the necessary correspondences arises. During the entire 17^{th} and 18^{th} centuries, for instance, mathematics had been understood as the science of quantity. As the new phenomena of electricity and magnetism had to be mathematized in the 19^{th} century, it become necessary to extend the concept of quantity to vector quantity (directed quantity) and to generalize the quantity operations correspondingly (e.g. by introducing a non-commutative vector product). One can as a rule not derive the axioms from the "essence" of the relata (from the meaning of the latter's concepts), simply because one may only know the things with regard to some few relations between them. All objective knowledge is, in fact, relational knowledge. Kant therefore insisted that we cannot know the things in themselves. Axiomatic theories then are synthetic in this sense, and their value lies in their fertility, not in objective or logical justification.

Russell was indeed not unaware of Hilbert's axiomatic method. For him, however, the interpretation of a deductive system always is of fundamental philosophical significance, as it shall serve to clarify how an empiric theory can be linked to perception, and how a mathematical theory, like the theory of arithmetic, can be linked to logic. Despite all his open-mindedness with regard to future developments of mathematics and of the exact natural sciences, Russell repeatedly stresses his interest in the absolutely invariant elements of cognition.

Russell deems the axiomatic method to be incomplete, as unspecified terms occur within the axioms. These uninterpreted terms must indeed be specified in a way, which permits to establish a connection to the intended application. An absolute or ultimate interpretation of mathematical concepts, however, is generally neither possible nor desirable. The axiomatic determination of mathematical concepts will always be incomplete, inasmuch one always has to take into account the possibility that a concept has an empty extension (inasmuch the axioms can be inconsistent), or that it is ambiguous (a property desirable under the aspect of application). If one intends, against that, to introduce all concepts by complete definitions, one must necessarily make metaphysical and psychological assumptions about the world, as it is in itself, which is a futile undertaking. The concept of set which Russell tries to use to specify the meaning of number now is certainly cognitively not less sophisticated than the number concept itself. Russell, however, deems the concept of set both logically more fundamental and empirically simpler.

Technically, there are two problems which have already been mentioned several times. Firstly, the contrast between constructive versus inductive foundation of the concepts or theories. With respect to arithmetics, Russell, like brands of constructivists (Hölder, for instance) prefers to base number on cardinality, defining the cardinal number of a given collection as the set of all equally numerous collections, rather than using the axiomatics of ordinal number.

Secondly, the problem of indeterminacy: the more we try to control our own concepts, and the more we codify and formalize our language, the more we restrict our notion of existence, the less we can maintain the claim to say *a priori* something about the applicability of our mathematical or scientific theories, or about their ontological foundations, for a logical language would not only be, as Russell himself says, a speaker's (resp. theory's) private language, but it would also be useless with regard to application and generalization. I think that this is the real meaning of Einstein's famous quip: "as far as the theorems of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality" (Einstein, quoted after the reprint in: Strubecker (ed), 1972, 414). This is not to mean that mathematics is dispensable, or not useful for application, but only that we cannot "explain" its applicability itself in a mathematical or logical theory. Theories are not at the same time theories of their own application.

They are not mere relational structures either. The meaning of our theoretical concepts depends rather more on the methods of their application, like methods of experimentation and measurement for instance, than on notions about the world. The axiomatic perspective, and in so far Russell is right, must always be supplemented by the perspective of the active subject, placed within a concrete context of life and activity. We must entertain other relationships to the objects of our cognitions than merely theoretico-conceptual ones.

VII

There is no possibility of ultimately determining the meaning of "number", not even within the framework of set theory. The widely known and repeatedly reprinted collection on the philosophy of mathematics (cf. Benacerraf/Putnam 1983), in which parts of Russell's "Introduction to Mathematical Philosophy" were reprinted as well, also contains an essay by Paul Benacerraf titled "What Number could not be" in which he shows that the number concept can be reduced to the concept of set in many, very different ways, without there being an opportunity to separate out the correct set theoretical interpretation from all the possible ones. Benacerraf hence concludes that numbers cannot be sets, or sets of sets, at all, as there exist very different accounts of the meaning and reference of number words in terms of set theory. And even Quine, who shared Russell's dislike for the "disinterpretation of mathematics" and who pointed out that words like "two" and "four" do not occur uninterpreted anywhere in our language, emphasizes that every set theoretical interpretation of number words – Frege's, von Neumann's, or Zermelo's – is used "opportunistically to suit the job in hand, if the job is one that calls for providing a version of number of all" (Quine 1960, 263).

Now either one understands set theory as a means of construction not unlike ruler and compass in Euclidian geometry, being thus prepared to accept, like in geometry, that things can be constructed in various ways and that every construction quite naturally highlights different characteristics of the thing constructed – Benacerraf's various set-theoretical reconstructions of the number system agree in overall structure while disagreeing when it comes to fixing the referents for the particular number terms –, or one would have to say that the concept of set does not represent the last, ultimate, adequate basis for establishing the existence of number. Existence cannot be provided by symbolic means and has to be assumed from the beginning.

One could thus come to see Russell's set theoretical reductionism, or should we now letter say: constructivism, on the one hand and the axiomatic method in Peano's and Hilbert's sense, on the other as complementary views on mathematics exactly like in classical geometry constructive and deductive method have been complementary to each other (cf. Casari 1974, 49-61; Putnam <1967>, 1975). This complementarity is also expressed in the notions of concept and object in mathematics.

Numbers, for instance, are universals, or general concepts. While from one point of view the general is conceived of as a predicative general, as a concept which (rightly or wrongly) assigns a certain predicate to (concrete or ideal) objects, the general also might appear on the other hand as an ideal object whose generality stems from its (relative) indeterminedness. Any proposition can be written as relation between concrete and universal objects. Instead of saying, for instance: "honey is sweet, one could say: honey posseses sweetness" and could than add sentences in which "sweetness" enters into the subject position like "the sweetness of honey is different from the sweetness of sugar". This implies that we have two types of universals (relations and relata). We claim, for instance, that numbers having certain properties exist, just like we speak of the existence of empirical concepts; like "energy" or "society", for instance. This is in line with the axiomatic view. A system of axiom never defines a determined, individual concept or object, but rather an undetermined universal object. Axiomatic theory refrains deliberately from establishing its field of application in detail – this being Russell's very complaint – but it possesses its "objects" in a general sense.

Let us take another empirical example, that of colors. Redness is to be understood predicatively, and at the same time as an object, as a color, which occurs in an infinite variety of hues. One can now ascribe the generality to its predicative character (that is to class), or to its relative indeterminedness. In the first case, the general is exclusively of predicative nature or, as Russell would say, it resides in propositional functions, and accordingly redness is to be understood as the propositional function "x is red". In the second case, one assumes the existence of general or ideal objects in a more or less Platonist sense, a view expressed in the fact that hypostatic abstractions like "redness" or "sweetness" assume in certain expressions the position of the subject.

In the first case, I have an attribution of things and judgments, in the second case, other judgments are inferred from given ones. In the first case, the generality is tied to the impossibility of finding a counter-example. In the second case, I can apply the law of contradiction only if I confine myself to the questions admissible within the theoretical context. It makes no sense to inquire into the color of the number 3 or into the spiritual life of the falling stone. As mathematics claims to establish objective truths as well, as it is essentially conceived very often also in terms of deductive reasoning and formal proof, both types of universals, functions and objects, are indispensable to its enterprise.

Where deductive inference is concerned, it is indeed not necessary for us to assume the existence of concrete objects determined in every respect. As mathematical deduction reduces to material implication which interprets the proposition "X implies Y", for two sentences X and Y to say that either X is false, or Y must be true (or both being the case), I need not assume the existence of an object x which would make Russell's propositional function "x is a general triangle" true. An index ensuring the reference is sufficient. General objects are nothing but relations or signs.

Let us consider the elementary geometrical concept of "general triangle". It has been the object of many discussions since Locke and Berkeley, which properties are due to a "general triangle". The essential point lies in that the "general triangle" does not possess any determinate properties at all, but that it represents merely a possibility to determine a triangle appropriate for a particular context. Just like the continuum contains the possibility of determining individual points, while not being constituted by points itself. A general triangle is a free variable, and not a collection of determinate triangles.

Which properties are due then to a "general triangle" in a certain situation depends on context. If the task, for instance, is to prove the theorem that the medians of a triangle intersect in one point, the triangle on which the proof is to be based can be assumed to be an equilateral one without loss of generality – because the theorem in case is a theorem of affine geometry and triangle is equivalent to an equilateral triangle under affine transformations. This fact considerably facilitates conducting the proof because of such a triangle's high symmetry. The truth of a proposition about the "general triangle" then means nothing but that this proposition is provable in a certain way, i.o. that a certain proof scheme applies. If one operates solely on the basis of the object's axiomatic characterizations one does not use the concepts involved referentially at all. Both mathematical and empirical theories as a rule use further resources to solve their problems, and this is how the problem of interpretation of deductive systems becomes significant.

VIII

Let us recapitulate what has been presented up to now. Empirical thought is initially objective thought without worries. It is about the properties of familiar objects, and about dealing with things. Mathematical thought, in contrast, as Aristotle already says, begins with the Pythagoreans, with "theoremata" like: "The product of two odd number is odd". Or: "If an odd number divides an even number without rest, it also divides half that number without rest". These are theorems which, as one says, go beyond what can be experienced concretely, because they state something about infinitely many objects. Actually, they do not state anything at all about objects (e.g. about numbers), but are about universals or ideal objects. They are analytic sentences, which unfold the meaning of certain concepts. This kind of conceptual inference finds its most exalted expression in the modern axiomatic method, a method that is not at all confined to mathematics and to logic.

Now in a period in which the problem of mathematics' applicability had become ever more significant, Immanuel Kant had put forward the claim that real mathematical cognition did not correspond to this structuralist conception, and that mathematics, while being *a priori*, is nevertheless, like any other cognition, both constructive and dependent on experience. Mathematical thought is objective, too, Kant says. Its objectivity shows precisely in intuition. Only, mathematics must construct its objects on the basis of certain conceptual determinations, being unable to take these abstractively from empirical reality because mathematical forms are the basis of all our scientific experience. And what is a precondition of experience cannot be its result. The definition of a mathematical concept, Kant say, thus has to be followed by making sure that the concept created is not "empty", but possesses possible applications. As the mathematical concepts proper designate connected actions, one must ascertain the possibility of their application in a mental intuition or anticipation.

How do we prove, for instance, those seemingly analytic propositions like the already quoted "the product of two odd numbers is odd"? We intuitively represent certain activities. We will say, for instance, if an odd number is divided by 2, there will per definition remain a rest of one. From this we now infer that there is for each odd number X another number N such as that X = 2N + 1 holds. If we now have two odd numbers represented in this way before us, and if we multiply these, the said theorem will result quasi automatically by applying the distributive and commutative laws. Mathematics typically proceeds by constructing (algebraic or geometric) diagrams and by observing them, rather than by analyzing the meanings of mathematical concepts. And this results from the fact that mathematics deals with relations and that relations are extrinsic to the relata, as Hume had taught Kant. Mathematics conceived of as diagrammatic reasoning is synthetic a priori, because it cannot itself further justify the rules according to which it proceeds (i. e. the commutative and distributive laws in the example given).

Now according to the structuralist or axiomatic approach one would define the concept exactly by the relations as specified by the axioms (distributive and associative laws) and thus would call the proof analytical. If in contrast one either shares Kant's epistemological approach, that means his fundamental affirmation about the objectivity of the subject, or his view that the objects in question are the individual numbers, rather than the structure itself, then one might be compelled to acknowledge that an equation involving large numbers is mediated by calculation and diagrammatic reasoning to check it and that hence the theorems of arithmetic do not flow directly from the nature of the numbers involved (an argument which Kant uses himself (B 16) such that it seems strange to see Frege apparently using it against Kant (Potter 2000, 60)), and one would have to understand the arithmetical laws (i.e. the arithmetical axioms) apodictically and objectively given and hence consider arithmetic as synthetic.

An empiricist like Russell finally might want to gain these laws inductively from examples and then has to proceed to arguing that for this purpose we need some principle of continuity or principle of the permanence of forms, and would then be forced to find out how this universal principle itself can be justified. According to Kant's prediction, he would have to accept it apodictically, which once more would show that arithmetic is in some sense synthetic.

Russell, on account of his realistic conception of the laws of logic, seems indeed near to some kind of Kantianism, but thinks that Kant restricts the validity of *a priori* statements too much inasmuch he seems to make them dependent on the subject. It could be objected against this that the logical principle of contradiction ultimately remains subject-referred, just like Kant's principle of pure intuition.

An essential objection against the analytical view results from the observation that mathematical structures as well as proofs do not exist or have no effect without being concretely represented or applied. Mathematical objects are intensional objects in the first place. Two mathematical objects can be extensionally identical, but intensionally different by being presented differently. This is why mathematical affirmations as a rule have the form of equations $A = B$. In our example we had: $X = 2N + 1$.

It is important to observe at this point that it does not make sense to ask of *any* two symbols whether they represent the same object or different ones. Identities of the form $A = B$ are context dependent referring to things from a certain perspective. Two commodities, like shoes and chairs, may be considered equivalent from the point of view of economical value. They are equivalent as representatives of some concept or universal. The law of the conservation of energy marks the most important discovery of natural science during the era of Industrial Revolution. Where from do we know, however, that motion and heat are different phenomenal forms of one and the same entity, we call energy? Is it really a case of a discovery, or is what we designate as energy merely a fiction? In our example we compared number symbols and would have to establish the number concept beforehand to establish the essential context. The approach of Russell or Frege in contrast demands a universal and uniform ontological furniture of the world. Logic would really have to deal directly with the universe itself. As Frege had put it:

> If we are to use the symbol A to signify an object, we must have a criterion for deciding in *all* cases whether B is the same as A, even if it is not always in our power to apply this criterion. (Frege 1884, § 62) (we shall come back to this in section XII)

IX

Russell felt the time when he made his discoveries, in 1900, after his visit to the Second International Congress of Mathematicians in Paris, to be a time of "intellectual excitation". He writes: "Intellectually the month of September 1900 was the highest point of my life" (Russell 1967, 145). It is well known that Russell's own discovery of the set theoretical antinomies one year later put an end to the belief in ultimate logical foundations. During the attempt to solve the logical problems arising from that effort and to bring the logicist program to a close – an attempt from which the monumental work "Principia Mathematica" evolved, which was published between 1910 and 1913 in three volumes – logicism had attained a culminating point, and simultaneously its definite crisis, as Russell found himself compelled to hypothetically accept non-logical axioms like that which ensures the existence of infinite sets. What proves to be problematic, however, is the reifying conception of set itself, say as a collection of any kind of things. By the paradoxes he had discovered, Russell saw himself prompted to found the concept of set logically.

Where set theory is concerned, Russell's familiar paradox can be avoided if we distinguish, in the sense of type theory, between the set and the totality of its elements, for in this case it is true for every set that it is not an element of itself, and Russell's "set of all sets which do not contain themselves as an element" would be the most comprehensive imaginable set indeed and would represent something like a largest cardinal number. Such a concept, however, is contradictory. Cantor had inferred this on the basis of his own power set axiom, and Leibniz already had connected the question of the infinite with the problem of the contradictory character of such universal concepts. To put it quite generally: our highest level concepts or principles cannot be unequivocally specified or defined.

In his letter of July 1899, in which Cantor pointed out to Dedekind the inconsistency in the latter's foundation of arithmetic, Cantor had already established the principle that only that can be called a set or a "consistent multiplicity" which lends itself to being combined to "one thing", whereas he calls multiplicities for which "the assumption that all of its elements are together leads to a contradiction", it being impossible to conceive of the multiplicity as of "one completed thing", absolutely infinite or inconsistent multiplicities. Independent of Cantor, Russell arrives at a similar conclusion, defining sets, as we shall see, finally as extensions of propositional functions.

From objects to relations or propositional functions, from construction to deduction, and from intuition to language and to logic, this is the path of mathematical philosophy, which Russell embodies in such a most exemplary way. This is how language and speech then become the "conditions of the possibility of cognition", the latter being nothing but linguistically modified Kantianism.

> "Analytic" philosophy is one more variant of Kantian philosophy, a variant marked principally by thinking of representation as linguistic rather than mental, and of philosophy of language rather than "transcendental critique", or psychology, as the discipline which exhibits the "foundations of knowledge". This emphasis on language [...] does not essentially change the Cartesian-Kantian problematic, and thus does not really give philosophy a new self-image. (Rorty 1979, p. 8)

From Russell's type theory it follows, as has been said, that sets cannot simply be what common sense imagines them to be and what everybody has learned who has been to school in the times of the "New Math" reform movement. Although Russell makes large efforts over extensive parts of the book to reduce the concept of number to the concept of set, treating the latter as a fundamental concept in doing so, the way he develops his own argument shows that such a conception must be revised. The last but one chapter hence is about sets, and there Russell's concern is to "realize why classes cannot be regarded as part of the ultimate furniture of the world [...] We cannot take classes in the *pure* extensional way as simply heaps or conglomerations" (pp. 182-183). If we conceived of sets extensionally as objects, Russell thinks, it would be impossible for us to understand

> how there can be such a class as the null-class, which has no members at all and cannot be regarded as a "heap"; we should also find it very hard to understand how it comes about that a class which has only one member is not identical with that one member. (p. 183)

Kurt Gödel has considered this an overreaction to the paradoxes, saying that Russell's argumentation at best shows "that the null class and the unit classes (as distinct from their only element) are fictions, not that all the classes are fictions" (Gödel 1944, p. 141). The two particular classes mentioned above could be treated, Gödel continues, like the points at infinity in geometry, and like similar mathematical generalizations. This, however, would require historically determined borderline concepts or general principles, like the principle of continuity, or the principle of permanence of relations, and would then lead to understanding the theory's context as the primary thing and foundation. With this, however, we should be back to the axiomatic view and then would again have to bother about Russell's' problem of how to determine the applicability of an axiomatized theory.

In the course of the argument of his book Russell gradually gives up his realistic attitude and the basic entities of his reconstruction become more and more turned into "logical fictions". In his penultimate chapter, Russell indeed defines sets as equivalence classes of propositional functions. Such a definition could be conceived of to be circular if one were to assume that the concept of function requires identity itself. A propositional function, Russell says, is "a function whose values are sentences". Russell, for instance, defines the zero class as the extension of a contradictory propositional function, i.e. a propositional function, which ascribes contradictory things to its subject. Here, the attempt at logical precision causes the tendency to push assumptions of existence and ontological notions as far back as possible, or to eliminate them altogether, and to advocate an intensional view of set theory. The ontological status of these intensions, concepts or functions, however remains somewhat unclear. In his chapter II, Russell had already expounded the advantages of an intensional conception of sets. The same tendency, however, characterized predominantly the axiomatic method, and thus set theory also came to be axiomatized by Zermelo, in order to exclude the paradoxical set formations. The problem of application arises anew.

From relations and functions (of axiomatic) to objects (sets) and from there back again to functions (propositional functions): this to and for path demonstrates the logician's desperate quest for a thing absolutely given and existent. On this path to make things ever more precise, Russell at times wanders into a position of extreme empirism, one that insists on regarding nothing but the immediate sensual data for real.

> Here we find a remarkable development in Russell's thought lasting two or tree decades. A seemingly straightforward epistemology is dragged by logic and a theory of meaning into one of the most exotic metaphysics yet presented. We call it "logical atomism". In the beginning Russell thought that the things with which I am acquainted are the immediate objects of experience. (Hacking 1975, 72)

But finally he ends up with pure sensual data. To make his point, Russell holds up a piece of chalk and says: "This is white [...] I do not wish you to be thinking of this piece of chalk in my hand, but of what you see if you direct your gaze to this piece of chalk." Hacking comments on the fact that the "this" in Russell's statement "this is white" is not intended to designate an object, but a sensual datum, by pointing out that Russell "almost adopted Berkeley's position" in this way. Russell himself, after having arrived at a position of extreme empirism, closes by assuming that empirical objects like chair, table, etc. only exist in one's head as well. This is in some way a Berkeleyism "without God", and hence without anybody who will still perceive my writing table, thus conveying existence to it, after I have turned my back on it. As Russell lacks Berkeley's God whose perception conveys duration to the world, his position is even more complicated or exquisite than the latter's.

X

We should like to briefly discuss the book's last chapter on the relationship between mathematics and logic, before we come back to Russell's notion of set. The chapter begins by stating that

> mathematics and logic, historically speaking, have been entirely distinct studies. Mathematics has been connected with science, logic with Greek. But both have developed in modern times: logic has become more mathematical and mathematics has become more logical [...] This view is resented by logicians who, having spent their time in the study of classical texts, are incapable of following a piece of symbolic reasoning, and by mathematicians who have learnt a technique without troubling to inquire into its meaning or justification. (p. 194)

Nevertheless, it is even truer today that the philosophy of mathematics means something different to mathematicians than to logicians.

Frege brought about the extension of logic, which forms the basis of all of Russell's work by drawing on the mathematical notion of function. This notion hence offers the opportunity of establishing a relationship between mathematics and logic, even if it is true that, in line with the separation of the fields of work proper to mathematics and to logic, different exemplary representations of the term of "function" exist in mathematics on the one hand, and logic on the other. For mathematics and the exact sciences, the concept of natural law yields the prototypical case of a function. For logic, against that, a function is either a propositional function, or an algebraic formula.

Notwithstanding these different intuitions, it makes sense to illustrate the difference between mathematics and logic by considering the problematic of the function concept. It is a fact that mathematics has always since the 17th century been busy with great emphasis to construct ever more general functions. Mathematics is essentially relational thinking, and logic did not begin to modernize itself before it learnt from mathematics during the 19th century to construct a logic of relations. Bolzano, de Morgan, and Jevons presumably were the first to treat propositions as relations, or functions.

Let us thus speak of functions quite generally. Functions initially were simply correspondences between a domain of inputs, that is the area of the function's arguments, and an area of values of the functional transformation, the so-called co-domain or outputs. Propositional functions assign sentences, resp. judgments about things to things, In this sense, one can also conceive of concepts as of abbreviated judgments, or functions.

But just as the user of a computer program is not interested in the details of programming, but rather in the function thus provided, mathematics, logic, and science as such are always concerned with dissociating themselves from the concrete modalities by which a function is given. Mathematics and logic worked at elaborating such a general concept of function throughout the entire 18th and 19th centuries, until they came to the conclusion to understand a function as an equivalence class of concrete representations of the same, the relation of equivalence being given by the "axiom of extensionality". This axiom says that functions are equivalent if they yield the same values when applied to the same arguments. Only to such equivalence classes of concrete representations can be attributed important properties, for instance the property of continuity, of differentiability etc. (cf. Otte, 1994, chapter X). And we are then even able to define algebraic or logical operations for such classes. In this way, the equivalence classes become the objects proper of mathematics and logic. This respective method of establishing functions or concepts in general is called "definition by abstraction".

Now it is possible to hold the view that mathematics has essentially to do with intensional objects, i.e. with objects, which are determined conceptually, resp. by their properties. One or another representation of the function can be of particular importance, according to the respective goal. If I have, for instance, two views of a house, and desire to know how the front looks like a picture of the front side will surely be more useful than one of the back side. The same is true with the ideal objects of mathematics and with logical fictions. Which of the properties are significant for the representation of a "general triangle" in elementary geometry is dependent on the task at hand. We have already commented on this. And this is where the entire problem lies which consists in finding a specific representation of such an equivalence class or of an ideal object, an extensionally equivalent, but intensionally quite different representation.

XI

In Frege's presentation of the distinction between meaning and reference, the same asymmetry occurs which we have established for the identity criteria of functions, and which is based on existence claims with respect to the arguments or objects, while concepts (intensions) or functions merely possess an identity. Russell has treated this problem in his famous theory of descriptions which he summarizes in its essential parts in the 16^{th} chapter of his "Introduction". He corrected and extended Frege's interpretation of A = B, resp. of A = A by introducing a distinction between designation and reference. Frege had treated the difference between these two forms of an equation by his own distinction between sense and meaning, concluding that singular descriptions function like designations as one usually understands them referentially. Russell considers this to be an error, for

> a proposition containing a description is not identical with what that proposition becomes when a name is substituted, even if the name names the same object as the description describes. "Scott is the author of *Waverley*" is obviously a different proposition from "Scott is Scott" (p. 174), and further on: "If 'x' is a name, x = x is not the same proposition as 'the author of *Waverley* is the author of *Waverley*'" (p. 176). x = x is always true because x being an index (a name) is taken as an existence claim, rather than as a description. In fact, propositions of the form "the so-and-so is the so-and-so" are not always true: it is necessary that the so-and-so should *exist*. It is false that the present king of France is the present king of France, or that a round square is a round square. (p. 176)

To speak of contradictions makes no sense without any existence claims: Green unicorns are yellow. Or aren't they? Even the paradox of the perpetually lying inhabitant of Crete or that of Russell's "barber who shaves exactly those men that do not shave themselves" can be reduced in this way to problems caused by implicit illegitimate existence claims (cf. L. Wen 2001). We cannot generate the existence of an objective reality by concepts and language alone, and without this reality, there are no contractions either, Kant would say (cf. *Critique of Pure Reason*, B 622).

Russell himself obviously passes here, where the concept of function is concerned, into the complementary extensional position as well (he had spoken of "two kinds of propositional functions with which we have to do"). In other terms: the "being" of functions now does indeed depend on the domain of the quantified argument variables, thus supposing an object classification in the function's field of application. In classical mathematics a theory of functions requires a theory of quantities first. Russell's theory of types alone imposes strong restrictions on the range of the quantified variables. In logic proper (first level predicate logic), the terms "all" and "there are" may only be applied to objects (subject variables), but in no case to predicate variables (concepts). This requirement is largely not met by Peano's axioms, for in his fifth axiom, there occurs predication over all properties.

Above all, however, Russell wishes to see established, before all science and mathematics, an absolute and universal ontology, to see "what there is". This kind of fundamentalism is of very traditional nature, and is being treated by Russell in an increasingly restrictive way. In modern mathematics, however, activity is primary, not ontology and Russell himself acts in this way. Russell himself remains silent with respect to his permanent assumptions concerning the existence or identity of concepts he employs. The questions of ontology are determined relatively, with reference to an activity, resp. to a theory. Sets, functions, concepts and the like cannot be conceived of in a purely extensional way. Mathematics was, during long periods of its history understood constructively.

Equations of the A=B kind can only be regarded within their context; the problem of individuation cannot be solved either in a purely logical way, nor empirically. Russell's famous theory of description shows that answers of any kind could be given to the question "what there is". It depends on the perspective taken. The theory or the system of description actually determines what there is, and what is admitted as possible. But even to-days structuralist philosophy of science which assumes that any decomposition of our world of experience into ontological categories is theory-dependent, conversely finds itself impelled, in the question as to the criteria of identity for theories, to accept pragmatic determinations. Let us take Newton's particle mechanics as an example.

> Of course, the initially purely formally characterized term, like that of "particle", must additionally be given an interpretation which is correct with regard to content; the "particles" mentioned here are to represent genuine particles, and no loving souls. However: what are genuine particles? [...] The answer given by the structuralist approach is very differentiated. The first thing established is that theories contain intended applications as constituents – in a comprehensive sense of application. [...] From this follows that it is the pragmatic relationship between the epistemic subject who tries to apply the theory, and the theory itself, which determines what is to be regarded as a real, or genuine, object. (Moulines 1994, 187f)

We have already pointed out the example of electromagnetism, and the connected reinterpretation of the concept of quantity. Logicians and mathematicians never operate only in one homogenous object field, but are intent on obtaining and opening ever new applications for their structures. If they are asked by what they let themselves be guided, the most frequent answers are: intuition, analogy, experience, etc. And surely we will have to understand these concepts in a pragmatic sense. What is "intuitive" in each case is quite relative, and human experience can never be fully expressed in words.

Logicians like Frege and Russell do not want to bother with intuition. If we intend to form equivalence classes, however, we must introduce a perspective from which we consider things as similar or equivalent. In a certain sense everything seems similar to everything. Ideal objects like number, energy, society, etc. are indeed not given to us in themselves, but only mediated by an activity and some feeling of "similarity" or "equality of species". The concept of similarity, however, resists "the reduction to less dubious concepts like that of logic or set theory" (Quine).

The subject matter of mathematics being defined in terms of similarity and difference as well, we have also to introduce strict divisions and classifications. Already Aristotle's thinking was riddled with two orientations diametrically opposed to each other. Aristotle is most often regarded as the great representative of a logic and mathematics, which rests on the assumption of the possibility of clear divisions and rigorous classification.

> But this is only half the story about Aristotle; and it is questionable whether it is the more important half. For it is equally true that he first suggested the limitations and dangers of classification, and the non-conformity of nature to those sharp divisions which are so indispensable for language [...] (Lovejoy 1964, 58),

and for mathematics, as one might add.

XII

Now logic and mathematics do not only differ as to their views about mathematics, but views differ within logic as well. We have already hinted at this in the examples given by Russell on the one hand and Peano on the other and have called these two orientations set-theoretical reductionism on the one hand and axiomatics on the other. J. van Heijenoort has characterized the two rather different conceptions of logic connected with these different foundational programs. The first considering logic as universal and as a language, the other viewing logic more as a part of an algebraic calculus in the sense of Boole, Grassmann, or Peano. Russell, who like many other authors was strongly influenced by Frege, rather more advocates the conception of logic as a language, which remains inevitably imbedded in our thinking.

> The universality of logic expresses itself in an important feature of Frege's system. As is well known, according to Frege, the ontological furniture of the universe, divides into objects and functions, (Heijenoort 1967, 325)

The other conception of logic does not know of the notion of a fixedly given universal ontology. Rather, the ontology can be changed.

> The universe of discourse comprehends only what we agree to consider at a certain time, in a certain context. For Frege it cannot be a question of changing universes. One could not even say that he restricts himself to *one* universe. Not necessarily the physical universe, of course, because for Frege some objects are not physical. Frege's universe consists of all that there is, and it is fixed. (Heijenoort 1967, 325)

Just as there were two views of logic in the 19th century, there were two different conceptions about what could be called the "foundation of mathematics". Besides the axiomatic movement, which climaxed with Hilbert's "Grundlagen der Geometrie" of 1899, there was the movement of the so-called "arithmetizing rigor" which strove to reduce all mathematics to numbers. Russell's focus, too, went increasingly towards "understanding number" after 1897, as he considered arithmetic to be the foundation and the starting point of all mathematics. He interpreted this in the sense of the goal of reducing arithmetic to logic, resp. to logic and set theory. With this, we have returned to our starting point.

This gives us occasion to sum up the presentation of Russell's philosophy of mathematics in a thesis which places the latter within a more general epistemological context. Russell's epistemology, and in particular his philosophy of mathematics, is based in all its particularities and difficulties on the attempt to specify the problem of application and the reference to reality of our (mathematical) cognition by a-priori means, and thereby to remove its independent dynamic and insecurities from the problem of application. This resides in the nature of a logical conception of the world. Russell's philosophy appears in some way as Kantianism turned inside out logically. In contrast to pure mathematics and to logical positivism, Russell is concerned with application. But he attributes absolute priority to logic, thus eliminating the connected problem of the interchange between cognition and reality.

The effect of this is that reading his texts is instructive in a twofold sense, both in his achievements and logical innovations and in his deficiencies and errors. Any introduction to mathematics and to mathematical philosophy should start as a commentary on Russell' project.

Bibliography

BATESON, G. (1973). *Steps to an Ecology of Mind.* Paladin, Frogmore UK.

BENACERRAF, P. and PUTNAM, H. (eds.) (1983). *Philosophy of Mathematics*. Cambridge UP.

BOCHNER, S. (1966). *The Role of Mathematics in the Rise of Science.* Princeton, Princeton University Press.

CARNAP, R. (1968). *The Logical Structure of the World.* London, Routledge & Kegan Paul.

CASSIRER, E. (1907). *Kant und die moderne Mathematik.* Kantstudien Bd. 12.

______ (1962). *Leibniz' System in seinen wissenschaftlichen Grundlagen*. Darmstadt, Wissenschaftliche Buchgesellschaft.

CASARI, E. (1974). Axiomatical and Set-theoretical Thinking. *Synthese* 27.

DEDEKIND, R. (1969). *Was sind und was sollen die Zahlen?* Vieweg, Braunschweig.

FREGE, G. (1884). *Grundlagen der Arithmetik.* Breslau.

GÖDEL, K. (1944). "Russell's Mathematical Logic". In: SCHILPP, P. A. (hgb.). *The Philosophy of Living Philosophers.*

HACKING, I. (1975). *Why does Language Matter to Philosophy*. Cambridge University Press.

HEIJENOORT, J. (1967). Logic as Calculus and Logic as Language. *Synthese*, v. 17, pp. 324-330.

HEINZMANN, G. (1993). *Zwischen Objektkonstruktion und Strukturanalyse.* Vandenoeck Ruprecht, Göttingen.

HEYTING, A. (1934). *Mathematische Grundlagenforschung, Intuitionismus. Beweistheorie.* Heidelberg, Springer Verlag.

HILBERT, D. (1964). Hilbertana. Wissenschaftliche Buchgesellschaft, Darmstadt

HÖLDER, O. (1899). Anschauung und Denken in der Geometrie. Inauguration Lecture, Leipzig.

KORIAKO, D. (1999). *Kants Philosophie der Mathematik.* Hamburg, Meiner Verlag.

KRAJEWSKI, W. (1977). *Correspondence Principle and Growth of Science.* Dordrecht, D. Reidel Publishing Company.

LOVEJOY, A. O. (1964). *The Great Chain of Being: A Study of the History of an Idea, (The William James Lectures 1933).* Harvard Univ. Pr, Cambridge, Mass.

MOULINES, C. U. (1994). "Wer bestimmt was es gibt?" In: Zeitschrift für philosophische Forschung, Bd. 2 (175-196).

OTTE, M. (1994). *Das Formale, das Soziale und das Subjektive.* Frankfurt/M, Suhrkamp Verlag.

POTTER, M. (2000). *Reasons Nearest Kin.* Oxford University Press.

PUTNAM, H. (1967-1975). Mathematics without Foundations in ders. Mathematics, Matter and Method, CUP Cambridge

QUINE, W. V. (1960). *Word and Object.* Cambridge, The Mit Press.

RORTY, R. (1979). *Philosophy and the Mirror of Nature.* Princeton University Press.

RUSSELL, B. (1954). *The Analysis of Matter.* London, Allen & Unwin.

______ (1967). *The Autobiography of Bertrand Russell.* London, Allen & Unwin.

______ (1970). *Introduction to Mathematical Philosophy.* London, Allen & Unwin.

______ (1971). *Logic and Knowledge: Essays 1901-1950.* MARSH, R. C. (ed.). London, Allen & Unwin.

______ (1971a). *The ABC of Relativity.* London, Allen & Unwin.

______ (1973). *Essays in Analysis.* LACKEY, D. (ed.), London, Allen & Unwin.

______ (1985). *The Philosophy of Logical Atomism.* La Salle Open Court Publ. Co.

______ (1997). *The Problems of Philosophy.* Oxford University Press.

SALMON, W. C. (ed.) (1970). *Zeno's paradoxes*. Indianapolis, Ind. Bobbs-Merrill.

SCHLICK, M. (1979). *Allgemeine Erkenntnislehre*. Frankfurt/M., Suhrkamp Verlag, 1. Aufl. 1925

STRUBECKER, K. (hgb.) (1972). *Geometrie*. Wissenschaftliche Buchgesellschaft, Darmstadt

WEN, L. (2001). Semantic Paradoxes as Equations. *Math. Intelligencer*, v. 23, pp. 43-49.

Recebido em abr./2000; aprovado em jun./2000

Can x=3 be the solution of an inequality? A study of Italian and Israeli students

PESSIA TSAMIR*
LUCIANA BAZZINI**

Abstract

This paper describes a study regarding Israeli and Italian students' solutions to algebraic inequalities. The findings presented here show similarities in students' correct and incorrect solutions, in both countries. Fischbein's notions of algorithmic, intuitive and formal knowledge are used to analyze the data. The findings indicate that students generally worked in an algorithmic manner, intuitively drawing analogies to the solutions of related equations. We conclude by suggesting some educational implications.

Key-words: inequalities resolutions; algorithmic knowledge; intuitive knowledge.

Resumo

Este artigo descreve um estudo que focaliza soluções dadas, por estudantes de Israel e da Itália, para inequações algébricas. Os resultados apresentados aqui mostram similaridades nas respostas corretas e, também, nas incorretas, nos dois países. As noções de Fischbein, sobre conhecimento algorítmico, intuitivo e formal, são usadas na análise de dados. Indicam que os estudantes usualmente trabalharam de um modo algorítmico, formando, intuitivamente, analogias com soluções de equações. Em conclusão sugerimos algumas implicações pedagógicas.

Palavras-chave: *resoluções de inequações; conhecimento algorítmico; conhecimento intuitivo.*

* Departamento de Ciência da Educação – Universidade de Tel Aviv. E-mail: pessia@post.tau.ac.il

** Professora da Universidade de Torino – Faculdade de Educação. Vice-presidente do PME. E-mail: luciana.bazzini@unito.it

Introduction

There is a wide call for using students' ways of thinking and their mistakes in teaching (e.g., NCTM, 2000). Even though, this recommendation seems to be "speaking for itself", any attempt to take it from theory to practice, shows how complex and demanding this is. Among the various prerequisites for such teaching, are familiarities with students' various correct and incorrect reactions to different related types of tasks, with possible reasons for students' errors, with different mathematical solutions to the given tasks and with available teaching approaches to be considered under specific circumstances. All of the above are needed, but do not guarantee that the student whom we taught would gain mathematical understanding. Thus, a critical reflection of the teaching process, i.e., planning, designing and carrying out instruction may contribute to a constructive use of students' ways of thinking when teaching.

May be it is needless to say that the teacher should be well acquainted with the relevant mathematical issues. As it is the students' conceptions and errors that are to be considered, there is naturally a need to have the

Inequalities play an important role in mathematics. They are part of various mathematical topics including algebra, trigonometry, linear planning and the investigation of functions (e.g, Chakrabarti & Hamsapriye, 1997; Mahmood & Edwards, 1999). They also provide a complementary perspective to equations. Accordingly, the American Standards documents specify that all students in Grades 9-12 should learn to represent situations that involve equations, inequalities and matrices (NCTM, 1989). They further recommend that students would "understand the meaning of equivalent forms of expressions, equations, inequalities and systems of equations and solve them with fluency" (NCTM, 2000, p. 296). To implement these NCTM recommendations it is crucial to consider students' ways of thinking about inequalities.

However, so far, research in mathematics education has paid only little attention to students' conceptions of inequalities (e.g., Dreyfus & Eisenberg, 1985; Linchevski & Sfard, 1991; Tsamir & Almog, in press). Most of the related articles dealt with teachers' and researchers' suggestions for instructional approaches, usually with no research support. They recommended, for instance, the sign-chart method (e.g., Dobbs &

Peterson, 1991), the number-line method (e.g., McLaurin, 1985; Parish, 1992), and various versions of the graphic method (e.g., Dreyfus & Eisenberg, 1985; Parish, 1992; Vandyk, 1990).

Those few studies, which have been published, tended to describe students' reactions to a few inequalities of the types commonly presented in class, and usually reported only one or two difficulties. For instance, studies pointed to students' tendency to make invalid connections between the solution of a quadratic equation and its related inequality (e.g., Linchevski & Sfard, 1991; Tsamir, Tirosh, & Almog, 1998). Other studies related to students' tendency to regard transformable inequalities as being equivalent. They further identified the need to use logical connectives (Parish, 1992), and found the solutions of inequalities with "R" or "f" results extremely difficult (Tsamir & Almog, 1999).

The present study was designed in order to extend the existing body of knowledge regarding students' ways of thinking and their difficulties when solving various types of algebraic inequalities. During discussions at conferences for the psychology of mathematics education (PME22, 1998; PME23, 1999), it was found that in both Italy and Israel, algebraic inequalities usually receive relatively little attention and are commonly discussed only with mathematics majors in the upper grades of secondary school. Discussions are usually limited, emphasizing the "practical" algorithmic perspective of algebraic manipulations. Attention is usually paid mainly to providing students with rules for solving, with no relation to "Why solve it this way?" "Are there additional ways to solve it?" or "How can I be sure that the solution I have reached is the correct solution?" Moreover, in both countries, the two researchers witnessed students' and teachers' frustration with the difficulties encountered when dealing with inequalities. Consequently, an Italian-Israeli collaborative study was designed to investigate students' ways of solving standard and non-standard tasks with similar, underlying mathematical ideas. The students were given six tasks, presented in the manner to which they were accustomed in their classes, i.e., "solve" tasks, designated as "standard tasks". They were also given nine tasks, related to the same mathematical issues, which were presented in a non-customary manner, and designated as "non-standard tasks".

In this paper we focus on 2 of the 15 tasks that were give to the students. Both tasks dealt with the same underlying mathematical situation, i.e., single-value solutions to inequality tasks. The main related research question was: Do Israeli and Italian secondary school students accept the expression x = a as the solution of an inequality – Once, presented in a standard multiple-choice "solve" task, and once as a "reversed order" task, asking whether a given set can be the truth sets (the solution) of any equation or of any inequality, and are the students' reactions to the two tasks consistent?

Methodology

Participants

One-hundred-and-seventy Italian high school students and 148 Israeli high school students participated in this study. Both the Italian and the Israeli participants were 16-17 year old mathematics majors. That is, in both countries we examined students who were aiming to take final mathematics examinations in high school. Success in these examinations is a condition for acceptance to academic institutions, such as universities.

In their previous algebra studies, the participating students had studied the topic of algebraic inequalities, including linear, quadratic, rational and absolute value inequalities. In both countries, the participating students were taught this topic in a conventional way, being presented with different methods for solving the different types of inequalities. For example, parabolas or the number line to solve quadratic inequalities, and "multiplying by the square of the denominator" for the solutions of rational inequalities.

Tools

A 15-task questionnaire was administered in both countries. Italian and Hebrew versions were given to the Italian and Israeli students respectively. The two tasks analyzed here are Task 1 (a non-standard task) and Task 9 (a standard task).

Task 1

Consider the set S= {xÎR: x=3} and check the following statement:

S can be the solution of both an equation and an inequality.

Explain your answer.

Task 9

Indicate which of the following is the truth set of $5x^4 \leq 0$

$A=\{x: x>0\}$ $\quad$ $B=R$ $\quad$ $C=\{x: x<-5\}$ $\quad$ $D=\{x: 0<x<1/5\}$

$E=\varnothing$ $\quad$ F $x=0$ $\quad$ $G=\{x: x\leq 0\}$

Task 1 demanded proving the existence of a case where x=3 is the solution of an equation, and also proving the existence of a case where x=3 is the solution of an inequality. The easiest way to go about this was by providing suitable examples. This kind of assignment, asking the students to examine the existence of a case where x=3 is the solution of either an equation or an inequality; then, if possible, to provide tasks to match a given solution, was not dealt with in either the Israeli or the Italian classes we investigated.

We expected the first part that related to the existence of a suitable equation to be easy, and the second part, where the students had to examine the existence of a case where x=3 is the solution of an inequality, to be problematic.

Task 9 was a standard task, similar to other tasks presented in Israeli and Italian classes. We assumed that most students would solve it correctly.

Procedure

In both countries, the mathematics teachers of the classes distributed the questionnaires, during mathematics lessons. The students in each of the countries were given approximately one hour to complete their solutions, which usually was enough time. The researchers analysed, categorised and summarised the different solutions. In two additional meetings the researchers decided on possible ways to present the data.

Results

The results will be presented in the following order. First, an analysis of Israeli and Italian students' responses to Task 1, then their responses to Task 9, to conclude with an analysis of the consistency in students' reactions to the two tasks.

Students' Reactions to Task 1

In both countries, none of the students had any problems in correctly responding that x = 3 can be the solution of an equation. Most of them accompanied their responses by an example, usually of a first-degree equation, such as 2x-6=0. This, however, was not the case with the participants' responses to the question whether x = 3 can be the solution of an inequality, in both Israel and Italy.

Table 1 – Frequencies of students' solutions and justifications to Task 1 (%)

	ISRAEL N=147	ITALY N=150
TRUE*	51.4	48.3
Valid explanation	5.4	2.0
A system of inequalities	15.5	0.7
X=3 belongs to the solution	3.3	3.0
Other**	27.2	42.5
FALSE	48.6	51.7
A solution of inequality is an inequality	19.5	22.0
Other**	29.1	29.7

* Correct response

** Irrelevant or missing justifications

Table 1 shows that in both countries, only about 50% of the students who responded to this task, correctly claimed that x = 3 can be the solution of an inequality. Still, most of them did not accompany their claims by any justification and only a few students, Israeli or Italian, gave valid explanations. These latter explanations were usually the presentation of the following example of the quadratic inequality (x-3) ²£ 0. More prevalently in Israel, but also in a few Italian cases, explained that the claim "x = 3 can be the solution of an inequality" is true, because

$x = 3$ can be the solution of *a system* of inequalities. Such justifications were often accompanied by an uncomplicated, linear example, such as

$$\begin{cases} 2x-6 \leq 0 \\ x-3 \geq 0 \end{cases}$$

Another type of interesting justification, given by a small number of Israeli and a small number of Italian participants was that "the claim is true, because $x = 3$ can belong to the set of solutions of an inequality." This justification was accompanied by illustrations, such as, $5x - 10 > 0$, further explaining that "the truth set (or solution) of this inequality is $\{x: x>2\}$, and 3 is one of the values that satisfies this condition, and therefore $x = 3$ belongs to the truth set of $5x - 10 > 0$."

Students' Reactions to Task 9

Only about 50% of both the Israeli and the Italian participants who responded to this task, correctly identified $x = 0$ as the solution of the inequality (see Table 2).

Table 2 – Frequencies of students' solutions to Task 9 (in %)

	ISRAEL N=128	ITALY N=168
$X = 0$*	53.3	51.2
$X \leq 0$	23.8	16.2
Phi	17.1	26.7
Other	5.8	0.9

* Correct solution

A substantial number of the participants claimed that the set of solutions was empty (Phi, or 'there is no solution to the given inequality'). Some of them volunteered the explanation that x^4 has an even power and thus it can never be negative, showing that they ignored the "zero-option". Another interesting phenomenon was the Israeli and Italian students' tendency to answer that the set of solutions of $5x^4 \leq 0$ was $x \leq 0$, which was further explained by a number of them, claiming, for instance, "I simply computed the fourth root of both sides of the inequality."

Examining the consistency in students' reactions to Tasks 1 and 9

As can be seen from Tables 1 and 2, and as mentioned before, about half of the participants from each of the two countries claimed that "$x = 3$ can be the solution of an inequality", and about half of the participants identified $x = 0$ as the solution of $5x^4 \leq 0$. That is, about half of the participating students pointed to the possibility of having $x = a$ as the solution of an inequality, either in Task 1 or in Task 9. A question that naturally arose was, were these the same students? That is to say, did the students consistently express their understanding that $x = a$ could be the solution of an inequality in their reactions to both tasks, by responding "true" to Task 1 and "$x = 0$" to Task 9? Table 3 shows that the answer to this question is no.

Table 3 – Frequencies of consistent and inconsistent reactions to Tasks 1 & 9 (in %)

		ISRAEL N=148	ITALY N=170
CONSISTENT		57.8	48.3
Task 1	*Task 9*		
True	Correct	29.36	23.9
False	Incorrect	28.44	24.4
INCONSISTENT		35.76	38.2
Task1	*Task 9*		
False	Correct	22	21
True	Incorrect	13.76	17.2
OTHER*		6.44	13.4

* Providing no response to at least one of the two tasks.

Only about 29% of the Israeli participants and about 24% of the Italian participants exhibited a general view that $x = a$ can be the solution of an inequality and also correctly reached this type of a solution in reaction to the "solve" drill in Task 9. It is also notable that a similar percentage of each group rejected the option of $x = a$ being the solution of an inequality, and did not reach the correct $x = 0$ solution in Task 9 as well.

More than 35% of the participants in each country were inconsistent in their reactions to the two tasks. Part of them correctly claimed that $x = 3$ could be the solution of an inequality, but did not identify $x = 0$ as the solution of the inequality in Task 9. More interesting were the inconsistent reactions of about 20% of both the Israeli and the Italian participants. On the one hand, they claimed that $x = 3$ can not be the solution of an inequality, usually explaining that "an inequality can only be the solution of an inequality". On the other hand, within the same questionnaire they reached an $x = 0$ solution to the inequality presented in Task 9.

Discussion

Our findings indicate that, as expected, all students in both countries were aware that $x = 3$ can be the solution of an equation, and that many of them encountered difficulties in identifying the possibility of $x = 3$ being the solution of an inequality. These findings can be examined by means of the Intuitive Rules Theory, formulated by Stavy and Tirosh (2000). Students expressed the views that "an equation-result can only be the solution of an equation task" or that "an inequality task must have an inequality-solution." These claims are in line with the intuitive rule Same A (equation / inequality relationship in the solution) – same B (equation / inequality relationship in the task).

Quite surprising were the findings showing students' difficulties in responding to the standard "solve" task. In both countries only about half of the participating students identified $x = 0$ as the solution of the inequality $5x^4 \leq 0$. It seems that, similar to previous studies, reporting "strange" solutions like Phi and R as problematic for students (Tsamir & Almog, 1999), this study identified that the $x = a$ type of solution is also problematic in cases of inequality-tasks, and should further be investigated. A wider analysis of students' reactions to this task, embedded in different theoretical frameworks (e.g., Fischbein, 1987; Arzrello, Bazzini, & Chiappini, 1993; Bazzini, 2000; Maurel & Sakur, 1998) will be provided in the oral presentation.

Most interesting were the findings related to the consistency of students' reactions to the two tasks. We should remember that both tasks were included in the same questionnaire and students were free to move back and forth among the different tasks. In this manner, students' correct solutions to Task 9 could have served as an example for correctly solving Task 1. Still, no student explicitly mentioned Task 9 when correctly responding to Task 1. Furthermore, a non-negligible number of students

(Italian and Israeli) responded to tasks 1 and 9 in a contradictory manner. They wrote, "an inequality can only be the solution of an inequality" (Task 1) and then, that $x = 0$ was the solution of the inequality $5x^4 \leq 0$ (Task 9). A possible explanation for this phenomenon could be that sometimes zero is regarded as a special number. Thus, students could accept $x = 0$, but reject $x = a$, when 'a' is other than zero, from being a solution of an inequality. Naturally, further research is needed to investigate such assumptions.

Fischbein Moreover, our findings call for interventions that deal with the specific issue of algebraic inequalities and with the general issue of consistency in mathematical reasoning. Questions that arise are, for instance, how to introduce inequalities? How to cope with inconsistencies in students' reactions to inequalities? and how to validate the correctness of specific solutions to inequalities? Suggestions for research based instruction will be presented and discussed in the oral presentation. Clearly, the impact of such interventions should be further investigated.

References

ARZRELLO, F.; BAZZINI, L. and CHIAPPINI, G. (1993). Cognitive processes in algebraic thinking: Toards a theoretical framework. Tsukuba, Japan. *Proceedings of PME 17*, v. I, pp. 138-145.

BAZZINI, L. (2000). *Cognitive processes in algebraic thinking and implications for teaching*. Paper presented at ICME 9. Tokyo, Japan.

CHAKRABARTI, A. and HAMSAPRIYE, A. (1997). On corrected summation of alternating series. *International Journal of Mathematical Education in Science and Technology*, v. 27, n. 4, pp. 565-582.

DOBBS, D. and PETERSON, J. (1991). The sign-chart method for solving inequalities. *Mathematics Teacher*, n. 84, pp. 657-664.

DREYFUS, T. and EISENBERG, T. (1985). A graphical approach to solving inequalities. *School Science and Mathematics*, n. 85, pp. 651-662.

FISCHBEIN, E. (1987). *Intuition in Science and Mathematics – An Educational Approach*. Reidel.

LINCHEVSKI, L. and SFARD, A. (1991). Rules without reasons as processes without objects – The case of equations and inequalities. Assisi, Italy. *Proceedings of PME 15*, v. II, pp. 317-324.

MAHMOOD, M. and EDWARDS, P. (1999). Some inequalities and sequences converging to e. *International Journal of Mathematical Education in Science and Technology*, v. 30, n. 3, pp. 430-434.

MAUREL, M. and SAKUR, C. (1998). Les inéquations en classe de seconde et l'expérience de la nécessité. Nice. *Proceedings of SFIDA XI*, pp. 35-46.

McLAURIN, S. C. (1985). A unified way to teach the solution of inequalities. *Mathematics Teacher*, n. 78, pp. 91-95.

NCTM - National Council of Teachers of Mathematics (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, Virginia.

NCTM - National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics*. Reston, Virginia.

PARISH, C.R. (1992). Inequalities, absolute value, and logical connectives. *Mathematics Teacher,* v. 85, pp. 756-757.

PIEZ, C. M. and VOXMAN, M. H. (1997). Multiple representations – using different perspective to form a clear picture. *Mathematics Teacher*, v. 90, n. 2, pp. 164-166.

PROJECT GROUP 1 (1999). "Algebra: Epistemology, cognition and new technologies". In: DROUHARD, J. P. and BELL, A. (coords.). *Proceedings of PME 23*. Haifa, Israel, v. I.

STAVY, R. and TIROSH, D. (2000). *How students (mis-)understand science and mathematics: Intuitive rules*. Teachers College Press, USA.

TSAMIR, P. and ALMOG, N. (1999). "No answer" as a problematic response: The case of inequalities. *Proceedings of PME 23*. Haifa, Israel, v. I.

TSAMIR, P.; TIROSH, D. and ALMOG, N. (1998). "Students" solutions of inequalities. *Proceedings of PME 22*. Stellenbosch, South Africa, v. VI, pp. 129-136.

VANDYK, R. P. (1990). Expressions, equations and inequalities. *Mathematics Teacher*, n. 83, pp. 41-42.

WORKING GROUP 2 (1998). "Algebra: Epistemology, cognition and new technologies". In: DROUHARD, J. P. and VALROSE, P. (coords.). *Proceedings of PME 22*. Stellenbosch, South Africa, v. I.

Recebido em maio/2000; aprovado em jul./2000

Propostas curriculares, planejamentos de ensino, práticas de classe e conhecimentos de alunos do ensino infantil sobre adição

MARIA CRISTINA SOUZA DE ALBUQUERQUE MARANHÃO*
MARIA SILVIA BRUMATTI SENTELHAS**

Resumo

Neste artigo, focalizamos o processo de passagem da contagem à adição, buscando entendimento de como o estudante de Educação Infantil constrói conhecimento sobre adição de números naturais e como o professor pode favorecer esse processo. Para tal, analisamos planejamentos de ensino, práticas de classe e uma proposta curricular municipal de ensino infantil de orientação construtivista, detectando certas inter-relações e diversas lacunas entre eles e, também, falta de atualização ante as propostas curriculares e os quadros teóricos recentes de mesma orientação. Analisamos também conhecimentos de estudantes da classe investigada, considerando, assim, algumas implicações pedagógicas. Usamos alguns elementos da dissertação de Sentelhas (2001) e acrescentamos outros, importantes para este estudo.

Palavras-chave: planos de ensino; práticas no ensino; passagem da contagem ao cálculo.

Abstract

In this article, we focused on the process of passage from counting to adding, aiming to understand how kindergarten students construct knowledge on addition of natural numbers and how the teacher can support this process. In order to achieve that, we analyzed constructivist teaching plans, classroom practices and a curricular proposal, detecting some inter-relations and several gaps among them, and also lack of modernization when compared to recent curricular proposals and theoretical references in the same orientation. We also analyzed the knowledge of students from the investigated class. Thus, certain pedagogical implications could be considered. We used some elements from Sentelhas (2001) and added others which we considered important for the present study.

Key-words: *teaching plans; teaching practices; passage from counting to adding.*

* Doutora em Psicologia da Educação Matemática, PUC-SP. E-mail: maranhao@uol.com.br

** Mestre em Educação Matemática, PUC-SP. E-mail: silviabs@osite.com.br

Introdução

Algumas conclusões dos trabalhos de Brzezinski e Garrido (2001) afirmam que a abordagem construtivista, que tem fundamentado grande parte das pesquisas sobre ensino de ciências e matemática, deixa abertas questões, que passam a ser objeto de estudos, relativas às diferenças entre *os conhecimentos cotidianos, trazidos pelo aluno, e a ciência ensinada nas escolas*. Consideram que ensinar passa a exigir do professor nova postura sobre a construção do conhecimento científico e exige também entendimento sobre como o estudante constrói o conhecimento e como o professor pode favorecer esse processo.

Essas pesquisadoras situam a urgência de trabalhos na orientação construtivista. Tendo em vista, também, a relativa carência de pesquisas brasileiras recentes, na Educação Infantil (Fiorentini, 2001), nessa orientação, no presente estudo realizamos uma investigação que focaliza *o entendimento de como o estudante constrói alguns conhecimentos sobre adição de números naturais e como o professor pode favorecer esse processo.*

A relevância dessa investigação se dá pelas implicações pedagógicas dela decorrentes, pois o estudo cada vez mais profundo e detalhado, entre alunos do ensino infantil, sobre conhecimentos desenvolvidos acerca de relações entre números e sobre a evolução de processos de contagem e de cálculo permite ao professor das séries iniciais ter indicações sobre quais conhecimentos poderá supor que seus alunos possuam e, além disso, ao do ensino infantil, entendimento de como os alunos constroem esses conhecimentos, proporcionando produção de melhores planejamentos de ensino.

Neste estudo visitamos a dissertação de mestrado de Sentelhas (2001), ora selecionando trechos pertinentes aos objetivos desta pesquisa, revisando-os, ora acrescentando elementos.

Naquele trabalho, foram analisadas, seqüencialmente, sem cruzamento de dados e sem a preocupação de discutir a atualidade, a proposta curricular de Educação Infantil da Rede Municipal de Santo André, planos de ensino (projetos escolares) de um setor, além do planejamento de ensino e certas práticas de uma classe dessa rede, de orientação construtivista. No presente trabalho, recuperamos alguns dados coletados naquela ocasião, mas que não tinham sido apresentados na referida dissertação, refizemos as análises e procedemos a seu cruzamento, *visando detectar inter-relações, ou lacunas, entre eles, bem como analisar sua atualização ante propostas curriculares e referenciais teóricos recentes de mesma orientação*.

Em Sentelhas (2001), foram investigados também alguns conhecimentos de alunos sobre números, antes e durante a aplicação de uma seqüência de ensino e aprendizagem. No presente trabalho, focalizamos *o processo de passagem da contagem à adição buscando entendimento de como o estudante de Educação Infantil constrói conhecimento sobre adição de números naturais e como o professor pode favorecer esse processo.* Utilizamos alguns dados daquele estudo e acrescentamos outros, relativos *ao processo de passagem da contagem à adição,* coletados na ocasião, mas que não tinham sido utilizados, por não ter sido esse o foco daquela investigação. Para aprofundamento das análises relativas a esse processo, utilizamos também novos referenciais teóricos e revisamos os utilizados naquele trabalho.

Pela discussão dos resultados à luz da revisão e dos novos referenciais, objetivamos também acrescentar algumas implicações pedagógicas àquela pesquisa.

Problemática e quadro teórico

Segundo Ermel (1991), trabalhos calcados na teoria de Piaget nos lembram que a conservação de quantidades é o preâmbulo para toda apresentação do número à criança. Segundo esse grupo de pesquisadores, o Movimento da Matemática Moderna (MMM) acentuou a importância das atividades pré-numéricas, como de classificação de objetos ou figuras, designando os conjuntos formados, ou de correspondência termo a termo entre conjuntos de objetos, alegando a necessidade de desenvolver um pensamento lógico e relacional antes de abordar o número. Uma das conseqüências, não desejada certamente, desses imperativos foi uma *linearização* das propostas de ensino, uma decomposição artificial do complexo em elementos simples. A questão que se colocou então foi saber se *a utilização dos números não poderia ser um meio de ajudar na formação da idéia de conservação de quantidades e das estruturas lógicas do pensamento.* Trabalhando antes sobre as *estruturas*, acaba-se por negligenciar as *funções* e esquece-se de que o saber, antes de se tornar autônomo, segue o desenvolvimento do saber fazer da ação e do pensamento, constituindo pouco a pouco as estruturas. É nessa idéia que repousa a introdução do termo *função social do número* – presente em propostas curriculares e em diversas obras internacionais dos anos 90. Segundo Ermal (ibid.), na série escolar CP[1], os

1 CP – Cours Préparatoire, corresponde ao ano escolar imediatamente anterior à primeira série do ensino fundamental.

números ganham sentido se servem para resolver problemas. Há duas funções do número, que os alunos podem reconhecer e utilizar. Uma delas é o número como memória, seja como "memória de quantidade", que permite ao aluno lembrar-se de uma quantidade sem que ela esteja presente e que corresponde ao aspecto cardinal do número, seja como "memória da posição na seqüência natural", que permite ao aluno lembrar-se do lugar que o número ocupa na seqüência numérica e que corresponde ao aspecto ordinal do número. Outra, é o número como possibilidade de antecipar resultados, para situações não presentes ou ainda não realizadas e sobre as quais dispõe-se de algumas informações que exigem o uso pelos alunos de procedimentos numéricos que envolvem cálculos ou contagem.

O Referencial Curricular Nacional para a Educação Infantil (1998), assim como fez Ermel (1991), na França, destaca, com base em pesquisas, certas práticas de ensino de números correntes no Brasil nos anos 70 e 80; interessa-nos citar que essas duas propostas curriculares apontam que é largamente privilegiada a construção do número como cardinal de uma classe de conjuntos eqüipotentes. As práticas de *contagem de objetos* são substituídas por correspondência entre elementos de conjuntos para comparação e equiparação de quantidades e seguidas de correspondência quantidade-número. As de memorização de seqüências numéricas são abandonadas.

Ermel (ibid.) cita ainda pesquisas que apontam prejuízos para a aprendizagem em numeração provocados por diversas daquelas práticas, propondo a utilização de outras, como as funções do número.

Do ponto de vista da psicologia cognitiva, ao enunciar a seqüência numérica, a criança pode situar-se em dois níveis diferentes:

a) no da simples recitação, em que diz as palavras que sabe que devem se suceder e, freqüentemente, pode se enganar. E, mesmo que saiba recitar, sem enganos, a seqüência dos n primeiros números, não significa que ela saiba contar objetos até n;

b) no da contagem propriamente dita, que implica em fazer acompanhar a recitação da seqüência numérica de gestos da mão e de movimentos dos olhos que mostram que a criança estabelece uma correspondência entre o conjunto contado e a seqüência numérica oral (Vergnaud, 1994, p. 81).

Quando o número é maior que nove, há a necessidade de, também, estabelecer a correspondência entre os agrupamentos realizados em um conjunto de objetos e a notação numérica. Além disso, Vergnaud considera que o que dá aos números sua característica essencial é a possibilidade de adicioná-los e de dar um sentido a essa adição.

Quanto à recitação, referida por Vergnaud no parágrafo anterior, Fayol (1996), citando Fuson, Richards e Briars, afirma que esses pesquisadores concluíram, em seus estudos, que, em experiências sucessivas, quando ocorre a recitação da seqüência numérica estimulada por questões do tipo *diga até quanto você sabe contar*, as respostas podem ser decompostas em três categorias: parte estável e convencional; parte estável e não convencional e parte nem estável nem convencional. Os autores nomearam de parte estável o intervalo da seqüência que a criança recitava quase sempre (ou sempre) em experiências sucessivas; chamaram de parte convencional o intervalo da seqüência recitado de modo ordenado e sem omissões.

Quanto à possibilidade de adicionar números, Fayol (1996) sintetiza diversos trabalhos de Baroody e Ginsburg, afirmando que esses autores fazem um importante balanço dos elementos das respostas de crianças relativas à passagem de atividades de enumeração de coleções ao cálculo mental. Distinguem cinco categorias de procedimentos de adição, que apresentamos a seguir, com redação e exemplos nossos:

a) *Adição concreta: contagem efetiva da totalidade dos elementos*.

Para resolver 2 + 4, as crianças representam, com seus dedos, ou com desenhos, 2, depois 4 e contam-nos desde 1, falando 1, 2; 3, 4, 5, 6.

b) *Adição mental*

b.1) Contar *tudo (desde 1) começando pelo primeiro termo*. O exemplo 2 + 4 é resolvido por intermédio de lembrança, falando, de um em um, 1, 2; 3, 4, 5, 6. Esse procedimento é mais sofisticado que (a) porque há simultaneamente aumento de um em um e conservação na lembrança do que já foi contado.

b.2) *Contar considerando o primeiro termo como já contado*. O exemplo 2 + 4 também é resolvido por intermédio de lembrança, mantendo o primeiro termo (2) como já contado, falando 3, 4, 5, 6. Esse processo alivia a carga de trabalho mental em relação ao (b.1) e é mais sofisticado porque considera o primeiro termo como já contado.

b.3) *Contar tudo (desde 1) começando pelo maior dos dois termos*. O exemplo 2 + 4 também é resolvido por intermédio de lembrança, começando pelo maior dos dois termos (4), falando **1, 2, 3, 4**, 5, 6. Esse procedimento reduz dois elementos na conservação da lembrança do já contado em relação ao (b.1) e menos sofisticado que (b.2), porque conta desde 1.

b.4) *Contar considerando o maior dos dois termos como já contado*. O exemplo 2 + 4 também é resolvido por intermédio de lembrança, considerando o primeiro termo como já contado (4), falando 5, 6.

Nesse último procedimento, a carga cognitiva pode ser diminuída encadeando os passos da contagem a partir do maior dos dois termos, em relação a (b.3). Segundo Fayol, (b.3) e (b.4) são mais sofisticados que (b.1) e (b.2) porque supõem uma *comutatividade em ação*, a que Vergnaud chama de *teorema em ação*.

Com o propósito de dar significado ao número e à sua escrita, Douady (1984) propõe um jogo denominado *Jogo do Alvo*, a alunos do CP. Esse jogo consiste em lançar dardos em um alvo comum, cujas regiões são assinaladas com os números 0, 3, 6 e 9. Na primeira fase, o jogo é individual e a proposta é classificar todos os jogadores pelo total de pontos obtidos em três lances. Na segunda fase, o jogo é realizado por equipes de quatro alunos, cada uma com 12 lances ao alvo, e a proposta é classificar todas as equipes pelo total de pontos obtidos.

Com o mesmo propósito de dar significado ao número e à sua escrita, Lerner e Sadovsky (1996) propõem um trabalho de exploração da escrita numérica, a fim de que o aluno reconheça as regularidades presentes na seqüência numérica natural. Essas pesquisadoras consideram que as crianças constroem hipóteses de escrita numérica com base nas regularidades que observam. A percepção dessas regularidades pode ser usada como apoio na leitura e escrita de números.

Realizaram entrevistas com crianças de cinco a oito anos, sob dois enfoques. O primeiro, centrado na *comparação de escritas numéricas*. O segundo, centrado na *escrita numérica produzida pelas crianças*.

Para observarem a *produção de escrita numérica pelas crianças*, as pesquisadoras solicitavam:

Pensem em um número muito alto e escrevam-no (Lerner e Sadovsky, 1996, p. 77).

Descreveram a produção de escritas de crianças das várias idades que investigaram, que consideraram não convencionais e interpretadas pelas autoras como correspondentes à numeração falada, isto é, ao escreverem o número trinta e quatro as crianças escreviam 304 e justificavam como sendo: o *trinta* e o *quatro*.

Douady apresentou alguns conhecimentos que os alunos tinham no início da aplicação do Jogo do Alvo. Eles sabiam enumerar objetos até 50, aproximadamente; adicionar números de zero a nove; *comparar dois* *números e, freqüentemente, justificar seus resultados, por cálculos aditivos*, mais precisamente, se $a < b$, eles calculavam c tal que $a + c = b$. As crianças entrevistadas por Lerner e Sadovsky apresentavam hipóteses de escrita numérica e de decomposições aditivas de números, *justificando as comparações numéricas pela posição do número na seqüência numérica natural*.

Ressaltamos que, enquanto os sujeitos pesquisados por Douady apresentavam justificativas calcadas no aspecto cardinal do número, *usando conhecimentos de adição*, os pesquisados por Lerner e Sadovsky apresentavam justificativas calcadas no aspecto ordinal do número, *não usando conhecimentos de adição*.

Nesse quadro e diante dos objetivos do presente trabalho, em relação a alunos de Educação Infantil, em série imediatamente anterior à primeira do Ensino Fundamental, questionamos:

- até que números recitam;
- quais procedimentos utilizam para resolver adições do tipo $a + b = c$, com $a, b \leq 9$, se apresentam-nos;
- quais hipóteses de leitura e escrita de números apresentam, se apresentam-nas;
- quais justificativas apresentam na comparação de números, se apresentam-nas.

Metodologia

Para respondermos aos objetivos da pesquisa e às questões anteriores, realizamos um estudo de caráter documental, a partir de elementos da dissertação de Sentelhas (2001) e de dados transcritos durante sua realização, os quais não tinham sido utilizados na ocasião.

Fase 1: Nesta fase, numa primeira etapa, analisamos o Projeto Pedagógico da Secretaria Municipal de Educação de Santo André de 1997, com implementações em 1998 e 1999.

Para conhecer o conteúdo de numeração desenvolvido na Educação Infantil de Santo André, numa segunda etapa desta fase foram analisados os projetos de todas as escolas de um setor do município, feitos pelas diretoras e professoras das escolas em discussão com as coordenadoras do setor. Nesse setor, havia trinta e duas professoras para a faixa etária de seis a sete anos, que foram consultadas a respeito do conteúdo que propunham em suas classes, entre as relacionadas nos projetos. Nessa etapa, recuperamos dados desses projetos que não tinham sido utilizados em Sentelhas (2001).

Fase 2: Nesta fase selecionamos os dados relevantes do trabalho de Sentelhas (2001) para o presente estudo, refazendo a análise sobre o conteúdo desenvolvido em uma classe em relação aos conteúdos desenvolvidos pelos professores do setor e ao Projeto Pedagógico da Secretaria Municipal de Educação de Santo André.

Esclarecemos que, em Sentelhas (2001), das escolas desse setor, foi escolhida uma em que a diretora permitiu acesso aos alunos no período de aula, liberou horários de trabalho pedagógico de uma professora para os encontros necessários e disponibilizou espaços e recursos da escola para a realização da pesquisa. Em contatos, gravados, realizados com a professora durante os meses de agosto, setembro e outubro de 1999, na escola, foram colhidas informações sobre o conteúdo desenvolvido em classe. As questões feitas à professora foram preparadas considerando os projetos das escolas e os conhecimentos necessários para aplicação do Jogo do Alvo em classe.

Fase 3: Nesta fase foram refeitas as análises de Sentelhas (ibid.) relativas a entrevistas com alunos da classe investigada, com o objetivo de obter respostas aos questionamentos do presente estudo à luz de novos referenciais teóricos. Foram utilizados dados escritos (em folha de papel), que tinham sido obtidos dos alunos pelas respostas às tarefas propostas, e, também, dados de intervenções orais e de conseqüentes respostas dos alunos no decorrer da solução das tarefas, que também estavam transcritos.

Acrescentamos que, em Sentelhas (ibid.), foram realizadas entrevistas individuais com doze alunos do grupo pesquisado, por indicação de sua professora, a nosso pedido, sendo quatro considerados fracos, quatro considerados médios e quatro considerados fortes. As entrevistas foram filmadas e gravadas na escola, durante o período de aula dos alunos, nos dias 27, 28, 29 e 30 de setembro de 1999, em uma sala especialmente cedida para esse fim.

Fase 4: Nesta fase, selecionamos três sessões de aplicação do jogo do alvo, de Sentelhas (ibid.), aquelas pertinentes aos objetivos do presente estudo. Esses objetivos nos conduziram também a buscar alguns dados que não tinham sido utilizados naquele trabalho, os quais foram recuperados das transcrições das sessões, que se deram com 23 alunos de seis a sete anos. As sessões, filmadas e acompanhadas por duas observadoras, ocorreram durante o período regular das aulas, nos dias 18, 22 e 25 de novembro de 1999, com duração média de cinqüenta minutos cada uma. Após a realização de cada sessão, observava-se a continuação da aula conduzida pela professora e houve oportunidade de discutir com ela o que tinha ocorrido na sala em termos de resultados obtidos e o que se deveria investir mais com os alunos, na sessão seguinte.

Resultados e análises

Proposta curricular e projetos escolares

A proposta curricular de Santo André

O Projeto Pedagógico da Secretaria Municipal de Educação de Santo André de 1997, com implementações em 1998 e 1999, apresenta uma concepção de planejamento que ultrapassa a visão de grade curricular ou de rol de conteúdos definidos com base apenas na prática do educador ou, ainda, "de pacotes fechados elaborados em gabinetes ou apenas em livros didáticos, de forma puramente mecânica" (Plano 1997, p. 23).

A proposta estabelece que se construa o projeto coletivo da escola para o qual: "os educadores devem estudar as características das classes e alunos e definir, coletivamente, os temas, conteúdos e estratégias a serem trabalhados" (Plano 1998, p. 10).

Em resumo, a proposta curricular do município fornece orientações gerais para a confecção dos planejamentos e propõe que cada escola elabore seu projeto de trabalho, definindo os temas e conteúdos a serem trabalhados.

Os projetos das escolas do setor 2

Nos projetos das escolas do setor, em relação ao ensino de matemática, em geral, são apresentadas listas de títulos chamados de *conteúdos,* seguidos de breves descrições de atividades a serem propostas em classe.

Sobre os conteúdos: A partir dos conteúdos, organizamos uma tabela, na qual mantivemos os títulos mais freqüentes entre os que encontramos. Em seguida, foram consultadas as 32 professoras do setor, sobre se desenvolviam (ou não) em classe os conteúdos relacionados na tabela. Os resultados são apresentados a seguir, na Tabela 1.

Tabela 1 – Conteúdos de numeração desenvolvidos no nível III[2] por professores de Educação Infantil do setor 2 do município de Santo André – 1999

Conteúdos	n_p / n_t *
Seqüência numérica até 10 – ordenação crescente e decrescente	32/ 32
Correspondência biunívoca até 10	14/ 32
Relação número-quantidade até 10	32/ 32
Adição e subtração de números até 9	32/ 32
Problemas envolvendo adição ou subtração de números até 9	22/ 32
Seqüência numérica até 31 – calendário	21/ 32

n_p corresponde ao número de professores que desenvolvem o conteúdo citado e n_t ao número total de professores consultados.

Sobre as atividades: Como afirmamos, os projetos continham breves descrições de atividades a serem propostas em classe. Essas atividades foram classificadas por nós segundo os títulos da Tabela 1.

No título *Seqüências numéricas até 10 – ordenação crescente e decrescente* classificaram-se as atividades nas quais os alunos ouvem as designações de números naturais até dez, na ordem natural, lêem-nos ou escrevem-nos. É priorizada a ordem crescente à decrescente. Não são propostas tarefas em que números de um certo intervalo numérico sejam fornecidos, para que os alunos organizem-nos na ordem natural crescente (ou decrescente). Assim, depreendemos que, em realidade, simplesmente o aluno toma contacto com a ordem numérica crescente (ou decrescente).

No título *Correspondência biunívoca até 10* classificaram-se atividades em que os alunos devem *ligar elementos de conjuntos para marcar os que tenham igual quantidade* de desenhos; não há referência à contagem ou a métodos de controle de contagem, como riscar o que já foi contado. São também propostas atividades de comparação de quantidades, através da correspondência biunívoca entre elementos de conjuntos de figuras desenhadas; não há referência à comparação numérica, pedindo justificativa da resposta.

2 Correspondente a alunos de 6 anos.

No título *Relação número-quantidade até 10* classificaram-se atividades em que os alunos devem escrever ou ligar o número correspondente a uma certa quantidade de desenhos agrupados e vice-versa.

No título *Adição e subtração de números até 9* classificaram-se atividades nas quais os alunos escrevem e calculam sentenças do tipo $a + b = c$ com a, b, c naturais e c até 10. Não há referência de jogos para esse conteúdo. Não há referência de atividades que promovam a passagem da contagem ao cálculo.

No título *Problemas envolvendo adição ou subtração de números até 9* classificaram-se problemas de enunciado curto, que se sugerem sejam lidos pelas professoras.

No título *Seqüências numéricas até 31 – calendário* classificou-se uma atividade de calendário com sugestão de que fossem usados em classe diariamente.

Não são propostas atividades de exploração de escritas de números maiores que 10, isto é, de hipóteses de leitura e escrita pela regularidade que os alunos possam perceber na seqüência numérica.

Os títulos e as atividades relativas a *Correspondência biunívoca até 10* e *Relação número-quantidade até 10* são apresentados nessa ordem e podem sugerir que a correspondência biunívoca deva preceder a relação número-quantidade. Não havendo referência à contagem, o processo da passagem da contagem ao cálculo parece ser considerado pouco importante ou desconhecido.

O título *Problemas envolvendo adição ou subtração de números até 9* figura nas listas de conteúdos depois do título *Adição e subtração de números até 9*. De fato, a descrição das atividades também sugere uma linearização seqüencial desses conteúdos. Isso pode induzir uma ordem no desenvolvimento curricular, pela qual os cálculos sejam ensinados antes dos problemas e desvinculados dos mesmos, conforme analisa Ermel (1991).

A partir das propostas *dos projetos escolares* e das consultas às professoras pudemos ter uma idéia *do que* é realizado em classe, mas não *como*. Isso justificou a análise do plano de ensino de ao menos uma professora do setor.

Atividades sobre números propostas pela professora dos alunos investigados

Essas atividades são apresentadas pela transcrição de trechos das conversas realizadas com a professora durante a entrevista livre. Nessa transcrição, usamos I para a investigadora e P para a professora.

I – *Quais atividades você costuma dar aos seus alunos como recurso para o ensino/aprendizagem de números?*

P – *Fazemos diariamente a contagem dos alunos presentes e costumamos jogar boliche e dado para que contem seus pontos. Cada aluno tem uma folha, representando o calendário do mês, que eles pintam diariamente.*

I – *Nesses jogos, eles marcam os pontos usando a escrita numérica?*

P – *Não, eles usam desenhos de bolinhas ou pauzinhos indicando os pontos obtidos. E eu nunca pedi que representassem usando a escrita numérica.*

I – *Nesses jogos, até que quantidade os alunos chegam a desenhar?*

P – *No boliche, usamos dez garrafas e eles marcam por partida. Nos jogos de dados, eles marcam os resultados de duas a três partidas.*

I – *Você costuma escrever números com diversos algarismos em suas atividades de sala de aula?*

P – *O maior número que escrevo com eles é o 1999, que usamos no cabeçalho. Na seqüência numérica, usamos até 31, no calendário.*

I – *Você já chegou a propor que os alunos escrevessem algum número que conhecessem, além dos que habitualmente usa em sala de aula?*

P – *Não, nunca fiz isso.*

I – *Seus alunos sabem fazer a adição de números menores que nove?*

P – *Fazemos algumas adições com tampinhas ou com desenhos. Estou começando a introduzir a adição.*

I – *Quando você dá um jogo no qual os jogadores acumulam pontos, os alunos sabem dizer quem ganhou?*

P – *Sabem.*

I – *E eles chegam a dizer com quantos pontos a mais ganharam?*

P – *Não sei se eles diriam isso. Esse tipo de questão, nunca fiz em nossos jogos.*

I – *Que tipo de atividade de comparação de quantidade você costuma propor a seus alunos?*

P – *Uma comparação que costumamos fazer é a de quantidade de meninas e quantidade de meninos que estão presentes e identificar os que faltaram. Propo-*

nho, também, algumas atividades com conjuntos de bolinhas, estrelinhas, etc., para marcarem onde tem mais, mas só uso de 1 a 9. Não é hábito meu questioná-los sobre como descobriram a resposta.

I – *Nessas atividades, eles usam a contagem ou outro procedimento?*

P – *... Quando a quantidade é pequena, menor que dez, não fazem mais a correspondência termo a termo, fazem a contagem.*

I – *Que tipo de atividade considera que faz bastante para o conhecimento de número?*

P – *As de relação número-quantidade, em que apresento um conjunto de objetos de um lado e números de outro para que relacionem o número à quantidade.*

I – *Em algum momento você chegou a propor atividades de comparação de dois números apresentados apenas por meio da sua escrita?*

P – *Não. Nunca propus algo assim para meus alunos.*

Na continuidade da entrevista com essa professora, ela afirmou que o grupo de alunos de sua classe compunha-se de 23 crianças de seis anos, os mais imaturos da escola, por tratar-se dos de menor faixa etária entre os que cursavam o nível III (série imediatamente anterior à primeira série do curso fundamental).

Sobre o relato da professora em face das atividades do planejamento escolar e dos conteúdos da Tabela 1.

a) No título *Seqüência numérica até 31 – calendário,* os alunos pintam uma folha que representa um calendário, diariamente. Assim, não obtivemos detalhes sobre se esse número é apenas identificado, lido ou também falado. O planejamento contendo apenas títulos e atividades tão concisos como o analisado não dá pistas sobre o que se pretende que seja ensinado, para termos uma base a respeito das inter-relações ou das lacunas entre ele e a prática em classe, nesse conteúdo. Mas o referencial teórico deste artigo nos dá base para ponderarmos que um calendário poderia ser utilizado em diversas atividades envolvendo *o processo de passagem da contagem à adição*, como as que devessem responder *quantos dias há em uma semana* ou *quantos dias há em duas semanas*.

b) As atividades de classe mais freqüentes são as dos títulos *Relação número-quantidade* e *Correspondência biunívoca até 10.* A comparação entre quantidades é proposta como parte do título *Correspondência biunívoca até 10*. O relato deixa transparecer que são feitas para números maiores que 10 e que essas atividades são vistas pela professora como precedentes à contagem.

c) Nos títulos *Adição e subtração de números até 9 e Problemas envolvendo adição ou subtração de números até 9,* ponderamos que apesar de a professora utilizar-se de jogos em classe, que requerem contagem de pontos e que podem envolver *adição (concreta ou mental)*, sendo que, por vezes, podem superar o número 10, não há explicitação sobre a relação entre essas atividades e a adição.

d) No título *Seqüência numérica até 10 – ordenação crescente e decrescente,* consideramos que não há atividade ou jogo no relato que requeira que os alunos coloquem números escolhidos pela professora na ordem numérica natural.

A nosso ver, essa entrevista confirmou vários aspectos de nossa análise dos planos do setor, acrescentando outros. Apesar das atividades de classe refletirem os itens do planejamento, em sua maior parte, queremos ressaltar que este último não faz menção a diversas práticas de classe.

O que consideramos mais importante dela foi o fato de ter revelado que havia preocupação da professora com o uso de tampinhas e de desenhos para marcar pontos em alguns jogos. Porém, essas atividades eram posteriores às de correspondência termo a termo, dado que a professora afirma que, para algumas quantidades, já não fazem mais a correspondência e, além disso, não eram relacionadas nem com escrita numérica e nem com adição, quando, pelo fato de serem feitas três rodadas dos referidos jogos, seria muito melhor, do ponto de vista didático, que envolvessem tanto a escrita numérica, para registro de pontos de cada rodada, como essa operação, para descoberta do vencedor ou para ordenação do número de pontos dos jogadores. Chamou-nos a atenção também o fato de a professora ter afirmado que estava *começando a introduzir a adição*. Pareceu-nos que poderia estar introduzindo-a por meio dos jogos referidos se a professora os relacionasse com essa operação. Aparentemente essa *introdução* é relacionada ao que está explicitado no plano de ensino.

Os procedimentos usados pelos alunos são diversos e incluem a contagem de objetos ou de desenhos, mas não obtivemos o nível de detalhe que perseguimos, sobre a passagem da contagem à adição. Obtivemos ainda que os alunos não estão acostumados a comparar números ou cálculos escritos nem a justificar suas produções e, além disso, que não há tratamento de hipóteses de leitura e de escrita de números maiores que 10. Isso nos conduziu à análise das entrevistas com os alunos.

As entrevistas com os alunos

Apresentamos a seguir cada tarefa proposta aos alunos, nas entrevistas, acompanhada de seu objetivo específico e dos resultados. Em transcrições de trechos de entrevistas usamos *I* para a investigadora e a inicial do nome de cada aluno.

Questão 1: Você sabe contar? Então conte até onde você sabe.

Um objetivo dessa questão foi saber até que número o aluno "recitava" a seqüência numérica sem auxílio, para ter dados para solicitar comparações, de modo que os alunos pudessem apoiar-se na seqüência numérica oral para dar a resposta.

Outro foi observar se, no momento em que o aluno interrompesse sua recitação, uma intervenção, que fornecesse o próximo número da seqüência, favoreceria a continuação da recitação, conforme sugere Ermel (1991). Um aluno que, ao recitar a seqüência numérica, pára no vinte e nove e continua a recitação, após ser auxiliado com a informação *trinta*, denota que está usando o fato de *trinta* permanecer enquanto varia a unidade de um a nove.

Resultados

Entre os doze alunos, um recitou a seqüência numérica natural até 10, dois recitaram-na até 24, seis recitaram-na até 29, dois recitaram-na até 39 e um, até 100.

Após intervenção do pesquisador, nove alunos continuaram as seqüências corretamente, conforme exemplo:

G – recita a seqüência até 29 e pára.

I – Vinte nove, trinta.

G – Trinta e um, trinta e dois, ..., trinta e nove.

Outros dois continuaram a seqüência, de um modo que conduziu a investigadora à intervenção prevista, como, por exemplo:

D – recita corretamente a seqüência até 24 e pára.

I – Vinte e quatro, vinte e cinco.

D – Trinta, trinta e um, trinta e dois, trinta e três, trinta e quatro, trinta e cinco, quarenta, vinte.

O aluno que contou até dez se recusou a continuar a seqüência, afirmando saber contar só até ali.

Questão 2: Veja todas essas canetas coloridas. Você poderia contá-las e me dizer quantas tem aí?

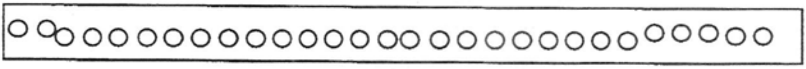
Para responderem a essa questão, os alunos dispunham de canetas coloridas. A quantidade fornecida a cada um dependeu da resposta dada à questão anterior, pois, segundo Vergnaud, para alunos dessa faixa etária, recitar a seqüência numérica até um número *n* não significa que eles contem até *n* objetos.

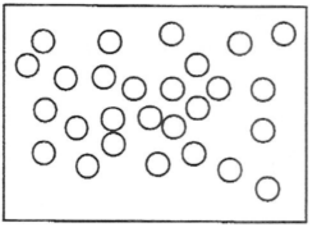
Nessa questão, havia interesse em observar o total de objetos que o aluno conta e o seu procedimento durante a contagem: (a) se havia sincronismo entre a recitação e o gesto da mão para efetuar contagem de objetos que se podem mover, pois tal procedimento fornece informação sobre o estabelecimento da correspondência um a um; (b) se os alunos alinhavam ou empilhavam os objetos contados com alguma organização que auxiliasse a contagem correta. Para nós, era possível que a falta de organização interferisse na contagem, levando o aluno a deixar de contar alguns objetos ou a contar mais de uma vez o mesmo objeto. O interesse na pesquisa sobre a questão se deu para uso em vários momentos em que fossem requeridos durante o jogo do alvo.

Resultados

Na contagem de objetos que se podem mover observamos que onze dos doze alunos estabeleceram sincronismo entre a fala e o gesto da mão, alinhando os objetos contados. Um aluno estabeleceu o sincronismo entre a fala e o gesto da mão apenas até o dez e empilhamento dos objetos contados.

Questão 3: Estas bolinhas foram desenhadas por uma criança. Você pode me dizer quantas bolinhas ela desenhou em cada quadro?





Para responderem a essa questão, os alunos dispunham de lápis, para que usassem, na contagem que iriam efetuar, caso necessitassem.

Nessa questão, também interessava observar os procedimentos usados para a realização das contagens pedidas: (a) se para efetuar a contagem estabeleciam a relação um a um pelo sincronismo entre a recitação e o gesto da mão; (b) se os alunos utilizavam alguma marca que auxiliasse a contagem assinalando, por exemplo, com o lápis fornecido, as bolinhas já contadas. Trata-se de situação diferente da anterior, por se referir à contagem de objetos desenhados e que, portanto, não se pode mover para separar os já contados.

O procedimento usado e o sucesso nessas contagens nos dariam pistas sobre a validade do emprego desse tipo de atividade em situação de aprendizagem que requeresse dos alunos alguma organização para controle de contagens já consideradas.

Resultados

Na contagem de objetos desenhados de modo alinhado, observamos que cinco dos doze alunos acertaram na contagem um a um, apenas apontando com o dedo os desenhos contados. Os outros sete erraram a contagem por pularem alguns dos desenhos, embora estivessem apontando com o dedo ou com um lápis conforme contavam.

Na contagem de objetos desenhados de modo não-alinhado apenas dois alunos acertaram na contagem, conforme exemplo:

L – conta um a um, primeiro, indicando com o dedo.

L – *Não..., espera* (pega o lápis, marca cada elemento contado e acerta).

Os outros doze alunos erraram na contagem, por deixarem de contar alguns ou por contarem duas vezes o mesmo desenho.

Questão 4: Você sabe escrever os números que você já sabe falar?
Escreva-os nesta folha.

Questão 5: Pense em um número muito alto e escreva-o
(na mesma folha em que responderam à questão anterior).

Questão 6: Como se lê esse número?
(Números apontados para leitura, em uma fita, contendo a seqüência numérica até 100: 10, 20, 30, 40, 50, 100, 11, 12, 13, 14, 19, 28, 31)

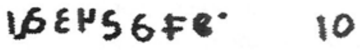
Nessas três questões, o conhecimento em jogo é o da escrita numérica e, por isso, as analisamos juntas.

Para a leitura de números, em fita numerada de 1 a 100, apontamos, inicialmente, os múltiplos de dez, acima listados, por serem esses números que Lerner e Sadovsky consideram os mais referidos pelas crianças para leitura e escrita de números. Além dos múltiplos de dez, interessou-nos saber se os alunos liam alguns números nos intervalos desses múltiplos. Assim, entre 10 e 20, indicamos os números 11, 12, 13 e 14, por serem números cuja denominação não apresenta qualquer similaridade com a escrita apresentada; 19, 28 e 31 por apresentarem similaridade com sua escrita. Se o aluno não reconhecesse algum dos números apontados, solicitávamos que fizesse a leitura de outro, menor, para termos informações sobre até que número reconhecia e qual o procedimento usado para realizar sua leitura.

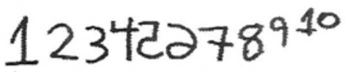
Em suas produções, queríamos verificar se faziam hipóteses sobre a escrita numérica, segundo Lerner e Sadovsky, e, para isso, fizemos a solicitação "desenhem um número bem alto", que foi a usada por estas pesquisadoras em suas entrevistas.

Resultados

A escrita da seqüência numérica até 10, de modo incompleto, foi feita por dois alunos, como no exemplo:



Sete alunos a apresentaram completa até 10:



Dois alunos a apresentaram completa até 19 e um aluno a apresentou completa até 100.

Apenas uma aluna apresentou hipótese de escrita, conforme Lerner e Sadovsky:

I – Você pode escrever um número bem alto?

A – Posso (escreve 1000).

I – E que número é esse?

A – O cem. Sei escrever o cento e um e o cento e dois também.

(A escreve 10001 e 10002)

I – E o duzentos como é que se escreve? (escreve 2000)

Nove alunos rejeitaram escrever números diferentes dos que haviam feito na questão quatro.

Dois alunos escreveram um numeral em tamanho grande:

I – Você pode escrever um número bem alto?

G – Escreva 10 bem grande na folha.

Na leitura de números em uma fita numérica, observamos que dois alunos recorreram à contagem para leitura de números menores que 10:

I – Você sabe que número é este? (indicando o 10)

S – Um zero.

I – Você sabe dizer o nome dele?

S – inicia contagem na fita desde o 1 até o 10.

I – Você sabe que número é este? (indicando o 8)

S – conta novamente desde o 1.

Seis alunos recorreram à contagem, desde o 1, para leitura de números entre 10 e 20, outros três se utilizaram desse recurso para a leitura de números entre 20 e 30 e apenas um aluno leu todos os números indicados.

Questão 7: Você acha que:

a) 6 é maior que 3? Como você sabe? Quanto maior?

b) 9 é maior que 6? Como você sabe? Quanto maior?

c) 12 é maior que 9? Como você sabe? Quanto maior?

Optamos pela comparação entre 3 e 6, considerando serem dois números sobre cujas quantidades todos os alunos poderiam ter controle, possibilitando respostas que evidenciassem o aspecto cardinal do número. A escolha de comparação entre o 9 e o 6 se deu pela semelhança de grafia e a de 12 e 9 para ultrapassarmos a faixa numérica que a professora relatou trabalhar com atividades que denominou relação número-quantidade.

Nossa intenção era verificar se os alunos comparavam números, considerando apenas os símbolos numéricos, e justificavam sua resposta, seja pela seqüência numérica, seja por conhecimentos de adição.

Resultados

Na comparação entre 3 e 6, dois alunos justificaram sua escolha com conhecimentos de adição, como no exemplo:

I – 6 é maior que 3?

R – É.

I – Como você sabe?

R – Porque 3 + 3 dá 6. O 3 é amigo do 6.

A justificativa pela seqüência numérica foi utilizada por 7 alunos, como mostra o exemplo:

I – 6 é maior que 3?

G – É.

I – Como você sabe?

G – Porque para contar até 6 passo do 3.

I – Quanto mais que o 3 ele tem?

G – Não sei.

Outros três alunos não souberam justificar sua escolha.

Na comparação entre 6 e 9, apenas um aluno justificou sua escolha com conhecimentos de adição:

I – Nove é maior que seis?

A – É.

I – Como você sabe?

A – Porque fiz a conta (conta nos dedos falando 7, 8, 9). *Tem mais três.*

Oito alunos justificaram pela seqüência numérica:

I – Nove é maior que seis?

R – É.

I – Como você sabe?

R – Quando conto para chegar no nove passo do seis.

I – Quanto mais que o seis ele tem?

R – *Dois* (apontando na fita os números 7 e 8).

Um aluno justificou pelo uso cultural:

I – Nove é maior que seis?

AC – É.

I – Como você sabe?

AC – Minha prima tem nove anos e eu tenho seis. Ela fala que é mais velha. Ela já passou do seis.

Dois alunos não souberam justificar.

Na comparação entre 9 e 12, um aluno justificou com conhecimentos de adição, cinco justificaram pela seqüência numérica:

I – Doze é maior que nove?

L – É.

I – Como você sabe?

L – Porque o doze vem depois do nove.

I – Quanto mais que o nove ele tem?

L – Esse e esse (Apontando na fita os números 10 e 11).

Questão 8: Você sabe quanto são:

a) 3 + 3? b) 3 + 6? c) 9 + 3? d) 6 + 6?

Optamos por propor adições envolvendo os números já tratados anteriormente na entrevista, de modo a podermos observar a estabilidade ou não do trato com a seqüência numérica ou com a quantidade que o número representa.

Nas entrevistas, a solicitação dessas adições *foi apenas oral, não sendo apresentadas aos alunos a sentença matemática equivalente.* Estávamos interessados em saber quais delas os alunos resolviam e quais os procedimentos usados. Para responderem a essa questão, eles poderiam utilizar-se de canetas ou de desenhos em uma folha que receberam.

Os números das questões foram escolhidos por propiciarem procedimentos variados de cálculo ou de contagem.

Resultados

Para a adição 3 + 3, quatro alunos usaram informação memorizada, seis alunos contaram nos dedos ou juntaram canetas, como no exemplo:

I – Você sabe quanto são 3 + 3?

E – É continha que você quer? (levanta três dedos em cada mão e conta-os desde o 1).

Dois alunos não resolveram o problema, afirmando *não sei fazer conta*. Tal afirmação não era esperada, tendo em vista os jogos efetuados previamente em classe. Isso nos intrigou, mesmo lembrando que a professora afirmou estar *introduzindo* a adição. Estariam esperando o ensino?

Na adição 3 + 6, um aluno deu o resultado usando informação memorizada, três alunos contam nos dedos ou juntam canetas, conforme exemplo:

I – Você sabe quanto são 3 + 6?

A – conta nos dedos, a partir de 7.

L – conta 6 canetas e, depois, junta outras, contando 7, 8, 9.

Cinco alunos não conseguiram realizar a contagem nos dedos, mas a realizaram com as canetas. Exemplo:

I – Você sabe quanto são 3 + 6?

E – Como vou colocar nos dedos? Não sei (levanta três dedos em uma mão e 4 na outra. Desiste desse procedimento e, em seguida, levanta 3 dedos em uma mão e 1 na outra).

I – Você pode pegar canetas, se quiser.

E – conta seis canetas, depois conta três, e reconta desde o um.

Três alunos afirmaram não saber fazer essa conta. Novamente nos perguntamos o porquê dessa recusa.

Na adição 9 + 3, quatro alunos contaram nos dedos ou juntaram canetas:

I – E 9 + 3, você sabe?

R – 12 (conta nos dedos a partir de 9).

Os outros oito alunos não resolveram o problema, alegando não saber essa conta. Pelas afirmações da professora na entrevista, esse resultado era esperado, pois os números ultrapassavam os que a professora vinha usando nos jogos efetuados, que envolviam, em grande parte, números até 10 e também os que usava no que chamou de *introdução da adição*.

Na adição 6 + 6, um aluno deu o resultado memorizado e três contaram nos dedos ou juntaram canetas:

P – E 6 + 6 quantos são?

AC – 12 (pega 6 canetas, contando uma a uma, depois pega mais 6. Reconta desde o um).

Os outros oito alunos não resolveram a adição, resultado já apresentado na adição anterior.

Realização das sessões em classe

A seguir, apresentamos as três primeiras sessões do estudo de Sentelhas (2001), acrescentando dados não utilizados naquele estudo, por focalizarmos aqui o processo de passagem da contagem à adição.

Na sessão1, introduzimos as regras da 1ª fase do Jogo do Alvo e o realizamos: cada jogador tem direito a três lançamentos ao alvo, demarcado com zonas de valor 3, 6 ou 9, como na Figura 1. O zero corresponde a não atingir uma das zonas do alvo. Vence o jogador que marcar o maior número de pontos. Ao final, em discussão coletiva, todos os jogadores são ordenados pelo número de pontos obtidos.

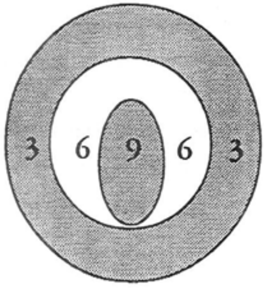


Figura 1

Os alunos realizaram lançamentos ao alvo e anotaram suas pontuações em papel; informaram a professora sobre o valor de cada pontuação obtida, compararam e ordenaram os resultados de todos os jogadores, em ordem decrescente. De início, alguns alunos escolheram o aluno que teve 9-0-0 como o de maior pontuação. Depois de questionamento da professora, escolheram os que apresentavam 6-6-0. Como muitos dos alunos não consideraram que os com pontuação 6-3-0 empatavam com os de 9-0-0, consideramos que comparavam cada um dos números obtidos nos lançamentos, sem usar a adição. Era, para nós, necessário investimento nessa operação. Isso estava de acordo com conhecimentos prévios para o jogo, apresentados por Douady (1984).

Quando a professora propôs que os alunos descobrissem o valor de 6 + 6, cinco alunos utilizaram-se dos dedos, iniciando a contagem a partir de 7, obtendo a resposta correta. Oito alunos utilizaram-se de desenhos (////// ou O O O), recontando todos, quatro deles desenharam 6 em uma linha e noutra linha, embaixo dos 6 primeiros, desenharam outro grupo de 6. Esses quatro alunos não perderam o controle sobre os desenhos contados. Os outros quatro dispuseram os desenhos em uma só linha e, ao recontarem, contaram alguns mais de uma vez ou deixaram de contar alguns, obtendo resultado incorreto. Dez alunos não realizaram a tarefa.

Ainda nessa sessão, os quatro alunos que haviam realizado a sobrecontagem nos dedos para a adição 6 + 6, na realização da adição 6 + 3 deram o resultado imediatamente. Três dos quatro alunos que haviam desenhado e acertado a contagem na adição 6 + 6, utilizaram-se dos dedos, esticando 6 e falando *seis*, depois, esticando um a um os outros

3, contando *sete, oito, nove.* Dez alunos utilizaram-se de desenhos, recontando todos, sendo que quatro deles confundiram-se na contagem. Seis alunos não realizaram a tarefa.

Na sessão 2, reinvestimos no jogo individual, para a familiarização com adições de duas parcelas envolvendo 0, 3, 6 e 9. Os alunos tinham que realizar novamente o jogo individual, com apenas dois lances; obter o total de seus pontos. Queríamos promover o uso de cálculos do tipo $a + b = c$, com $a, b \leq 9$, requeridos para a junção dos pontos obtidos no jogo. Na obtenção do total de seus pontos, os alunos representaram os pontos de cada um de seus lances para, em seguida, realizar a contagem geral. Os procedimentos observados foram os mesmos da sessão 1.

Na sessão 3 propusemos a realização da 2ª fase do Jogo do Alvo em equipes de quatro alunos. Cada aluno tinha direito a três lançamentos ao alvo. As equipes anotaram suas pontuações em papel e, em discussão coletiva, foram ordenadas pelo número de nove pontos obtidos. Durante a ordenação das equipes, com o grupo-classe, a adição foi proposta, mas os alunos não puderam realizar todas as adições envolvidas porque algumas tabelas continham demasiadas pontuações de números 9, 6 e 3.

Nessa sessão, a professora propôs novamente a adição 6 + 3, quando observamos seis alunos que responderam imediatamente de modo correto; doze alunos que utilizaram os dedos dando o resultado correto, quatro deles esticando 3 dedos após dizerem *seis*, falando em seguida *sete, oito, nove* e os outros esticando 6 dedos, sem contar, depois esticando outros 3, sem contar, recontando todos ao final. Três alunos não realizaram a tarefa. Na adição 6 + 6, os procedimentos foram similares, com exceção de quatro alunos que utilizaram desenhos, mas que passaram a fazer marcas nos desenhos contados, acertando a contagem.

Discussão dos resultados e implicações pedagógicas

Sobre os planos de ensino, a proposta curricular e as práticas em classe

O conteúdo de numeração planejado pela maioria das escolas é compatível com a proposta do município, uma vez que esta sugere que cada uma elabore, coletivamente, seu projeto de trabalho, definindo os temas e conteúdos a serem trabalhados. Essa proposta é bastante aberta, por clamar pela liberdade dos educadores das escolas e consideramos que tenta assegurar a coesão entre os diversos planejamentos de cada escola, pedindo sua feitura *coletiva*. Essa coesão é assegurada porque cada escola tem um projeto feito pela diretora e pelas professoras, discutido com as coordenadoras do setor. Disso resulta também a grande similaridade nos projetos das escolas desse setor. Os projetos para o ensino de matemática, em geral, são constituídos apenas de uma listagem de certos títulos, chamados de *conteúdos* propostos, seguidos de atividades sugeridas, sem orientações didáticas, o que pode favorecer lacunas entre o que se pretende que seja ensinado e o que é praticado em classe.

Comparando os itens do planejamento com o relato da professora entrevistada, constatamos a existência de certas similaridades e de diversas lacunas entre o que é explicitado no planejamento e o que é desenvolvido em classe. Pareceu-nos que a ausência de orientações didáticas nos planejamentos é responsável, em parte, pelas diferenças.

Com base em Ermel (1991), as atividades dos planos, na ordem em que figuram, seriam provenientes do Movimento da Matemática Moderna (MMM), calcadas na teoria de Piaget, pois incluem atividades pré-numéricas, sobre comparação de quantidades e sobre correspondência termo a termo entre elementos de conjuntos, antes de atividades que envolvam escritas numéricas, contagem ou cálculos aditivos. Esse grupo afirma que pesquisas dos anos 90 em diante vêm: a) apontando prejuízos para a aprendizagem em numeração causadas pelo emprego de diversas daquelas práticas utilizadas no MMM; b) propondo a utilização de práticas que envolvem escrita numérica, como o desenvolvimento de hipóteses de leitura e escrita de números, da comparação numérica com justificativa pelo aspecto cardinal ou pelo aspecto ordinal do número, das que promovem a adição pelo processo de contagem ao cálculo. Os resultados atestam que os planos não trazem orientações didáticas para as referidas práticas. Por essa análise, consideramos que seriam adequados cursos de formação de professores e de educadores nesse setor, visando à atualização dos projetos e das práticas.

Sobre as entrevistas com os alunos

Apresentamos as discussões, em grande parte, por grupos de questões, cruzando os dados para aprofundamento do que já foi analisado em cada questão. Usamos aqueles referenciais teóricos e introduzimos outros, visando à discussão dos resultados.

Na questão 1: Segundo Fayol (1996), a respeito de Fuson, Richards e Briars (1982), a questão feita aos alunos permite analisar se a seqüência recitada apresenta uma parte convencional e uma parte não convencional. Notamos que onze dos doze alunos apresentaram, na recitação da seqüência numérica, uma parte convencional até o vinte e quatro, sendo que se estende até trinta e nove para dois desses alunos e até cem para um deles. Os dois alunos que recitaram até 39, continuaram a seqüência, de um modo que nos conduz a dizer que, na recitação, apresentaram uma parte convencional e outra não convencional.

O fato de nove dos alunos continuarem corretamente a seqüência numérica oral (depois de pararem em trinta e nove ou vinte e nove), após intervenção em que se fornece o próximo número da seqüência (quarenta ou trinta), indica-nos a percepção da regularidade do sistema decimal, por parte desses alunos. Mais que isso, dá-nos indicações de intervenções pedagógicas eficientes para a evolução da memorização da seqüência numérica natural. Embora esse trabalho não se aprofunde no sistema de numeração decimal, esse resultado fornece indicações para a abordagem de seus rudimentos.

Nas questões 2 e 3: Comparando os procedimentos e os desempenhos dos alunos, nessas questões, temos indicação da maior dificuldade da questão 3 em relação à questão 2.

Na contagem de desenhos dispostos de maneira alinhada, cinco alunos fizeram a contagem correta, estabelecendo a correspondência um a um, e sete alunos deixaram de contar alguns, mesmo que apontando com um lápis a correspondência um a um não foi estabelecida. Quando os desenhos estavam dispostos de modo não-alinhado, dez alunos deixaram de contar alguns; contavam outros, duas ou mais vezes; apontavam com um dedo ou lápis, mas não marcavam os já contados.

Notamos que a organização dos elementos a serem contados tem, para uma grande parcela de alunos, influência sobre o controle da relação um a um. Além disso, como os alunos não apresentaram marcas para controle dos objetos já contados, indica-nos que em sala de aula podemos lançar à classe a necessidade de se estabelecer um modo para esse controle e desenvolver meios para tal.

Nas questões 4, 5 e 6: Comparando o desempenho dos alunos na questão 4 com o da questão 1, atesta-se que a parte convencional na recitação não corresponde ao que poderia ser chamado de parte convencional na escrita numérica, de 1 a 10, que foi produzida por todos os alunos.

Um aluno apresentou hipótese de escrita de números. Embora tivesse reconhecido o número 100 na fita numérica, ao escrevê-lo, colocou três zeros, escrevendo 10001 para cento e um e 10002 para cento e dois e apresentando, assim, alguma regularidade na hipótese de escrita desses números. Nenhum outro aluno apresentou hipóteses de escrita como as encontradas por Lerner e Sadovsky. Vale lembrar que esse tipo de proposta não é habitual por parte da professora, conforme entrevista com ela.

Notamos que houve interferência do uso cotidiano da palavra *alto* na produção de dois alunos, ao desenharem um número em tamanho grande na folha diante da solicitação "desenhem um número bem alto", a que Lerner e Sadovsky não se referem em suas pesquisas.

Para o reconhecimento de números, onze alunos recorreram à fita numérica e à contagem dos números desde o 1, sendo que três deles utilizaram-se desse recurso para números entre 20 e 30 e seis, para números entre 10 e 20. Isso nos mostra que esses alunos só conseguem fazer a leitura dos números vinculados à recitação da seqüência numérica desde o número 1.

Esses resultados nos conduziram a refletir sobre a importância do estudo de referenciais de pesquisa, como as citadas neste trabalho, para esses professores, porque há clara desvinculação entre o que essas pesquisas afirmam serem os alunos capazes de realizar, mesmo os que tenham dificuldades, e o que era proposto que realizassem na classe investigada, com prováveis implicações em seu desenvolvimento.

Na questão 7: Todos os alunos souberam dizer qual era o maior número, ao compararem 3 e 6; 6 e 9; 9 e 12. Queremos ressaltar que, em cada uma das comparações propostas, a parcela de alunos que utilizou a posição que o número ocupa na seqüência numérica natural (aspecto ordinal do número) foi bastante significativa. Sete alunos, ao compararem 3 e 6, e nove alunos, ao compararem 6 e 9. Isso nos sugere que as ações de sala de aula, de professores que se considerem construtivistas, devam levar em conta tal tendência dos alunos (seus conhecimentos prévios e modos de produzir conhecimento). Além disso, que se propiciem situações em que sejam levados a utilizar o aspecto cardinal do número para justificar comparações, pois, conforme Ermel (1991), esses dois aspectos devem ser desenvolvidos desde o ensino infantil, por serem indissociáveis.

Na questão 8: A maioria dos alunos acertou o resultado das adições 3 + 3 e 3 + 6: dez e nove, respectivamente. As adições 6 + 6 e 9 + 3 não foram resolvidas por oito dos doze alunos. Além disso, o procedimento usado pela maioria dos alunos que realizaram as adições foi o de recontagem de todos os dedos ou de todas as canetas após junção das duas quantidades a serem somadas.

Segundo Vergnaud, esses alunos não estão colocando em jogo a adição de dois números. Para ele, a adição só terá sua verdadeira significação quando, por exemplo, ao realizar a adição 3 + 3, o aluno retiver um três, em seus dedos ou com objetos, e, em seguida, contar quatro, cinco, seis.

Tendo em vista o processo de passagem da contagem à adição, segundo Fayol (1996), citando Baroody e Ginsburg, esses alunos usaram procedimentos da categoria *adição concreta,* em que há contagem efetiva da totalidade dos elementos, pois contaram um primeiro grupo de objetos, depois contaram um segundo grupo e, então, contaram todos.

Observamos ainda que a adição 3 + 3 apresentou-se como *adição mental* – de lembrança para quatro alunos que responderam imediatamente o resultado dessa adição. A adição 3 + 6 era dessa categoria para apenas um aluno e notamos que três alunos realizaram a *contagem considerando o maior dos termos como já contado*. Esse procedimento se manteve para esses três alunos na adição 9 + 3, mas mostrou-se instável para um deles, na adição 6 + 6, em que voltou à *contagem de todos, desde 1*.

Sobre as sessões em classe

Nos encontros, a maioria dos alunos fez a correspondência entre o número e a quantidade de figuras que ele representa. Consideramos que esse aspecto do significado do número não fora decorrente da seqüência elaborada por nós. Ponderamos, no entanto, que *a memória de quantidade ficou preservada* no decorrer da seqüência, embora tenhamos privilegiado o trato de números escritos.

Quando a professora propôs que os alunos descobrissem o valor de 6 + 6, cinco alunos utilizaram-se dos dedos, iniciando a contagem a partir de 7, obtendo a resposta correta. Ainda nessa sessão, esses mesmos alunos, na realização da adição 6 + 3, deram o resultado imediatamente. Esses alunos mostraram-nos que o uso dos procedimentos de *adição concreta e adição mental*, descritos por Baroody e Ginsburg, *dependem dos números utilizados*, o que não foi descrito por esses pesquisadores.

Na adição 6 + 6, quatro alunos utilizaram-se de desenhos, recontando todos. Desenharam 6 em uma linha e 6 noutra linha, embaixo dos 6 primeiros. Esses quatro alunos não perderam o controle sobre os desenhos contados. Na adição 6 + 3, três desses alunos utilizaram-se dos dedos, esticando 6 e falando *seis*, depois, esticando um a um os outros 3, contando *sete, oito, nove*. Esses três alunos mostraram uso de um procedimento parecido com *contagem considerando o maior dos termos como já* descrito por Baroody e Ginsburg. Mas, nesse autor, esse procedimento está na categoria *adição mental* e o que observamos foi uma mistura de lembrança do já contado com a contagem concreta, o que não foi contemplado naquelas categorias e que consideramos importante no estudo do processo de passagem da contagem ao cálculo.

Ainda, na adição 6 + 6, quatro alunos dispuseram os desenhos em uma só linha e, ao recontarem, contaram alguns mais de uma vez ou deixaram de contar alguns, obtendo resultado incorreto. Na adição 6 + 3, dez alunos utilizaram-se de desenhos alinhados, recontando todos, sendo que quatro deles confundiram-se na contagem.

O cruzamento dos resultados relativos aos desenhos, indicou-nos que a organização de desenhos emparelhados é um recurso interessante para ser utilizado em classes para melhor controle da contagem, o qual não tínhamos considerado em nossas análises anteriores.

Na adição 6 + 6, dez alunos não realizaram a tarefa. Na 6 + 3, seis alunos não realizaram a tarefa.

Na terceira sessão, quando a professora propôs novamente a adição 6 + 3, observamos seis alunos que responderam imediatamente de modo correto, estando na categoria *adição mental*; quatro alunos que esticaram 3 dedos após dizerem *seis*, falando em seguida *sete, oito, nove,* mostraram-nos que não poderiam ser encaixados nas categorias descritas por Baroody e Ginsburg e que usavam o mesmo procedimento misto de *lembrança do já contado* com *contagem concreta*, que aparecera na primeira sessão. Os oito que esticaram 6 dedos, sem contar, depois esticando outros 3, sem contar, recontando todos ao final apresentaram o procedimento *conta todos* da categoria *adição concreta*.

Na adição 6 + 6, os procedimentos foram similares, com exceção de quatro alunos, que utilizaram desenhos, mas passaram a fazer marcas nos desenhos contados, acertando a contagem. Esse procedimento reforça que nossas considerações sobre uso de marcas para o já contado é procedente para alunos do ensino infantil.

O que encontramos aqui foi a existência de uma categoria diferente das descritas por Baroody e Ginsburg, no processo de contagem ao cálculo, *que chamamos de sobrecontagem considerando o maior dos termos, envolvendo lembrança do já contado e adição concreta*. Esse procedimento representa uma conquista da criança nesse processo, por envolver lembrança do já contado, por revelar uso da comutatividade da adição e por trazer importante economia nos cálculos, tornando-os menos custosos. De fato, diversas pesquisas estrangeiras, como a de Lerner e Sadovsky (1996), citam *sobrecontagem* considerando-a como *adição mental*. Além disso, procedimentos envolvendo *adição concreta* e *lembrança do já contado,* foram observados em estudos brasileiros, como o de Mesquita (2001) e representam maior sofisticação que a *contagem efetiva de todos* e menor sofisticação que a *adição mental*. Portanto, provavelmente, podem ser detectados nas práticas de estudantes da Educação Infantil, em grupos diferentes do investigado.

Enfim, cruzando os dados das entrevistas com os alunos e sessões de classe com o relato da professora, observamos que entre seus alunos havia os que se encontravam adiantados no processo de passagem da contagem ao cálculo, embora fossem considerados imaturos. Essa percepção foi confirmada pela professora, que assistiu aos filmes das entrevistas, mostrando-se surpresa com desempenhos de seus alunos, não esperados por ela, ora além, ora aquém de suas expectativas. Mostrou interesse em ficar com as folhas de tarefa aplicada aos alunos e com cópia dos filmes, para estudo futuro, no que foi atendida.

O entrelaçamento das considerações em cada plano de estudo deste trabalho – as propostas curriculares e as pesquisas, o plano municipal, os projetos escolares, a entrevista com a professora sobre suas práticas, as entrevistas com os alunos, a seqüência realizada em classe – leva-nos a considerar que, apesar de haver possibilidade de serem desenvolvidos conhecimentos de adição de números entre os alunos investigados e de parte de esse trabalho ter mostrado isso – a falta de atualização de professores e de educadores do ensino infantil do setor pesquisado parece não favorecer a evolução adequada dos alunos relativa a esses conhecimentos. Conduz-nos a afirmar também a necessidade de atualização de professores e educadores no setor.

Bibliografia

BRASIL – Ministério da Educação e do Desporto (1998). *Referencial curricular nacional para a educação infantil: matemática.* Brasília, SEF.

BRZEZINZKI, I. e GARRIDO, E. (2001). Análise dos trabalhos do GT Formação de professores: o que revelam as pesquisas do período 1992-1998. Rio de Janeiro, Autores Associados. *Revista Brasileira de Educação*, n. 18, pp. 82-100.

DOUADY, R. (1984). *Jeux de cadres et dialectique outil-objet dans l'enseignemant des mathématiques.* These de doctorat d'etat. Universitè Paris VII, Paris.

ERMEL (1991). *Apprentissages numériques – CP*. Institut National de Recherche Pedagogique. Paris, Hatier.

FIORENTINI, D. (2001). Relação de teses e dissertações de mestrado e doutorado em Educação Matemática produzidas no Brasil nos anos de 1998 a 2001. Campinas, Cempem – FE/Unicamp. *Zetetiké*, v. 9, n. 15-16.

FAYOL, M. (1996). *A criança e o número: da contagem à resolução de problemas.* Porto Alegre, Artes Médicas.

LERNER, D. e SADOVSKY, P. (1996). "O sistema de numeração: um problema didático". In: PARRA, C. e SAIZ, I. (orgs.). *Didática da Matemática*. 1 ed. Porto Alegre, Artes Médicas.

MARANHÃO, M. C. (1999). "Dialética ferramenta objeto". In: MACHADO, S. (org.). *Educação Matemática: uma introdução*. 1 ed. São Paulo, Educ.

MESQUITA, M. (2001). *Pré-escola: um estudo a respeito da sobrecontagem na resolução de problemas aditivos.* Dissertação de mestrado. São Paulo, Pontifícia Universidade Católica.

SANTO ANDRÉ – Secretaria Municipal de Educação e Formação Profissional (1997). *Educação Inclusiva – Plano 1997: manual de orientação.* Santo André.

SANTO ANDRÉ – Secretaria Municipal de Educação e Formação Profissional (1998). *Estação Gente: Educação Inclusiva – Plano 1998: manual de orientação*. Santo André.

SANTO ANDRÉ – Secretaria Municipal de Educação e Formação Profissional (1999). *Estação Gente: Educação Inclusiva – Plano 1999: manual de orientação.* Santo André.

SENTELHAS, M. S. B. (2001). *Um estudo de numeração com alunos de 6 a 7 anos da rede municipal de Santo André.* Dissertação de mestrado. São Paulo, Pontifícia Universidade Católica.

VERGNAUD, G. (1994). *L'enfant, la mathématique et la réalité.* 5 ed. (1 ed. 1981). Berna, Peter Lang.

Recebido em mar./2000; aprovado em jun./2000

Dissertações defendidas no primeiro semestre de 2001*

BARRETO, I. M. A. *Problemas verbais multiplicativos de quarta-proporcional: a diversidade de procedimentos de resolução.*
Palavras-chave: quarta proporcional, problemas verbais, procedimentos pré-multiplicativos.
Key-words: proportion, word problems, pre-multiplicative procedures.

BEZERRA, F. J. B. *Introdução do conceito de número fracionário e de suas representações: uma abordagem criativa para a sala de aula.*
Palavras-chave: educação de jovens e adultos; ensino de matemática; plano de curso.
Key-words: education of adolescents and adults; mathematics teaching, teaching programs.

CARDOSO, E. A. *Uma análise da perspectiva do professor sobre o currículo de Matemática na EJA.*
Palavras-chave: educação de jovens e adultos; ensino de matemática; plano de curso.
Key-words: education of adolescents and adults; mathematics teaching, teaching programs.

CASTRO, S. C. *Os vetores do plano e do espaço e os registros de representação.*
Palavras-chave: ensino/aprendizagem; geometria analítica; vetor; representação semiótica.
Key-words: teaching/learning; analytic geometry; vector; semiotic representation.

---

* As dissertações completas estão no site: www.pucsp.br/pos/edmat.

ESTEVES, I. *Investigando os fatores que influenciam o raciocínio combinatório em adolescentes de 14 anos – 8ª série do Ensino Fundamental.*
Palavras-chave: formação de conceito; análise combinatória; ensino fundamental.
Key-words: combinatory analysis; school education; concept formation.

FÁBREGA, E. P. *Espaço representativo: um estudo das habilidades de alunos da 4ª série.*
Palavras-chave: representação; espaço tridimensional; ensino fundamental.
Key-words: representation; three dimensional space; secondary school teaching.

MEGA, E. *Ensino/Aprendizagem da rotação na 5ª série.*
Palavras-chave: isometria; estratégia de ensino; ensino fundamental.
Key-words: isometry; teaching strategy; secondary school teaching.

MESQUITA, M. M. B. *Pré-Escola: um estudo a respeito da sobrecontagem na resolução de problemas aditivos.*
Palavras-chave: sobrecontagem; problemas aditivos; pré-escola.
Key-words: counting on; additive problems; pre-school.

NUNES, M. N. F. *Seqüências Numéricas: Um estudo da convergência através de atividades.*
Palavras-chave: construtivismo de Piaget; licenciatura em matemática; limite; infinito.
Key-words: Piaget's constructivism; limit; infinit; pre-service teacher's graduation.

PERROTTA, S. G. M. *Reflexões sobre simetria em dois contextos: Matemática e Física.*
Palavras-chave: simetria axial; estratégias de ensino; ensino de matemática/física.
Key-words: axial symmetry; teaching strategies; teaching of mathematics and physics.

PEREIRA, M. R. O *A Geometria Escolar: uma análise dos estudos sobre o abandono de seu ensino.*

Palavras-chave: ensino de geometria; níveis fundamental e médio.

Key-words: geometry teaching; school mathematics.

PERROTTA, R. C. *O ensino de área e perímetro através de modelagem.*

Palavras-chave: seqüência de ensino; área e perímetro; materiais manipulativos.

Key-words: teaching sequence; area and perimeter; manipulative materials.

RIBEIRO, A. J. *Analisando o desempenho de alunos do Ensino Fundamental em Álgebra, com base em dados do Saresp.*

Palavras-chave: procedimentos de resolução; álgebra elementar; ensino fundamental.

Key-words: solution procedures; elementary algebra; school mathematics.

RODRIGUES, R. N. *Relação com o saber: um estudo sobre o sentido da matemática em escola pública.*

Palavras-chave: relação com o saber; sentido da aprendizagem em matemática; escola pública.

Key-words: relating to knowledge; meaning of mathematics learning; public school.

RODRIGUES, W. S. *Base dez: o grande tesouro matemático e sua aparente simplicidade.*

Palavras-chave: escrita numérica; ensino fundamental; proposta didática.

Key-words: written numbers; school mathematics; didactic proposal.

SENTELHAS, M. S. B. *Um estudo de numeração com alunos de 6 a 7 anos da rede Municipal de Santo André.*

Palavras-chave: significado de número; escrita numérica com dois algarismos; pré-escola.

Key-words: number meaning; pre-school; two-digit's numbers writting.

SILVEIRA, E. C. *Uma seqüência didática para aquisição/construção da noção de taxa de variação média de uma função.*

Palavras-chave: didática da Matemática, ensino de função, taxa de variação média.

Key-words: didactics of mathematics; teaching of functions; average rate of change.

Normas para publicação

Pesquisadores interessados em contribuir com publicação nesta revista deverão preparar o texto e enviá-lo segundo as regras que se seguem.

Preparação para envio – uma cópia do texto **em disquete(s)** com os nomes dos autores e sem numeração de página. Outras quatro (4) cópias **impressas**, sendo que uma deve ser idêntica à(s) do(s) disquete(s) e as outras três (3) devem ter numeração de página e não trazer os nomes dos autores.

Versão – programa Word 6.0 for Windows, para ser lido em PC.

Formatação

Título – centralizado, em letras maiúsculas e em negrito.

Nomes dos autores – em uma só das vias impressas e no disquete, separar os nomes dos autores do título por um espaço simples entre linhas. Os dados de cada autor deverão ser colocados conforme exemplo, abaixo do título.

Ex: Maria Dolores da Silva
Mestre em Educação Matemática – PUC-SP
Professora do Curso de Matemática – PUC-SP
e-mail: dolores@pucsp.br

Resumo – em português e inglês ou francês, com, no máximo, 10 linhas, espaço duplo, mesma fonte do texto, em itálico, acompanhado de três palavras-chave.

Corpo do texto – Papel tamanho A4
Margem superior e inferior com 2,5 cm
Margem direita e esquerda com 3,0 cm
Fonte Times New Roman,
Tamanho da letra 12 pontos.
Espaçamento entre linhas 1,5 linha
Alinhamento justificado

Referências bibliográficas – de acordo com as normas da ABNT em vigor.

Exemplos:

- Livro

  GOMES, L. G. F. (1998). *Novela e sociedade no Brasil.* Niterói, EdUFF. (Coleção Antropologia e Ciência Política, 15).

- Tese

  BARCELOS, M. F. P. (1998). *Ensaio tecnológico, bioquímico e sensorial de Soja e guandu enlatados no estádio verde e maturação de colheita*. Tese de doutorado em Nutrição, Campinas, Faculdade de Engenharia de Alimentos, Universidade Estadual de Campinas.

- Artigo de revista

  GURGEL, C. (1997). Reforma do Estado e segurança pública. *Política e Administração*, Rio de Janeiro, v. 3, n. 2, pp. 15-21, set.

Citações no texto – citações no texto devem vir acompanhadas de sobrenomes(s) do(s) autor(es) em corpo menor e entre parênteses, acrescido do ano de publicação e página.

Tabelas e gráficos – deverão ter como elementos: número, título, data de referência, fonte e nota.

Impressão – em jato de tinta ou em laser. Páginas impressas só numa face.

Os trabalhos devem ser enviados para:

Revista Educação Matemática Pesquisa
Programa de Estudos Pós-Graduados em Educação Matemática
Rua Marquês de Paranaguá, 111 - Consolação - SP - CEP 01303-050
Fone: (11) 3256-1622 ramal 202
Fax: (11) 3159-0189
e-mail: pgedmat@exatas.pucsp.br

Impressão de miolo e acabamento:

Gráfica da PUC-SP
Rua Ministro Godói, 965 – Perdizes – SP
Tel.: 3670-8366

SUMÁRIO